Method of lines for nonlinear first order partial functional differential equations

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Abstract

Classical solutions of initial problems for nonlinear functional differential equations of Hamilton–Jacobi type are approximated by solutions of associated differential difference systems. A method of quasilinearization is adopted. Sufficient conditions for the convergence of the method of lines and error estimates for approximate solutions are given. Nonlinear estimates of the Perron type with respect to functional variables for given operators are assumed. The proof of the stability of differential difference problems is based on a comparison technique. The results obtained here can be applied to differential integral problems and differential equations with deviated variables.

1 Introduction

The method of lines for evolution functional differential equations is obtained by replacing partial derivatives with respect to spatial variables with difference expressions. Differential equations contain functional variables which are elements of the set of continuous functions defined on subsets of a finite dimensional space. Then we need some interpolating operators. This leads to initial problems for systems of ordinary functional differential equations. Such obtained differential difference problems satisfy consistency condition on sufficiently regular solutions of original problems. The main question in these considerations

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is to find sufficient conditions for the stability of the numerical method of lines. Methods of differential inequalities and comparison techniques are used in the investigations of the stability.

There is an ample literature on the numerical method of lines for evolution differential or functional differential equations. The monographs [9], [10], [11], [20], [21], [22], [26] contain a large bibliography on theoretical investigations and applications.

The papers [5], [15] initiated a theory of the numerical method of lines for functional differential equations. Nonlinear parabolic functional differential equations with initial boundary value conditions were investigated in [13], [14], [16], [18], [27]. Results concerning the stability of the method of lines were obtained in these papers by using a comparison technique.

The papers [1], [2], [6], [7], [12], [28] concern equations with first order partial derivatives. Initial problems with solutions defined on the Haar pyramid and initial boundary value problems were considered. Error estimates implying the convergence of the method are obtained by using a method of differential inequalities. It is assumed that given operators satisfy nonlinear estimates of the Perron type with respect to functional variables.

The monograph [11] contains an exposition of the method of lines for hyperbolic functional differential problems.

The method is also treated as a tool for proving existence theorems for differential problems corresponding to parabolic equations [22] - [24] or hyperbolic problems [3], [4], [8], [17], [19].

The aim of the paper is to construct a method of lines for nonlinear first order partial functional differential equations with initial conditions and solutions defined on the Haar pyramid. Our results are based on the following idea. The original problem is transformed into a system of quasilinear functional differential equations for an unknown function and for their partial derivatives with respect to spatial variables. The numerical method of lines is constructed for systems such obtained.

All the results on the numerical method of lines given in [1], [2], [5] - [7], [12] - [14], [27], [28] have the following property. The authors have assumed that given operators satisfy the Lipschitz condition or satisfy nonlinear estimates of the Perron type with respect to functional variables and these conditions are global with respect to all variables. Our assumptions on regularity of given functions are more general. We construct estimates of solutions of initial problems for first order partial functional differential equations and solutions of differential difference systems. We assume that nonlinear estimates of the Perron type and suitable inequalities are local with respect to functional variables. It is clear that there are differential equations with deviated variables and differential integral equations such that local estimates of the Perron type hold and global inequalities are not satisfied.

We use in the paper general ideas for functional differential equations and inequalities which were introduced in [11], [25].

We formulate our functional differential problems. For any metric spaces X and Y, by $C(X,Y)$ we denote the class of all continuous functions from X into Y. We use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components.

Let E be the Haar pyramid

$$E = \{(t,x) \in \mathbb{R}^{1+n} : \ t \in [0,a], \ -b + Mt \le x \le b - Mt\}$$

where $a > 0$, $b, M \in \mathbb{R}^n_+$, $b = (b_1, \ldots, b_n)$, $M = (M_1, \ldots, M_n)$ and $b > Ma$. Write

$$E_0 = [-b_0, 0] \times [-b, b].$$

For $(t,x) \in E$ we define

$$D[t,x] = \{(\tau, y) \in \mathbb{R}^{1+n} : \ \tau \le 0, \ (t+\tau, x+y) \in E_0 \cup E\}.$$

Then the set $D[t,x]$ is a sum of the following sets

$$D_0[t,x] = \{(\tau,y) \in \mathbb{R}^{1+n} : \ -b_0 - t \le \tau \le -t, \ -x - b \le y \le -x + b\},$$

$$D_\star[t,x] = \{(\tau,y) \in \mathbb{R}^{1+n} : \ -t \le \tau \le 0, \\ -b - x + M(t+\tau) \le y \le b - x - M(t+\tau)\}.$$

Let $B = [-b_0 - a, 0] \times [-2b, 2b]$ then $D[t,x] \subset B$ for $(t,x) \in E$. For a function $z : E_0 \cup E \to \mathbb{R}$ and for a point $(t,x) \in E$ we define $z_{(t,x)} : D[t,x] \to \mathbb{R}$ by

$$z_{(t,x)}(\tau,y) = z(t+\tau, x+y), \ \ (\tau,y) \in D[t,x].$$

The function $z_{(t,x)}$ is the restriction of z to the set $(E_0 \cup E) \cap ([-b_0, t] \times \mathbb{R}^n)$ and this restriction is shifted to the set $D[t,x]$.

Let $\phi_0 : [0,a] \to \mathbb{R}$ and $\phi : E \to \mathbb{R}^n$, $\phi = (\phi_1, \ldots, \phi_n)$, be given functions. The requirements on ϕ_0 and ϕ are that $0 \le \phi_0 \le t$ for $t \in [0,a]$ and $(\phi_0(t), \phi(t,x)) \in E$ for $(t,x) \in E$. Write $\varphi(t,x) = (\phi_0(t), \phi(t,x))$ on E.

Put $\Omega = E \times C(B,\mathbb{R}) \times C(B,\mathbb{R}) \times \mathbb{R}^n$ and suppose that $f : \Omega \to \mathbb{R}$, $\psi : E_0 \to \mathbb{R}$ are given functions. We will say that f satisfies condition (V) if for each $(t,x,q) \in E \times \mathbb{R}^n$ and for v, $\widetilde{v}$, w, $\widetilde{w} \in C(B,\mathbb{R})$ such that $v(\tau,y) = \widetilde{v}(\tau,y)$ for $(\tau,y) \in D[t,x]$ and $w(\tau,y) = \widetilde{w}(\tau,y)$ for $(\tau,y) \in D[\varphi(t,x)]$ we have $f(t,x,v,w,q) = f(t,x,\widetilde{v},\widetilde{w},q)$. Note that the condition (V) means that the value of f at the point $(t,x,v,w,q) \in \Omega$ depends on (t,x,q) and on the restrictions of v and w to the sets $D[t,x]$ and $D[\varphi(t,x)]$ only.

Let z be an unknown function of the variables $(t,x) = (t, x_1, \ldots, x_n)$. We consider the functional differential equation

$$\partial_t z(t,x) = f(t,x,z_{(t,x)}, z_{\varphi(t,x)}, \partial_x z(t,x)), \tag{1}$$

with the initial condition

$$z(t,x) = \psi(t,x) \ \text{ on } \ E_0 \tag{2}$$

where $\partial_x z = (\partial_{x_1} z, \ldots, \partial_{x_n} z)$. In the paper we assume that f satisfies the condition (V) and we consider classical solutions of (1), (2).

Our concern is the method of lines for problem (1), (2). In the first step we construct a quasilinear system of functional differential equations for z and $u = \partial_x z$. We use a discretization with respect to spatial variable x for such obtained system. Then we associate with (1), (2) a net of Cauchy problems for ordinary functional differential equations. Solutions of such systems are considered as approximate solution of (1), (2). Then we estimate the difference between the exact and approximate solutions of (1), (2) and, as a consequence, we prove that approximate solutions converge to the classical solution of (1), (2). We present a complete convergence analysis for the method and we give numerical examples.

The paper is organized as follows. In Section 2 we formulate a numerical method of lines for (1), (2). In the next section we prove that there exists exactly one solution of the Cauchy problem for differential difference equations generated by (1), (2). We give estimates of solutions of (1), (2) and solutions of ordinary functional differential equations. A convergence result and an error estimate of approximate solutions are presented in Section 4. Examples are given in the last part of the paper.

2 Differential difference problems

We denote by $M_{n\times n}$ the class of all $n \times n$ matrices with real elements. If $U \in M_{n\times n}$ then U^T is the transpose matrix. For $x, y \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and $U \in M_{k\times n}$, $U = [u_{ij}]_{i,j=1,\ldots,n}$, we put

$$x \diamond y = (x_1 y_1, \ldots, x_n y_n), \quad \|x\| = \sum_{i=1}^{n} |x_i|,$$

$$\|x\|_\infty = \max\{|x_i| : 1 \le i \le n\}, \quad \|U\| = \max\{\sum_{j=1}^{n} |u_{ij}| : \ 1 \le i \le n\}.$$

We denote by $CL(B, \mathbb{R})$ the set of all linear and continuous real functions defined on $C(B, \mathbb{R})$ and by $\|\cdot\|_\star$ the norm in $CL(B, \mathbb{R})$ generated by the maximum norm in $C(B, \mathbb{R})$.

