Positive bounded solutions for semilinear elliptic equations in smooth domains

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Abstract

We are concerned with the following semilinear elliptic equation $\Delta u = \lambda f(x,u)$ in D, subject to some Dirichlet conditions, where $\lambda \geq 0$ is a parameter and D is a smooth domain in $\mathbb{R}^n$ $(n \geq 3)$. Under some appropriate assumptions on the nonnegative nonlinearity term $f(x,u)$, we show the existence of a positive bounded solution for the above semilinear elliptic equation. Our approach is based on Schauder's fixed point Theorem.

1 Introduction and Main result

Let $D \subset \mathbb{R}^n$ $(n \geq 3)$, be a $C^{1,1}$-domain with compact boundary and a, α nonnegative fixed constants such that $a + \alpha > 0$. Consider the following boundary value problem:

$$\begin{cases} \Delta u = \lambda f(x,u) & \text{in } D, \text{ (in the sense of distributions)} \\ u > 0 & \text{in } D, \\ u = a\varphi & \text{on } \partial D, \\ \lim\limits_{|x|\to\infty} u(x) = \alpha & \text{(whenever } D \text{ is unbounded)}, \end{cases} \tag{1.1}$$

where λ is a nonnegative real number and φ is a nontrivial nonnegative continuous function on ∂D.

When the nonlinearity f is negative, there exist a lot of works related to this subject; see for example, the papers of Alves, Carriao and Faria [1], de Figueiredo,

Received by the editors in May 2012.
Communicated by P. Godin.
2010 *Mathematics Subject Classification* : 34B27, 35J65.
Key words and phrases : Green function, positive solutions, Schauder's fixed point theorem.

Girardi and Matzeu [5], Ghergu and Radulescu [6,7], Lair and Wood [9], Zhang [11] and references therein. In all these papers, the main tools used are Galerkin method, sub-supersolution method and variational techniques. Here, we show that the Schauder's fixed point theorem allows us to find solutions to (1.1) for nonnegative nonlinearity f.

More precisely, we assume that $f : D \times [0,\infty) \to [0,\infty)$ is Borel measurable function satisfying

(H_1) f is continuous and nondecreasing with respect to the second variable.

(H_2) $\forall c > 0$, $f(.,c) \in K(D)$,

where the Kato class $K(D)$ is defined by means of the Green function $G_D(x,y)$ of the Dirichlet Laplacian in D as follows

Definition 1.1. *A Borel measurable function q in D belongs to the Kato class $K(D)$ if*

$$\lim_{r\to 0}(\sup_{x\in D}\int_{(|x-y|\le r)\cap D}\frac{\rho_D(y)}{\rho_D(x)}G_D(x,y)|q(y)|dy) = 0 \tag{1.2}$$

and satisfies further

$$\lim_{M\to\infty}(\sup_{x\in D}\int_{(|y|\ge M)\cap D}\frac{\rho_D(y)}{\rho_D(x)}G_D(x,y)|q(y)|dy = 0)\ (\textit{whenever } D \textit{ is unbounded}), \tag{1.3}$$

where $\rho_D(x) = \dfrac{\delta(x)}{\delta(x)+1}$ and $\delta(x)$ denotes the Euclidean distance from x to the boundary of D.

This class was introduced and studied in [2] for unbounded domains and in [10] for the bounded ones. It is quite rich, it contains for example any function belonging to $L^p(D)\cap L^1(D)$, with $p > \frac{n}{2}$.

Throughout this paper, we denote by $H_D\varphi$ the unique harmonic function u in D with boundary value φ and satisfying further $\lim_{|x|\to\infty} u(x) = 0$ whenever D is unbounded. We also denote by $h = 1 - H_D 1$ and we remark that $h \equiv 0$ if D is bounded. Let $\omega(x) := aH_D\varphi(x) + \alpha h(x)$, for $x \in D$.

It is clear that ω is the solution of the problem

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u > 0 & \text{in } D, \\ u = a\varphi & \text{on } \partial D, \\ \lim\limits_{|x|\to\infty} u(x) = \alpha & (\text{whenever } D \text{ is unbounded}). \end{cases} \tag{1.4}$$

Here, we study a perturbation to the problem (1.4) , that is problems of the form (1.1) and we obtain a solution which its behavior is not affected by the perturbed term. A fundamental role will be played by the number

$$\lambda_0 := \inf_{x\in D}\frac{\omega(x)}{V(f(.,\omega))(x)}, \tag{1.5}$$

where V is the potential kernel associated to Δ ($i.e$ $V = (-\Delta)^{-1}$).

Our main result is the following.

Theorem 1.2. *Assume that $\lambda_0 > 0$ and f satisfies (H_1)-(H_2). If $\lambda \in [0, \lambda_0)$, then the problem (1.1) has a continuous bounded solution u such that*

$$(1 - \frac{\lambda}{\lambda_0})\omega \leq u \leq \omega \ \ in\ D.$$

Remark 1.3. *Let $\lambda \geq 0$ and $D = B(0,1)$ be the unit ball of $\mathbb{R}^n$ $(n \geq 3)$. Then the solution of the problem*

$$\begin{cases} \Delta u = \lambda & in\ B, \\ u = 1 & on\ \partial B, \end{cases} \tag{1.6}$$

is given by

$$u(x) = 1 - \lambda V1(x) = 1 - \lambda \frac{(1 - |x|^2)}{2n},\ for\ x \in B. \tag{1.7}$$

Hence we deduce that

$$u > 0,\ in\ B \Leftrightarrow 0 \leq \lambda < 2n = \inf_{x \in B} \frac{1}{\frac{(1-|x|^2)}{2n}} = \lambda_0.$$

This implies that λ_0 is optimal.