For each $(t, x) \in E$ we define the sets $I_0[t, x]$, $I_-[t, x]$, $I_+[t, x] \subset \{1, \ldots, n\}$ as follows:

$$-b_i + M_i t < x_i < b_i - M_i t \ \text{ for } \ i \in I_0[t, x],$$

$$x_i = -b_i + M_i t \ \text{ for } \ i \in I_-[t, x], \quad x_i = b_i - M_i t \ \text{ for } \ i \in I_+[t, x],$$

$$I_0[t, x] \cup I_-[t, x] \cup I_+[t, x] = \{1, \ldots, n\}, \quad I_-[t, x] \cap I_+[t, x] = \emptyset.$$

We need assumptions on φ, f and ψ.

Assumption $H[\varphi]$. The functions $\phi_0 : [0, a] \to \mathbb{R}$ and $\phi : E \to \mathbb{R}^n$, $\phi = (\phi_1, \ldots, \phi_n)$, are continuous and

1) $0 \le \phi_0 \le t$, for $t \in [0, a]$ and $\varphi(t, x) = (\phi_0(t, x), \phi(t, x))$ for $(t, x) \in E$,

2) partial derivatives $\partial_x \phi = [\partial_{x_i} \phi_j]_{i,j=1,\ldots,n}$ exist on E and $\partial_x \phi \in C(E, M_{n\times n})$,

3) $Q \in \mathbb{R}_+$ is defined by the relation $\|\partial_x \phi(t, x)\| \le Q$ on E.

Assumption $H[f,\psi]$. The function $f : \Omega \to \mathbb{R}$ of the variables (t,x,v,w,q), $x = (x_1,\dots,x_n)$, $q = (q_1,\dots,q_n)$, is continuous and satisfies the condition (V), moreover

1) the partial derivatives $(\partial_{x_1} f(P),\dots,\partial_{x_n} f(P)) = \partial_x f(P)$, $(\partial_{q_1} f(P),\dots,\partial_{q_n} f(P)) = \partial_q f(P)$ and the Fréchet derivatives $\partial_v f(P)$, $\partial_w f(P)$ exist for $P = (t,x,v,w,q) \in \Omega$,

2) $\partial_x f, \partial_q f \in C(\Omega,\mathbb{R}^n)$, $\partial_v f, \partial_w f \in CL(B,\mathbb{R})$,

3) the function $\partial_q f$ satisfies the conditions:

   (i) if $x \in [-b,b] \setminus [-b+Ma, b-Ma]$ and $(t,x,v,w,q) \in \Omega$ then

$$x \diamond \partial_q f(t,x,v,w,q) \le 0_{[n]} \tag{3}$$

   where $0_{[n]} = (0,\dots,0) \in \mathbb{R}^n$,

   (ii) if $x \in [-b+Ma, b-Ma]$ then the function

$$\text{sign}\, \partial_q f(\cdot,x,\cdot) : [0,a] \times C(B,\mathbb{R}) \times C(B,\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n, \tag{4}$$

$$\text{sign}\, \partial_q f(\cdot,x,\cdot) = (\text{sign}\, \partial_{q_1} f(\cdot,x,\cdot),\dots,\text{sign}\, \partial_{q_n} f(\cdot,x,\cdot)),$$

   is constant,

4) $\psi : E_0 \to \mathbb{R}$ is of class C^2.

We define a mesh on the set $E_0 \cup E$ with respect to the spatial variable. Let $h = (h_1,\dots,h_n)$, $h_i > 0$ for $1 \le i \le n$, stand for steps of the mesh. Let us denote by H the set of all h such that there is $K = (K_1,\dots,K_n) \in \mathbb{N}^n$ with the property $K \diamond h = b$. For $h \in H$ and for $m \in \mathbb{Z}^n$, $m = (m_1,\dots,m_n)$, we put

$$x^{(m)} = m \diamond h, \;\; x^{(m)} = (x_1^{(m_1)},\dots,x_n^{(m_n)}).$$

Write

$$R^{1+n}_{t.h} = \{(t,x^{(m)}) : \; t \in \mathbb{R}, \; m \in \mathbb{Z}^{1+n}\}$$

and

$$E_h = E \cap R^{1+n}_{t.h}, \;\; E_{h.0} = E_0 \cap R^{1+n}_{t.h}, \;\; B_h = B \cap R^{1+n}_{t.h}.$$

Elements of the set $E_{0.h} \cup E_h$ will be denoted by $(t,x^{(m)})$ or (t,x). By $\mathbb{F}_c(B_h,\mathbb{R})$ we denote the class of all $w : B_h \to \mathbb{R}$ such that $w(\cdot,x^{(m)}) \in C([-b_0-a,0],\mathbb{R})$ for $-K \le m \le K$. In a similarly way we define the space $\mathbb{F}_c(B_h,\mathbb{R}^n)$. For a functions $z : E_{0.h} \cup E_h \to \mathbb{R}$, $u : E_{0.h} \cup E_h \to \mathbb{R}^n$, $u = (u_1,\dots,u_n)$, we write $z^{(m)}(t) = z(t,x^{(m)})$, $u^{(m)}(t) = u(t,x^{(m)})$.

Suppose that Assumption $H[f,\psi]$ is satisfied. For $x^{(m)} \in (-b,b)$ we put

$$J_+[m] = \{i \in \{1,\dots,n\} : \partial_{q_i} f(\cdot,x^{(m)},\cdot) \ge 0\},$$

$$J_-[m] = \{1,\dots,n\} \setminus J_+[m].$$

We construct the numerical method for (1), (2). Write $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$ with 1 standing on the $i - th$ place. For functions $z : E_{0.h} \cup E_h \to \mathbb{R}$, $u : E_{0.h} \cup E_h \to \mathbb{R}^n$, $u = (u_1, \dots, u_n)$, we write

$$\delta_i z^{(m)}(t) = \frac{1}{h_i}[z^{(m+e_i)}(t) - z^{(m)}(t)] \text{ for } i \in J_+[m], \tag{5}$$

$$\delta_i z^{(m)}(t) = \frac{1}{h_i}[z^{(m)}(t) - z^{(m-e_i)}(t)] \text{ for } i \in J_-[m], \tag{6}$$

$$\delta_i u^{(m)}(t) = \frac{1}{h_i}[u^{(m+e_i)}(t) - u^{(m)}(t)] \text{ for } i \in J_+[m], \tag{7}$$

$$\delta_i u^{(m)}(t) = \frac{1}{h_i}[u^{(m)}(t) - u^{(m-e_i)}(t)] \text{ for } i \in J_-[m], \tag{8}$$

and we put $i = 1, \dots, n$ in above definitions. Set

$$\delta z^{(m)}(t) = (\delta_1 z^{(m)}(t), \dots, \delta_n z^{(m)}(t)), \;\; \delta u^{(m)}(t) = [\delta_j u_i^{(m)}(t)]_{i,j=1,\dots,n}.$$

Since equation (1) contains the functional variables $z_{(t,x)}$ and $z_{\varphi(t,x)}$ which are elements of the spaces $C(D[t,x], \mathbb{R})$ and $C(D[\varphi(t,x)], \mathbb{R})$ then we need an interpolating operator $T_h : \mathbb{F}_c(B_h, \mathbb{R}) \to C(B, \mathbb{R})$. For a simplicity we write $T_h z_{[t,m]}$ instead of $T_h z_{(t,x^{(m)})}$ and $T_h z_{\varphi[t,m]}$ instead of $T_h z_{\varphi(t,x^{(m)})}$ where $z : E_{0.h} \cup E_h \to \mathbb{R}$. Let us denote

$$P[z,u]^{(m)}(t) = (t, x^{(m)}, T_h z_{[t,m]}, T_h z_{\varphi[t,m]}, u^{(m)}(t)).$$

Write

$$F_h[z,u]^{(m)}(t) = f(P[z,u]^{(m)}(t)) + \sum_{i=1}^{n} \partial_{q_i} f(P[z,u]^{(m)}(t))(\delta_i z^{(m)}(t) - u_i^{(m)}(t))$$

and

$$\begin{aligned} G_h[z,u]^{(m)}(t) &= \partial_x f(P[z,u]^{(m)}(t)) + \partial_v f(P[z,u]^{(m)}(t)) \star T_h u_{[t,m]} \\ &+ \Big[\partial_w f(P[z,u]^{(m)}(t)) \star T_h u_{\varphi[t,m]}\Big] \partial_x \phi^{(m)}(t) + \partial_q f(P[z,u]^{(m)}(t)) \Big[\delta u^{(m)}(t)\Big]^T, \end{aligned}$$

where $G_h = (G_{h.1}, \dots, G_{h.n})$ and

$$\partial_v f(P) \star T_h u_{[t,m]} = \Big(\partial_v f(P) \star T_h (u_1)_{[t,m]}, \dots, \partial_v f(P) \star T_h (u_n)_{[t,m]}\Big)$$

and

$$\begin{aligned} &\Big[\partial_w f(P) \star T_h u_{\varphi[t,m]}\Big] \partial_x \phi^{(m)}(t) \\ &= \Big(\sum_{i=1}^{n} \partial_w (P) T_h (u_i)_{\varphi[t,m]} \partial_{x_1} \phi_i^{(m)}(t), \dots, \sum_{i=1}^{n} \partial_w (P) T_h (u_i)_{\varphi[t,m]} \partial_{x_n} \phi_i^{(m)}(t) \Big) \end{aligned}$$

where $P \in \Omega$. We consider the system of functional differential equations

$$\frac{d}{dt} z^{(m)}(t) = F_h[z,u]^{(m)}(t), \tag{9}$$

$$\frac{d}{dt}u^{(m)}(t) = G_h[z,u]^{(m)}(t) \tag{10}$$

with initial conditions

$$z^{(m)}(t) = \psi_h^{(m)}(t), \;\; u^{(m)}(t) = \Psi_h^{(m)}(t) \;\text{ on }\; E_{0.h} \tag{11}$$

where $\psi_h : E_{0.h} \to \mathbb{R}$ and $\Psi_h : E_{0.h} \to \mathbb{R}^n$, $\Psi_h = (\Psi_{h.1}, \dots, \Psi_{h.n})$, are given functions. The differential difference problem (9) - (11) is called a method of lines for (1), (2). This method is obtained in the following way.

We use a method of quasilinearization for (1), (2). It means that we transform the nonlinear initial problem (1), (2) into a system of quasilinear differential equations with unknown functions (z, u) where $u = \partial_x z$. Suppose that Assumption $H[f, \psi]$ is satisfied. We consider the following linearization of equation (1) with respect to u

$$\partial_t z(t,x) = f(U[t,x]) + \sum_{i=1}^{n} \partial_{q_i} f(U[t,x])(\partial_{x_i} z(t,x) - u_i(t,x)) \tag{12}$$

where $U[t,x] = (t, x, z_{(t,x)}, z_{\varphi(t,x)}, u(t,x))$. Differential equations for u are obtained by differentiating equation (1) with respect to the spatial variable

$$\begin{aligned}\partial_t u(t,x) &= \partial_x f(U[t,x]) + \partial_v f(U[t,x]) \star u_{(t,x)} \\ &\quad + \Big[\partial_w f(U[t,x]) \star u_{\varphi(t,x)}\Big]\partial_x \phi(t,x) + \partial_q f(U[t,x])[\partial_x u(t,x)]^T,\end{aligned} \tag{13}$$

where

$$\partial_v f(U[t,x]) \star u_{(t,x)} = (\partial_v f(U[t,x]) \star (u_1)_{(t,x)}, \dots, \partial_v f(U[t,x]) \star (u_n)_{(t,x)})$$

and

$$\Big[\partial_w f(U[t,x]) \star u_{\varphi(t,x)}\Big]\partial_x \phi(t)$$

$$= \Bigg(\sum_{i=1}^{n} \partial_w (U[t,x])(u_i)_{\varphi(t,x)} \partial_{x_1} \phi_i(t), \dots, \sum_{i=1}^{n} \partial_w (P)(u_i)_{\varphi(t,x)} \partial_{x_n} \phi_i(t) \Bigg).$$

With equations (12), (13) we consider the following initial condition

$$z(t,x) = \psi(t,x), \;\; u(t,x) = \Psi(t,x) \;\text{ on }\; (t,x) \in E_0. \tag{14}$$

Under natural assumptions on given functions the above problem has the properties:

(i) If $(\tilde{z}, \tilde{u})$ is a solution of (12) - (14) then $\partial_x \tilde{z} = \tilde{u}$ and $\tilde{z}$ is a solution of (1), (2).

(ii) If $\tilde{z}$ is a solution of (1), (2) and $\partial_x \tilde{z} = \tilde{u}$ then $(\tilde{z}, \tilde{u})$ is a solution of (12) - (14).

The differential difference problem (9) - (11) is discretization with respect to the spatial variable of (12) - (14).

3 Solutions of functional differential problems

For functions $z \in C(E_0 \cup E, \mathbb{R})$, $u \in C(E_0 \cup E, \mathbb{R}^n)$ and $z_h \in \mathbb{F}_c(E_{0.h} \cup E_h, \mathbb{R})$, $u_h \in \mathbb{F}_c(E_{0.h} \cup E_h, \mathbb{R}^n)$ we define

$$\|z\|_t = \max\{|z(\tau,s)| : (\tau,s) \in E_0 \cup E,\ \tau \le t\},$$

$$\|z_h\|_{h.t} = \max\{|z_h(\tau,s)| : (\tau,s) \in E_{0.h} \cup E_h,\ \tau \le t\},$$

$$[|u|]_t = \max\{\|u(\tau,s)\|_\infty : (\tau,s) \in E_0 \cup E,\ \tau \le t\},$$

$$[|u_h|]_{h.t} = \max\{\|u_h(\tau,s)\|_\infty : (\tau,s) \in E_{0.h} \cup E_h,\ \tau \le t\}$$

where $t \in [0,a]$. We need the following assumptions.
Assumption $H[T_h]$. The operator $T_h : \mathbb{F}_c(B_h, \mathbb{R}) \to C(B, \mathbb{R})$ satisfies the conditions

1) for $w, \bar{w} \in \mathbb{F}_c(B_h, \mathbb{R})$ we have

$$\|T_h w - T_h \bar{w}\|_B \le \|w - \bar{w}\|_{B_h},$$

2) if $w : B \to \mathbb{R}$ is of class C^1 and w_h is the restriction of w to B_h then there is $\gamma : H \to \mathbb{R}_+$ such that

$$\|T_h w_h - w\|_B \le \gamma(h), \quad \lim_{h \to 0} \gamma(h) = 0,$$

3) if $\theta_h \in \mathbb{F}_c(B_h, \mathbb{R})$ is given by $\theta_h(\tau,y) = 0$ on B_h then $(T_h \theta_h)(\tau,y) = 0$ for $(\tau,y) \in B$.

Example of the interpolating operator which satisfies the above assumptions can be found in [11], Chapter VI.
Assumption $H[f,\varrho]$. The functions φ and f, ψ satisfy Assumptions $H[\varphi]$ and $H[f,\psi]$, moreover

1) there is $\varrho \in C([0,a) \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ such that

$$\|\partial_x f(P)\|_\infty \le \varrho(t, \max\{\|v\|_B, \|w\|_B\}, \|q\|) \text{ on } \Omega$$

and the function ϱ is nondecreasing with respect to the last two variables,

2) the constant $A \in \mathbb{R}_+$ is defined by the relation

$$|f(t,x,\theta,\theta,0_{[n]})| \le A, \quad (t,x) \in E,$$

where $\theta \in C(B,\mathbb{R})$ and $\theta(\tau,s) = 0$ for $(\tau,s) \in B$,

3) there is $A_0 \in \mathbb{R}$ such that for a point $P = (t,x,v,w,q) \in \Omega$ we have

$$\|\partial_v f(P)\|_\star, \ \|\partial_w f(P)\|_\star \le A_0,$$

4) for $P = (t,x,v,w,q) \in \Omega$ we have

$$(|\partial_{q_1} f(P)|, \ldots, |\partial_{q_n} f(P)|) \le (M_1, \ldots, M_n) = M,$$

5) for every $(\mu, \nu) \in \mathbb{R}_+ \times \mathbb{R}_+$ there exists on $[0, a]$ the maximal solution $(\omega_0(\cdot, \mu, \nu), \omega(\cdot, \mu, \nu))$ of the problem

$$\lambda'(t) = A + 2A_0\lambda(t) + 2\|M\|\eta(t), \tag{15}$$

$$\eta'(t) = \varrho(t, \lambda(t), \eta(t)) + A_0(1+Q)\eta(t), \tag{16}$$

$$(\lambda(0), \eta(0)) = (\mu, \nu), \tag{17}$$

6) $\psi_h : E_{0.h} \cup E_h \to \mathbb{R}$, $\Psi_h : E_{0.h} \cup E_h \to \mathbb{R}^n$ and there are α_0, $\alpha : H \to \mathbb{R}_+$ such that

$$|\psi(t,x) - \psi_h(t,x)| \le \alpha_0(h) \ \text{ and } \ \|\partial_x \psi(t,x) - \Psi_h(t,x)\|_\infty \le \alpha(h) \ \text{ on } \ E_{0.h} \tag{18}$$

and

$$\lim_{h\to 0} \alpha_0(h) = 0, \ \lim_{h \to 0} \alpha(h) = 0.$$

Suppose that Assumption $H[f, \varrho]$ is satisfied. Let $\bar{\mu}$, $\bar{\nu} \in \mathbb{R}_+$ be defined by the relations

$$|\psi(t,x)| \le \bar{\mu}, \ \|\partial_x\psi(t,x)\|_\infty \le \bar{\nu} \ \text{ on } \ E_0, \tag{19}$$

$$|\psi_h(t,x)| \le \bar{\mu}, \ \|\Psi_h(t,x)\|_\infty \le \bar{\nu} \ \text{ on } \ E_{0.h}, \tag{20}$$

we will assume nonlinear estimates of Perron type for $\partial_x f$, $\partial_v f$, $\partial_w f$, $\partial_q f$ on subspace of Ω. Now we construct this subspace.

Suppose that Assumption $H[f, \varrho]$ is satisfied and $\bar{\mu}$, $\bar{\nu} \in \mathbb{R}_+$ are defined by (19), (20). Let us denote by $(\omega_0(\cdot, \bar{\mu}, \bar{\nu}), \omega(\cdot, \bar{\mu}, \bar{\nu}))$ the maximal solution of (15) - (17) with $\mu = \bar{\mu}$, $\nu = \bar{\nu}$. Set $\bar{c} = \omega_0(a, \bar{\mu}, \bar{\nu})$, $\tilde{c} = \omega(a, \bar{\mu}, \bar{\nu})$, $C = (\bar{c}, \tilde{c})$ and

$$\Omega[C] = \{(t,x,v,w,q) \in \Omega : \ \|v\|_B \le \bar{c}, \ \|w\|_B \le \bar{c}, \ \|q\|_\infty \le \tilde{c}\}.$$

Assumption $H[f, \sigma]$. The functions φ and f, ψ satisfy Assumptions $H[\varphi]$, $H[f, \psi]$, $H[f, \varrho]$ and

1) $\sigma : [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and it is nondecreasing with respect to the second variable,

2) for each $c \ge 1$ the maximal solution of the Cauchy problem

$$\omega'(t) = c[\omega(t) + \sigma(t, \omega(t))], \ \ \omega(0) = 0 \tag{21}$$

is $\tilde{\omega}(t) = 0$ for $t \in [0, a]$,

3) the expressions

$$\|\partial_x f(t,x,v,w,q) - \partial_x f(t,x,\tilde{v},\tilde{w},\tilde{q})\|, \ \ \|\partial_q f(t,x,v,w,q) - \partial_q f(t,x,\tilde{v},\tilde{w},\tilde{q})\|,$$

$$\|\partial_v f(t,x,v,w,q) - \partial_v f(t,x,\tilde{v},\tilde{w},\tilde{q})\|_\star, \ \ \|\partial_w f(t,x,v,w,q) - \partial_w f(t,x,\tilde{v},\tilde{w},\tilde{q})\|_\star$$

are estimated on $\Omega[C]$ by $\sigma(t, \max\{\|v - \tilde{v}\|_B, \|w - \tilde{w}\|_B, \|q - \tilde{q}\|\})$.

Remark 3.1. *It is important that we have assumed nonlinear estimates of Perron type for $\partial_x f$, $\partial_v f$, $\partial_w f$, $\partial_q f$ on $\Omega[C]$. There are differential equations with deviated variables and differential integral equations such that condition 3) of Assumption $H[f,\sigma]$ is satisfied and global estimates for $\partial_x f$, $\partial_v f$, $\partial_w f$, $\partial_q f$ are not satisfied. We give comments on such equations.*

Set $\tilde{\Omega} = E \times \mathbb{R}^2 \times \mathbb{R}^n$ and suppose that the function $G : \tilde{\Omega} \to \mathbb{R}$ of the variables (t, x, p, r, q) satisfies the conditions

1) $G \in C(\tilde{\Omega}, \mathbb{R})$ and for each $(t, x) \in E$ the function $G(, t, x, \cdot) : \mathbb{R}^2 \times \mathbb{R}^n \to \mathbb{R}$ is of class C^2,
2) there is $\tilde{A} \in \mathbb{R}_+$ such that $|\partial_p G(P)| \le \tilde{A}$, $|\partial_v G(P)| \le \tilde{A}$ and
$$(|\partial_{q_1} G(P)|, \ldots, |\partial_{q_n} G(P)|) \le (M_1, \ldots, M_n)$$
where $P = (t, x, v, w, q) \in \tilde{\Omega}$,
3) there is $\varrho \in C([0, a] \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ such that
   (i) for each $t \in [0, a]$ the function $\varrho(t, \cdot, \cdot)$ is nondecreasing,
   (ii) condition 5) of Assumption $H[f, \varrho]$ is satisfied and
$$\|\partial_x G(t, x, v, w, q)\|_\infty \le \varrho(t, \max\{|v|, |w|\}, \|q\|_\infty) \text{ on } \tilde{\Omega}.$$

Consider the operator f defined by
$$f(t, x, v, w, q) = G(t, x, v(0, 0_{[n]}), w(0, 0_{[n]}), q) \text{ on } \Omega. \tag{22}$$

Then (1) reduces to the differential equation with deviated variables
$$\partial_t z(t, x) = G(t, x, z(t, x), z(\varphi(t, x)), \partial_x z(t, x)).$$

Then there is $L \in \mathbb{R}_+$ such that the operator f given by (22) satisfies Assumption $H[f, \sigma]$ for $\sigma(t, p) = Lp$, $(t, p) \in [0, a] \in \mathbb{R}_+$.

Set
$$f(t, x, v, w, q) = G(t, x, \int_{D[t,x]} v(\tau, y)dyd\tau, w(0, 0_{[0]}), q) \text{ on } \Omega. \tag{23}$$

Then (1) reduces to the functional differential equation
$$\partial_t z(t, x) = G(t, x, \int_{D[t,x]} z(t + \tau, x + y)dyd\tau, z(\varphi(t, x)), \partial_x z(t, x)).$$

There is $L \in \mathbb{R}_+$ such that the operator given by (23) satisfies Assumption $H[f, \sigma]$ for $\sigma(t, p) = Lp$, $(t, p) \in [0, a] \times \mathbb{R}_+$.

It is important in the above examples that we do not assume that the partial derivatives of the second order of $G(t, x, \cdot)$ are bounded on $\tilde{\Omega}$.

We give estimates of solutions of (12) - (14).

Lemma 3.1. *Suppose that Assumption $H[f,\varrho]$ is satisfied and $(\bar{z},\bar{u}) : E_0 \cup E \to \mathbb{R}^{1+n}$, $\bar{u} = (\bar{u}_1,\ldots,\bar{u}_n)$, are the solutions of (12) - (14) then*

$$\|\bar{z}\|_t \leq \omega_0(t,\bar{\mu},\bar{v}), \;\; [|\bar{u}|]_t \leq \omega(t,\bar{\mu},\bar{v}), \tag{24}$$

where $(\omega_0(\cdot,\bar{\mu},\bar{v}),\omega(\cdot,\bar{\mu},\bar{v}))$ is the maximal solution of (15)-(17) with $(\mu,v) = (\bar{\mu},\bar{v})$.

Proof. Write

$$\bar{\lambda}(t) = \|\bar{z}\|_t, \;\; \bar{\eta}(t) = [|\bar{u}|]_t, \;\; t \in [0,a].$$

Let us denote by $(\omega_0(\,\cdot\,,\bar{\mu},\bar{v},\varepsilon),\omega(\,\cdot\,,\bar{\mu},\bar{v},\varepsilon))$ the maximal solution of the initial problem

$$\lambda'(t) = A + 2A_0\lambda(t) + 2\|M\|\eta(t) + \varepsilon, \tag{25}$$

$$\eta'(t) = \varrho(t,\lambda(t),\mu(t)) + A_0(1+Q)\eta(t) + \varepsilon \tag{26}$$

$$(\lambda(0),\eta(0)) = (\bar{\mu}+\varepsilon,\bar{v}+\varepsilon) \tag{27}$$

where $\varepsilon > 0$. There is $\tilde{\varepsilon} > 0$ such that for $0 < \varepsilon < \tilde{\varepsilon}$ the solution of (25) - (27) is defined on $[0,a]$ and

$$\lim_{\varepsilon\to 0}(\omega_0(t,\bar{\mu},\bar{v},\varepsilon),\omega(t,\bar{\mu},\bar{v},\varepsilon)) = (\omega_0(t,\bar{\mu},\bar{v}),\omega(t,\bar{\mu},\bar{v})) \tag{28}$$

uniformly on $[0,a]$. We prove that for $0 < \varepsilon < \tilde{\varepsilon}$ we have

$$\bar{\lambda}(t) < \omega_0(t,\bar{\mu},\bar{v},\varepsilon), \quad \bar{\eta}(t) < \omega(t,\bar{\mu},\bar{v},\varepsilon) \tag{29}$$

where $t \in [0,a]$.

It is clear that there is $\tilde{t} \in (0,a]$ such that inequalities (29) are satisfied on $[0,\tilde{t})$. Suppose by contradiction that estimates (29) are not satisfied on $[0,a]$. Then there is $t \in (0,a]$ such that

$$\bar{\lambda}(\tau) < \omega_0(\tau,\bar{\mu},\bar{v},\varepsilon) \;\text{ and }\; \bar{\eta}(\tau) < \omega(\tau,\bar{\mu},\bar{v},\varepsilon) \;\text{ for }\; \tau \in [0,t)$$

and

$$\bar{\lambda}(t) = \omega_0(t,\bar{\mu},\bar{v},\varepsilon) \;\text{ or }\; \bar{\eta}(t) = \omega(t,\bar{\mu},\bar{v},\varepsilon).$$

Suppose that $\bar{\eta}(t) = \omega(t,\bar{\mu},\bar{v},\varepsilon)$. Then we have

$$D_-\bar{\eta}(t) \geq \omega'(t,\bar{\mu},\bar{v},\varepsilon). \tag{30}$$

There are $(\bar{t},x) \in E$, $\bar{t} \leq t$, and $j \in \{1,\ldots,n\}$ such that $\bar{\eta}(t) = |\bar{u}_j(\bar{t},x)|$. Suppose that $\bar{t} < t$. Then $D_-\bar{\eta}(t) = 0$ which contradicts (30). If $\bar{t} = t$ then we have (i) $\bar{\eta}(t) = \bar{u}_j(t,x)$ or (ii) $\bar{\eta}(t) = -\bar{u}_j(t,x)$. Let us consider the first case. Then we have

$$\partial_{x_i}\bar{u}_j(t,x) = 0 \;\text{ for }\; i \in I_0[t,x], \tag{31}$$

$$\partial_{x_i}\bar{u}_j(t,x) \leq 0 \;\text{ for }\; i \in I_-[t,x], \tag{32}$$

$$\partial_{x_i}\bar{u}_j(t,x) \geq 0 \;\text{ for }\; i \in I_+[t,x]. \tag{33}$$

Let us consider the function $\gamma : [0,t] \to R^n$, $\gamma = (\gamma_1,\ldots,\gamma_n)$, defined as follows:

$$\gamma_i(\tau) = x_i \;\text{ for }\; i \in I_0[t,x],$$

$$\gamma_i(\tau) = -b_i + M_i\tau \ \text{ for } \ i \in I_-[t,x],$$
$$\gamma_i(\tau) = b_i - M_i\tau \ \text{ for } \ i \in I_+[t,x].$$

Set $\xi(\tau) = \bar{u}_j(\tau, \gamma(\tau))$ for $\tau \in [0,t]$. Then we have $\xi(\tau) \le \bar{\eta}(\tau)$ for $\tau \in [0,t)$ and $\xi(t) = \bar{\eta}(t)$. This gives

$$D_-\bar{\eta}(t) \le \xi'(t)$$

and

$$\xi'(t) = \partial_t \bar{u}_j(t,x) + \sum_{i\in I_-[t,x]} M_i \partial_{x_i}\bar{u}_j(t,x) - \sum_{i\in I_+[t,x]} M_i \partial_{x_i}\bar{u}_j(t,x).$$

Set

$$\bar{U}(t,x) = \big(t, x, \bar{z}_{(t,x)}, \bar{z}_{\varphi(t,x)}, \bar{u}(t,x)\big).$$

It follows from (13) that

$$\xi'(t) = \partial_{x_j} f(\bar{U}(t,x)) + \partial_v f(\bar{U}(t,x))(u_j)_{(t,x)}$$
$$+ \sum_{i=1}^{n} \partial_w f(\bar{U}(t,x))(u_i)_{\varphi(t,x)} \partial_{x_j}\phi_i(t,x) + \sum_{i=1}^{n} \partial_{q_i} f(\bar{U}(t,x)) \partial_{x_i}\bar{u}_j(t,x)$$
$$+ \sum_{i\in I_-[t,x]} M_i \partial_{x_i}\bar{u}_j(t,x) - \sum_{i\in I_+[t,x]} M_i \partial_{x_i}\bar{u}_j(t,x)$$

It follows from condition 1) - 4) Assumption $H[f,\varrho]$ and from (31) - (33) that

$$D_-\bar{\eta}(t) \le \xi'(t) < \varrho(t, \bar{\lambda}(t), \bar{\eta}(t)) + A_0(1+Q)\bar{\eta}(t) + \varepsilon = \omega'(t, \bar{\mu}, \bar{\nu}, \varepsilon)$$

which contradicts (30). The case (ii) can be treated in a similar way.

We can use the same reasoning for the relation $\bar{\lambda}(t) = \omega_0(t, \bar{\mu}, \bar{\nu}, \varepsilon)$.

From (29) we obtain in the limit, letting ε tend to zero, inequalities (24). This completes the proof.

Lemma 3.2. *If Assumptions $H[f,\sigma]$ and $H[T_h]$ are satisfied then there exists exactly one solution $(z_h, u_h) : E_{0.h} \cup E_h \to \mathbb{R}^{1+n}$, $u_h = (u_{h.1}, \ldots, u_{h.n})$, of the Cauchy problem (12) - (14) and*

$$\|z_h\|_{h.t} \le \omega_0(t, \bar{\mu}, \bar{\nu}), \ \ [|u_h|]_{h.t} \le \omega(t, \bar{\mu}, \bar{\nu}) \tag{34}$$

where $(\omega_0(t, \bar{\mu}, \bar{\nu}), \omega(t, \bar{\mu}, \bar{\nu}))$ is the maximal solution of (15) - (17) with $(\mu, \nu) = (\bar{\mu}, \bar{\nu})$.

Proof. It is clear that there is $\tilde{\varepsilon} > 0$ such that the solution (z_h, u_h) of (12) - (14) is defined on $(E_{0.h} \cup E_h) \cap ([-b_0, \tilde{\varepsilon}] \times \mathbb{R}^n)$. Suppose that (z_h, u_h) is defined on $(E_{0.h} \cup E_h) \cap ([-b_0, \tilde{a}) \times \mathbb{R}^n)$, $\tilde{a} > 0$, and it is non continuable. For $\varepsilon > 0$ we denote by $(\omega_0(\cdot, \bar{\mu}, \bar{\nu}, \varepsilon), \omega(\cdot, \bar{\mu}, \bar{\nu}, \varepsilon))$ the maximal solution of (15) - (17). There is $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the functions $(\omega_0(\cdot, \bar{\mu}, \bar{\nu}, \varepsilon), \omega(\cdot, \bar{\mu}, \bar{\nu}, \varepsilon))$ are defined on $[0, \tilde{a})$ and condition (28) is satisfied. Set

$$\xi_h(t) = \|z_h\|_{h.t}, \ \ \chi_h(t) = [|u_h|]_{h.t}, \ \ t \in [0, \tilde{a}).$$

We prove that

$$\xi_h(t) < \omega_0(t, \bar{\mu}, \bar{\nu}, \varepsilon) \ \text{ and } \ \chi_h(t) < \omega(t, \bar{\mu}, \bar{\nu}, \varepsilon) \tag{35}$$

where $t \in [0, \tilde{a})$. It is clear that there is $\tilde{t} > 0$ such that estimates (35) are satisfied on $[0, \tilde{t})$. Suppose by contradiction that (35) fails to be true on $[0, \tilde{a})$. Then there is $t \in (0, \tilde{a})$ such that

$$\xi_h(\tau) < \omega_0(\tau, \bar{\mu}, \bar{\nu}, \varepsilon) \text{ and } \chi_h(\tau) < \omega(\tau, \bar{\mu}, \bar{\nu}, \varepsilon) \text{ for } \tau \in (0, t)$$

and

$$\xi_h(t) = \omega_0(t, \bar{\mu}, \bar{\nu}, \varepsilon) \text{ or } \chi_h(t) = \omega(t, \bar{\mu}, \bar{\nu}, \varepsilon).$$

Suppose that $\chi_h(t) = \omega(t, \bar{\mu}, \bar{\nu}, \varepsilon)$. Then we have

$$D_-\chi_h(t) \geq \omega'(t, \bar{\mu}, \bar{\nu}, \varepsilon). \tag{36}$$

There are $(\bar{t}, x^{(m)}) \in E_h$, $\bar{t} \leq t$, and $j \in \{1, \ldots, n\}$ such that $\chi_h(t) = |u_{h.j}^{(m)}(\bar{t})|$. If $\bar{t} < t$ then $D_-\chi_h(t) = 0$ which contradicts (36). Let us consider the case when $\bar{t} = t$. Then we have (i) $\chi_h(t) = u_{h.j}^{(m)}(t)$ or $\chi_h(t) = -u_{h.j}^{(m)}(t)$. We consider the first case. Then we have

$$D_-\chi_h(t) \leq \frac{d}{dt} u_{h.j}^{(m)}(t)$$

$$= \partial_{x_j} f(P[z_h, u_h]^{(m)}(t)) + \partial_v f(P[z_h, u_h]^{(m)}(t)) \star T_h(u_h)_{[t,m]}$$

$$+ \sum_{i=1}^{n} \partial_w f(P[z_h, u_h]^{(m)}(t)) T_h(u_{h.i})_{\varphi[t,m]} \partial_{x_j} \phi_i^{(m)}(t) + \sum_{i=1}^{n} \partial_{q_i} f(P[z_h, u_h]^{(m)}(t)) \delta_i u_{h.j}^{(m)}(t).$$

It follows from condition 3) of Assumption $H[f, \psi]$ and from (7), (8) that

$$\sum_{i=1}^{n} \partial_{q_i} f(P[z_h, u_h]^{(m)}(t)) \delta_i u_{h.j}^{(m)}(t) \leq 0.$$

We thus get

$$D_-\chi_h(t) \leq \varrho(t, \omega_0(t, \bar{\mu}, \bar{\nu}, \varepsilon), \omega(t, \bar{\mu}, \bar{\nu}\varepsilon)) + A_0(1 + Q)\omega(t, \bar{\mu}, \bar{\nu}, \varepsilon) < \omega'(t, \bar{\mu}, \bar{\nu}, \varepsilon),$$

which contradicts (36). The case (ii) can be treated in a similar way.

A similar proof remains valid for case $\xi_h(t) = \omega_0(t, \bar{\mu}, \bar{\nu}, \varepsilon)$. The inequalities (35) are satisfied on $[0, \tilde{a})$. From (35) we obtain in the limit, letting ε tend to zero, that

$$\|z_h\|_{h.t} \leq \omega_0(t, \bar{\mu}, \bar{\nu}) \text{ and } [|u_h|]_{h.t} \leq \omega(t, \bar{\mu}, \bar{\nu}) \text{ for } t \in (0, \tilde{a}). \tag{37}$$

Suppose that $(t, x^{(m)})$, $(\bar{t}, x^{(m)}) \in E_h$, $t, \bar{t} \in (0, \tilde{a})$. It follows from Assumptions $H[T_h]$, $H[f, \varrho]$ and from (37) that there are $\tilde{\omega}_{h.0}, \tilde{\omega}_h \in C([0, \tilde{a}], \mathbb{R})$ such that

$$|z_h^{(m)}(\bar{t}) - z_h^{(m)}(t)| = \left| \int_t^{\bar{t}} \frac{d}{dt} z_h^{(m)}(\tau) d\tau \right| \leq |\tilde{\omega}_{h.0}(\bar{t}) - \tilde{\omega}_{h.0}(t)|$$

and

$$\|u_h^{(m)}(\bar{t}) - u_h^{(m)}(t)\|_\infty = \left\| \int_t^{\bar{t}} \frac{d}{dt} u_h^{(m)}(\tau) d\tau \right\|_\infty \leq |\tilde{\omega}_h(\bar{t}) - \tilde{\omega}_h(t)|.$$

Then there are the limits

$$\lim_{\substack{t \to \tilde{a} \\ t < \tilde{a}}} z_h^{(m)}(t) = z_h^{(m)}(\tilde{a}), \quad \lim_{\substack{t \to \tilde{a} \\ t < \tilde{a}}} u_h^{(m)}(t) = u_h^{(m)}(\tilde{a}).$$

Then the solution (z_h, u_h) is defined on $(E_{0.h} \cup E_h) \cap ([-b_0, \tilde{a}] \times \mathbb{R}^n)$. This contradicts our assumption that (z_h, u_h) is defined on $(E_{0.h} \cup E_h) \cap ([-b_0, \tilde{a}), \mathbb{R}^n)$ and it is non continuable.

If $\tilde{a} < a$ then there is $\varepsilon > 0$ such that the solution (z_h, u_h) exists on $(E_{0.h} \cup E_h) \cap ([-b_0, \tilde{a} + \varepsilon) \times \mathbb{R}^n)$ and inequalities (37) are satisfied for $t \in [0, \tilde{a} + \varepsilon)$. It follows from the above considerations that (z_h, u_h) is defined on $E_{0.h} \cup E_h$ and estimates (34) are satisfied.

Suppose that (z_h, u_h) and $(\tilde{z}_h, \tilde{u}_h)$ are solutions of (12) - (14). Then we have

$$(t, x^{(m)}, T_h(z_h)_{[t,m]}, T_h(z_h)_{\varphi[t,m]}, u_h^{(m)}(t)) \in \Omega[C], \tag{38}$$

$$(t, x^{(m)}, T_h(\tilde{z}_h)_{[t,m]}, T_h(\tilde{z}_h)_{\varphi[t,m]}, \tilde{u}_h^{(m)}(t)) \in \Omega[C]. \tag{39}$$

Set

$$\tilde{\lambda}_h(t) = \|z_h - \tilde{z}_h\|_{h.t}, \;\; \tilde{\zeta}_h(t) = [|u_h - \tilde{u}_h|]_{h.t}, \;\; t \in (0, a],$$

and $\bar{\omega}_h = \tilde{\lambda}_h + \tilde{\zeta}_h$. It follows from condition 3) of Assumption $H[f, \sigma]$ and from (38), (39) that there is $c_h \geq 1$ such that the function $\bar{\omega}_h$ satisfies the differential inequality

$$D_{-}\bar{\omega}_h(t) \leq c_h[\bar{\omega}_h(t) + \sigma(t, \bar{\omega}_h(t))], \;\; t \in (0, a],$$

and $\bar{\omega}_h(0) = 0$. It follows from condition 2) of Assumption $H[f, \sigma]$ that $\bar{\omega}_h(t) = 0$ for $t \in (0, a]$. Then $(z_h, u_h) = (\tilde{z}_h, \tilde{u}_h)$. This completes the proof of the lemma.

4 Convergence of the method of lines

Now we formulate the main result of the paper.

Theorem 4.1. *Suppose that Assumptions $H[f, \sigma]$ and $H[T_h]$ are satisfied and*

1) *$\bar{z} : E_0 \cup E \to \mathbb{R}$ is a solution of (1), (2) and $\bar{z}$ is of class C^2,*

2) *$\bar{u} = \partial_x \bar{z}$ and $(\bar{z}_h, \bar{u}_h)$ is the restriction of $(\bar{z}, \bar{u})$ to $E_{0.h} \cup E_h$.*

Then there is exactly one solution $(z_h, u_h) : E_{0.h} \cup E_h \to \mathbb{R}^{1+n}$ of (12) - (14) and there is $\tilde{\beta} : H \to \mathbb{R}_+$ such that

$$\|z_h - \bar{z}_h\|_{h.t} + [|u_h - \bar{u}_h|]_{h.t} \leq \tilde{\beta}(h) \;\; \text{for} \;\; t \in [0, a] \tag{40}$$

and

$$\lim_{h \to 0_{[n]}} \tilde{\beta}(h) = 0. \tag{41}$$

Proof. The existence and uniqueness of a solution of (12) - (14) follows from Lemma 3.2. Let $\Gamma_{h.0} : E_{0.h} \to \mathbb{R}$, $\Gamma_h : E_h \to \mathbb{R}^n$ be defined by the relations

$$\frac{d}{dt}\bar{z}_h^{(m)}(t) = F_h[\bar{z}_h, \bar{u}_h]^{(m)}(t) + \Gamma_{h.0}^{(m)}(t),$$

$$\frac{d}{dt}\bar{u}_h^{(m)}(t) = G_h[\bar{z}_h, \bar{u}_h]^{(m)}(t) + \Gamma_h^{(m)}(t).$$

There are $\gamma_0, \gamma : H \to \mathbb{R}_+$ such that

$$|\Gamma_{h.0}^{(m)}(t)| \leq \gamma_0(h) \text{ and } \|\Gamma_h^{(m)}(t)\|_\infty \leq \gamma(h) \text{ on } E_h$$

and

$$\lim_{h\to 0_{[n]}} \gamma_0(h) = 0, \quad \lim_{h\to 0_{[n]}} \gamma(h) = 0.$$

Let $c^\star \in \mathbb{R}_+$ be defined by the relation

$$c^\star = \max\{\|\partial_{xx}\bar{z}(t,x)\|_{n\times n} : (t,x) \in E\}$$

where $\partial_{xx}\bar{z}(t,x) = [\partial_{x_i x_j}\bar{z}(t,x)]_{i,j=1,\dots,n}$. It follows from Lemma 3.1 and 3.2 and from Assumption $H[T_h]$ that for $(t, x^{(m)}) \in E_h$ we have

$$|T_h(z_h)_{[t,m]}(\tau,s)| \leq \bar{c}, \quad |T_h(\bar{z}_h)_{[t,m]}(\tau,s)| \leq \bar{c}, \quad (\tau,s) \in D[t,x^{(m)}] \tag{42}$$

and

$$|T_h(z_h)_{\varphi[t,m]}(\tau,s)| \leq \bar{c}, \quad |T_h(\bar{z}_h)_{\varphi[t,m]}(\tau,s)| \leq \bar{c}, \quad (\tau,s) \in D[\varphi(t,x^{(m)})] \tag{43}$$

and

$$\|u_h^{(m)}(t)\|_\infty \leq \tilde{c}, \quad \|\bar{u}_h^{(m)}(t)\|_\infty \leq \tilde{c}. \tag{44}$$

For functions $\zeta, \chi : [0,a] \to \mathbb{R}_+$ we define

$$\mathbb{L}_{h.0}[\zeta,\chi](t) = 2A_0\zeta(t) + 2\|M\|\chi(t) + 2\tilde{c}\sigma(t,\zeta(t)+\chi(t)) + \gamma_0(h),$$

$$\mathbb{L}_h[\zeta,\chi](t) = A_0(1+Q)\chi(t) + \bar{a}\sigma(t,\zeta(t)+\chi(t)) + \gamma(h)$$

where $\bar{a} = 1 + \tilde{c}(1+Q) + c^\star$. Let us denote by $(\omega_{h.0}(\cdot,\varepsilon), \omega_h(\cdot,\varepsilon))$ the maximal solution of the Cauchy problem

$$\zeta'(t) = \mathbb{L}_{h.0}[\zeta,\chi](t) + \varepsilon, \quad \chi'(t) = \mathbb{L}_h[\zeta,\chi](t) + \varepsilon, \tag{45}$$

$$\zeta(0) = \alpha_0(h) + \varepsilon, \quad \chi(0) = \alpha(h) + \varepsilon \tag{46}$$

where $\alpha_0, \alpha : H \to \mathbb{R}^+$ are given by (18). It follows from condition 2) of Assumption $H[f,\sigma]$ that there is $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the functions $(\omega_{h.0}(\cdot,\varepsilon), \omega(\cdot,\varepsilon))$ are defined on $[0,a]$ and

$$\lim_{\varepsilon\to 0} \omega_{h.0}(t,\varepsilon) = \omega_{h.0}(t), \quad \lim_{\varepsilon\to 0} \omega_h(t,\varepsilon) = \omega_h(t) \text{ uniformly on } [0,a]$$

where $(\omega_{h.0}, \omega_h)$ is the maximal solution (45), (46) with $\varepsilon = 0$. Write

$$\lambda_h(t) = \|\bar{z}_h - z_h\|_{h.t}, \quad \xi_h(t) = [|\bar{u}_h - u_h|]_{h.t}, \quad t \in (0,a].$$

We prove that for each $0 < \varepsilon < \varepsilon_0$ we have

$$\lambda_h(t) < \omega_{h.0}(t,\varepsilon) \text{ and } \xi_h(t) < \omega_h(t,\varepsilon) \tag{47}$$

where $t \in [0, a]$. It is clear that is $\tilde{t} \in (0, a]$ such that inequalities (47) are satisfied on $[0, \tilde{t})$. Suppose by contradiction that (47) fails to be true on $[0, a]$. Then there is $t \in (0, a]$ such that

$$\gamma_h(\tau) < \omega_{h.0}(\tau, \varepsilon) \ \text{ and } \ \xi_h(\tau) < \omega_h(\tau, \varepsilon) \ \text{ for } \ \tau \in [0, t)$$

and

$$\lambda_h(t) = \omega_{h.0}(t, \varepsilon) \ \text{ or } \ \xi_h(t) < \omega_h(t, \varepsilon).$$

Suppose that $\lambda_h(t) = \omega_{h.0}(t, \varepsilon)$. Then we have

$$D_{-}\lambda_h(t) \geq \omega'_{h.0}(t, \varepsilon). \tag{48}$$

There is $(\bar{t}, x^{(m)}) \in E_h$, $\bar{t} \leq t$, such that $\lambda_h(t) = |z_h^{(m)}(\bar{t}) - \bar{z}_h^{(m)}(\bar{t})|$. If $\bar{t} < t$ then $D_{-}\lambda_h(t) = 0$ which contradicts (48). Suppose that $\bar{t} = t$. Then we have (i) $\lambda_h(t) = z_h^{(m)}(t) - \bar{z}_h^{(m)}(t)$ or (ii) $\lambda_h(t) = -[z_h^{(m)}(t) - \bar{z}_h^{(m)}(t)]$. Let us consider the case (i). It follows from condition 3) of Assumption $H[f, \sigma]$ and from (42) - (44) that

$$\|\partial_q f(P[z_h, u_h]^{(m)}(t)) - \partial_q f(P[\bar{z}_h, \bar{u}_h]^{(m)}(t))\| \leq \sigma(t, \omega_{h.0}(t, \varepsilon) + \omega_h(t, \varepsilon)).$$

Then we have

$$D_{-}\lambda_h(t) \leq \frac{d}{dt}(z_h^{(m)}(t) - \bar{z}_h^{(m)}(t)) = F_h[z_h, u_h]^{(m)}(t) - F_h[\bar{z}_h, \bar{u}_h]^{(m)}(t) + \Gamma_{h.0}^{(m)}(t)$$

$$\leq \mathbb{L}_{h.0}[\omega_{h.0}(\cdot, \varepsilon), \omega_h(\cdot, \varepsilon)](t) + \sum_{i=1}^{n} \partial_{q_i} f(P[z_h, u_h]^{(m)}(t)) \delta_i (z_h - \bar{z}_h)^{(m)}(t) + \gamma_0(h).$$

It follows from conditions 3), 4) of Assumption $H[f, \psi]$ and from (5) , (6) that

$$\sum_{i=1}^{n} \partial_{q_i} f(P[z_h, u_h]^{(m)}(t)) \delta_i (z_h - \bar{z}_h)^{(m)}(t) \leq 0.$$

This gives

$$D_{-}\lambda_h(t) < \mathbb{L}_{h.0}[\omega_{h.0}(\cdot, \varepsilon), \omega_h(\cdot, \varepsilon)](t) + \varepsilon = \omega'_{h.0}(t, \varepsilon)$$

which contradicts (48). The case (ii) can be treated in a similar way. Suppose that $\xi_h(t) = \omega_h(t, \varepsilon)$. Then we have

$$D_{-}\xi_h(t) \geq \omega'_h(t, \varepsilon). \tag{49}$$

There are $(\bar{t}, x^{(m)}) \in E_h$, $\bar{t} \leq t$ and $j \in \{1, \ldots, n\}$ such that $\xi_h(t) = |u_{h.j}^{(m)}(\bar{t}) - \bar{u}_h^{(m)}(\bar{t})|$. Suppose that $\bar{t} < t$. Then $D_{-}\xi_h(t) = 0$ which contradicts (49). Suppose that $\bar{t} = t$. Then we have (i) $\xi_h(t) = u_{h.j}^{(m)}(t) - \bar{u}_{h.j}^{(m)}(t)$ or (ii) $\xi_h(t) = -[u_{h.j}^{(m)}(t) - \bar{u}_{h.j}^{(m)}(t)]$. Let us consider the case (i). We deduce from condition 3) of Assumption $H[f, \sigma]$ and from (42) - (44) that the expressions

$$\|\partial_x f(P[z_h, u_h]^{(m)}(t)) - \partial_x f(P[\bar{z}_h, \bar{u}_h]^{(m)}(t))\|,$$

$$\|\partial_v f(P[z_h, u_h]^{(m)}(t)) - \partial_v f(P[\bar{z}_h, \bar{u}_h]^{(m)}(t))\|_\star,$$
$$\|\partial_w f(P[z_h, u_h]^{(m)}(t)) - \partial_w f(P[\bar{z}_h, \bar{u}_h]^{(m)}(t))\|_\star$$

may be estimated by $\sigma(t, \omega_{h.0}(t,\varepsilon) + \omega_h(t,\varepsilon))$. Then we have

$$D_-\xi_h(t) \leq \frac{d}{dt}(u_{h.j}^{(m)}(t) - \bar{u}_{h.j}^{(m)}(t)) = G_{h.j}[z_h, u_h]^{(m)}(t) - G_{h.j}[\bar{z}_h, \bar{u}_h]^{(m)}(t) + \Gamma_{h.j}^{(m)}(t)$$

$$\leq \mathbb{L}_h[\omega_{h.0}(\cdot,\varepsilon), \omega_h(\cdot,\varepsilon)](t) + \sum_{i=1}^{n} \partial_{q_i} f(P[z_h, u_h]^{(m)}(t))\delta_i(u_{h.j} - \bar{u}_{h.j})^{(m)}(t) + \gamma(h).$$

It follows from condition 3) of Assumption $H[f, \psi]$ and from (7), (8) that

$$\sum_{i=1}^{n} \partial_{q_i} f(P[z_h, u_h]^{(m)}(t))\delta_i(u_{h.j} - \bar{u}_{h.j})^{(m)}(t) \leq 0.$$

Then we obtain

$$D_-\xi_h(t) < \mathbb{L}_h[\omega_{h.0}(\cdot,\varepsilon), \omega_h(\cdot,\varepsilon)](t) = \omega_h'(t,\varepsilon)$$

which contradicts (49). The case (ii) can be treated in a similar way. Then inequalities (47) are satisfied on $[0, a]$. From (47) we obtain in the limit, letting ε tend to zero, the estimates

$$\|z_h - \bar{z}_h\|_{h.t} \leq \omega_{h.0}(t), \;\; [|u_h - \bar{u}_h|]_{h.t} \leq \omega_h(t), \;\; t \in (0, a] \tag{50}$$

where $(\omega_{h.0}, \omega_h)$ is the maximal solution of (45), (46) with $\varepsilon = 0$. Let us denote by $\bar{\omega}_h$ the maximal solution of the Cauchy problem

$$\omega'(t) = d\omega(t) + (\bar{a} + 2\tilde{c})\sigma(t, \omega(t)) + \gamma_0(h) + \gamma(h), \tag{51}$$

$$\omega(0) = \alpha_0(h) + \alpha(h) \tag{52}$$

where $d = \max\{A_0(3+Q), 2\|M\|\}$. We conclude from (50) that

$$\|z_h - \bar{z}_h\|_{h.t} + [|u_h - \bar{u}_h|]_{h.t} \leq \bar{\omega}_h(t), \;\; t \in (0, a].$$

It follows that conditions (40), (41) are satisfied for $\tilde{\beta}(h) = \bar{\omega}_h(a)$. This completes the proof of the theorem.

Remark 4.1. *Suppose that all the assumptions of Theorem 4.1 are satisfied with* $\sigma(t, p) = Lp$, $(t, p) \in [0, a] \in \mathbb{R}_+$, *where* $L \in \mathbb{R}_+$ *Then we have the following error estimate*

$$||z_h - \bar{z}_h||_{h.t} + [|u_h - \bar{u}_h|]_{h.t} \leq \tilde{\omega}_h(t), \;\; t \in (0, a]$$

where

$$\tilde{\omega}_h(t) = (\alpha_0(h) + \alpha(h))e^{\tilde{L}t} + \frac{\gamma_0(h) + \gamma(h)}{\tilde{L}}(e^{\tilde{L}t} - 1)$$

and $\tilde{L} = d + (\bar{a} + 2\tilde{c})L$. *The above inequality is obtained by solving problem (51), (52) with* $\sigma(t, p) = Lp$.

Remark 4.2. *It is assumed in [11] that the right hand sides of functional differential equations satisfy global estimates of Perron type. It follows from Theorem 4.1 that local estimates are sufficient for the convergence of the method of lines.*

5 Numerical Examples

Example 5.1. *Put $n = 2$ and*

$$E = \{(t, x, y) \in \mathbb{R}^3 : \ t \in [0, 1], \ |x| \le 2 - t, \ |y| \le 2 - t\},$$
$$E_0 = \{0\} \times [-2, 2] \times [-2, 2].$$

Consider the differential integral equation

$$\partial_t z(t, x, y) = -\frac{1}{2} \arctan \Big[x \partial_x z(t, x, y) + y \partial_y z(t, x, y) \Big] + \frac{1}{2} \arctan \Big[t(x - y) z(t, x, y) \Big]$$
$$+ t \int_0^x z(t, s, y) ds + t \int_0^y z(t, x, s) ds + f(t, x, y), \tag{53}$$

with the initial condition

$$z(0, x, y) = 1 \ \ for \ \ (x, y) \in [-2, 2] \times [-2, 2] \tag{54}$$

where

$$f(t, x, y) = e^{-ty} - e^{tx} + (x - y) e^{t(x-y)}.$$

The solution of the above problem is known, it is $\bar{z}(t, x, y) = e^{t(x-y)}$. Let us denote by $(\tilde{z}_h, \tilde{z}_{h.x}, \tilde{z}_{h.y})$ approximate solutions of ordinary functional differential equations corresponding to (53), (54). They are obtained by using the explicit Euler difference method. Nodal points on $[0, 1]$ are obtained by $t^{(r)} = rh_0$, $r = 0, 1, \ldots, N_0$.

Set

$$\varepsilon_h^{(r)} = \max\{|(\bar{z}_h - \tilde{z}_h)(t^{(i)}, x^{(m_1)}, y^{(m_2)})| : \ (t^{(i)}, x^{(m_1)}, y^{(m_2)}) \in E, \ 0 \le i \le r\} \tag{55}$$

and

$$\varepsilon_{h.x}^{(r)} = \max\{|(\partial_x \bar{z}_{h.x} - \tilde{z}_h)(t^{(i)}, x^{(m_1)}, y^{(m_2)})| : \ (t^{(i)}, x^{(m_1)}, y^{(m_2)}) \in E, \ 0 \le i \le r\}, \tag{56}$$

$$\varepsilon_{h.y}^{(r)} = \max\{|(\partial_y \bar{z}_h - \tilde{z}_{h.y})(t^{(i)}, x^{(m_1)}, y^{(m_2)})| : \ (t^{(i)}, x^{(m_1)}, y^{(m_2)}) \in E, \ 0 \le i \le r\} \tag{57}$$

where $0 \le r \le N_0$. Let us denote by $\hat{z}_h$ an approximate solution of (53), (54) which is obtained by using the Lax difference scheme. Set

$$\hat{\varepsilon}_h^{(r)} = \max\{|(\bar{z}_h - \hat{z}_h)(t^{(i)}, x^{(m_1)}, y^{(m_2)})| : \ (t^{(i)}, x^{(m_1)}, y^{(m_2)}) \in E, \ 0 \le i \le r\} \tag{58}$$

where $0 \le r \le N_0$. In the Table 1 we give experimental values of the errors $(\varepsilon_h, \varepsilon_{h.x}, \varepsilon_{h.y})$ and $\hat{\varepsilon}_h$ for $h_0 = 0.001, h_1 = h_2 = 0.05$.

Note that errors of the classical difference method $\hat{\varepsilon}_h^{(r)}$ are larger then the errors obtained by discretization of the numerical method of lines $\varepsilon_h^{(r)}$. This is due to the fact that Lax difference scheme has the following property: we approximate partial derivatives of z with respect to spatial variables by difference expressions which are calculated by using previous values of the approximate solutions. In our approach we approximate the partial derivatives $\partial_x z$ and $\partial_y z$ by using difference equations which are generated by the original problem.

Table 1

$t^{(r)}$	$\varepsilon_h^{(r)}$	$\varepsilon_{h.x}^{(r)}$	$\varepsilon_{h.y}^{(r)}$	$\hat{\varepsilon}_h^{(r)}$
0.5	0.000935	0.000656	0.000609	0.143086
0.6	0.001491	0.000998	0.000865	0.184319
0.7	0.002155	0.001512	0.001218	0.228142
0.8	0.002915	0.002222	0.001668	0.272560
0.9	0.003755	0.003151	0.002213	0.315301
1.0	0.004657	0.004316	0.002849	0.354176

Example 5.2. *For $n = 2$ we put*

$$E = \{(t,x,y) \in \mathbb{R}^3 : \; t \in [0, 0.5], \; |x| \leq 2.5 - 2t, \; |y| \leq 2.5 - 2t\},$$

$$E_0 = \{0\} \times [-2.5, 2.5] \times [-2.5, 2.5].$$

Consider the differential equation with deviated variables

$$\partial_t z(t,x,y) = -x\partial_x z(t,x,y) - y\partial_y z(t,x,y) + \cos\left[x\partial_x z(t,x,y) - y\partial_y z(t,x,y)\right] + z(t, 0.5(x+y), 0.5(x-y)) \sin z(t, 0.5x, 0.5y) + f(t,x,y), \quad (59)$$

with the initial condition

$$z(0,x,y) = 1 \;\; \textit{for} \;\; (x,y) \in [-2.5, 2.5] \times [-2.5, 2.5] \quad (60)$$

where

$$f(t,x,y) = xy(1+2t)\exp\{txy\} - 1 - \exp\left\{\frac{t}{4}(x^2 - y^2)\right\} \sin \exp\left\{\frac{t}{4}xy\right\}.$$

The solution of the above problem is $\bar{z}(t,x,y) = e^{txy}$. Let us denote by $(\tilde{z}_h, \tilde{z}_{h.x}, \tilde{z}_{h.y})$ approximate solutions of ordinary functional differential equations corresponding to (59), (60). They are obtained by using the implicit Euler method. Let $(\varepsilon_h, \varepsilon_{h.x}, \varepsilon_{h.y})$ be defined by (55) - (57).

Let us denote by $\hat{z}_h$ an approximate solution of (53), (54) which is obtained by using the Lax difference scheme. Denote by $\hat{\varepsilon}_h$ errors of the method given by (58). In the Table 2 we give experimental values of the above defined errors for $h_0 = 0.01, h_1 = h_2 = 0.01$.

In theorems on the convergence of explicit difference schemes for (1), (2) we need assumptions on the mesh. They are called the (CFL) condition.

The (CFL) condition for (59) and for the Lax difference method has the form

$$h_0 \leq 0.1 h_i, \;\; i = 1, 2.$$

Note that the steps $h_0 = 0.01$, $h_1 = h_2 = 0.01$ do not satisfy the above condition and classical Lax difference scheme is not applicable.

Table 2

$t^{(r)}$	$\varepsilon_h^{(r)}$	$\varepsilon_{h.x}^{(r)}$	$\varepsilon_{h.y}^{(r)}$	$\hat{\varepsilon}_h^{(r)}$
0.25	0.004843	0.004770	0.004637	0.296195
0.30	0.005078	0.005130	0.004961	0.417914
0.35	0.005109	0.005279	0.005082	6.239350
0.40	0.004971	0.005238	0.005019	257.9430
0.45	0.004702	0.005038	0.004807	5300.330
0.50	0.004341	0.004716	0.004483	46673.60

Remark 5.1. *The result presented in the paper can be extended on weakly coupled functional differential systems*

$$\partial_t z_i(t,x) = f_i(t,x,z_{(t,x)}, z_{\varphi(t,x)}, \partial_x z_i(t,x)), \quad i = 1,\ldots,k,$$

with the initial condition

$$z(t,x) = \psi(t,x), \quad (t,x) \in E_0,$$

where $z = (z_1,\ldots,z_k)$, $\partial_x z_i = (\partial_{x_1} z_i,\ldots,\partial_{x_n} z_i)$ *and* $f = (f_1,\ldots,f_k) : E \times C(B,\mathbb{R}^k) \times C(B,\mathbb{R}^k) \times \mathbb{R}^n \to \mathbb{R}^k$, $\psi = (\psi_1,\ldots,\psi_k) : E_0 \to \mathbb{R}^k$ *are given functions.*

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