As usual, let $B^+(D)$ be the set of nonnegative Borel measurable functions in D. We denote by $\partial^\infty D = \partial D$ if D is bounded and $\partial^\infty D = \partial D \cup \{\infty\}$ whenever D is unbounded, $C_0(D)$ the set of continuous functions in $\overline{D}$ vanishing at $\partial^\infty D$. Note that $C_0(D)$is a Banach space with respect to the uniform norm $\|u\|_\infty = \sup_{x \in D} |u(x)|$. The letter C will denote a generic positive constant which may vary from line to line. When two positive functions f and g are defined on a set S, we write $f \approx g$ if the two sided inequality $\frac{1}{C} g \leq f \leq Cg$ holds on S. Finally, for $f \in B^+(D)$, we denote by

$$Vf(x) := \int_D G_D(x,y) f(y) dy$$

and by

$$\|f\|_D := \sup_{x \in D} \int_D \frac{\rho_D(y)}{\rho_D(x)} G_D(x,y) f\,(y)\, dy.$$

Next we collect some properties of the Green function $G_D\,(x,y)$ and functions belonging to the Kato class $K(D)$, which are useful to establish our main result. For the proofs we refer to [2] and [10].

Proposition 1.4. *For each $x, y \in D$, we have*

$$G_D(x,y) \approx \frac{\lambda_D(x)\lambda_D(y)}{|x-y|^{n-2}\left(|x-y|^2+\lambda_D(x)\lambda_D(y)\right)}, \tag{1.8}$$

where $\lambda_D(x) = \delta(x)(\delta(x)+1)$.
Moreover, for $M > 1$ and $r > 0$ there exists a constant $C > 0$ such that for each $x \in D$ and $y \in D$ satisfying $|x-y| \geq r$ and $|y| \leq M$, we have

$$G_D(x,y) \leq C\frac{\rho_D(x)\rho_D(y)}{|x-y|^{n-2}}. \tag{1.9}$$

We remark that in the case where D is bounded, we have

$$\rho_D(x) \approx \delta(x) \approx \lambda_D(x).$$

Proposition 1.5. *Let q be a function in $K(D)$, then we have*
(i) $\|q\|_D < \infty$,
(ii) Let h be a positive superharmonic function in D. Then there exists a constant $C_0 > 0$ such that

$$\int_D G_D(x,y)h(y)|q(y)|dy \leq C_0 \|q\|_D\, h(x). \tag{1.10}$$

Furthermore, for each $x_0 \in \overline{D}$, we have

$$\lim_{r\to 0}(\sup_{x\in D}\frac{1}{h(x)}\int_{B(x_0,r)\cap D} G_D(x,y)h(y)|q(y)|dy) = 0 \tag{1.11}$$

and

$$\lim_{M\to\infty}(\sup_{x\in D}\frac{1}{h(x)}\int_{(|y|\geq M)\cap D} G_D(x,y)h(y)|q(y)|dy) = 0 \text{ (whenever } D \text{ is unbounded)}. \tag{1.12}$$

(iii) The function $x \to \frac{\delta(x)}{(|x|^{n-1}+1)}q(x)$ is in $L^1(D)$.

2 Proof of Theorem 1.2.

Let $\lambda \in [0, \lambda_0)$ and Λ be the nonempty closed bounded convex set given by

$$\Lambda = \{v \in C(\overline{D}\cup\{\infty\}) : \ (1-\frac{\lambda}{\lambda_0})\omega \leq v \leq \omega\}.$$

We define the operator T on Λ by

$$Tv(x) = \omega(x) - \lambda\int_D G_D(x,y)f\,(y, v(y))\,dy. \tag{2.1}$$

We shall prove that the family $T\Lambda$ is relatively compact in $C(\overline{D}\cup\{\infty\})$. First, we claim that the family

$$\{\int_D G_D(x,y)f\,(y,v(y))\,dy,\ \ v \in \Lambda\}, \tag{2.2}$$

is relatively compact in $C_0(D)$.
Indeed, observe that from (H_1)-(H_2), (1.10) and Proposition 1.5 (i), we have for each $v \in \Lambda$ and $x \in D$,

$$\int_D G_D(x,y)f(y,v(y))\,dy \le \int_D G_D(x,y)f(y,\|\omega\|_\infty)\,dy \le C_0 \|f(.,\|\omega\|_\infty)\|_D < \infty.$$

So the family $\{\int_D G_D(.,y)f(y,v(y))\,dy,\ v \in \Lambda\}$, is uniformly bounded.
Next we aim at proving that the family $\{\int_D G_D(.,y)f(y,v(y))\,dy,\ v \in \Lambda\}$, is equicontinuous on $\overline{D} \cup \{\infty\}$. Let $x_0 \in \overline{D}$ and $\varepsilon > 0$. By (H_2), (1.11) and (1.12), there exist $r > 0$ and $M > 1$ such that

$$\sup_{z\in D} \int_{B(x_0,2r)\cap D} G_D(z,y)f(y,\|\omega\|_\infty)\,dy \le \frac{\varepsilon}{4}$$

and

$$\sup_{z\in D} \int_{(|y|\ge M)\cap D} G_D(z,y)f(y,\|\omega\|_\infty)\,dy \le \frac{\varepsilon}{4}.$$

Let $x, x' \in B(x_0,r)\cap D$, then for each $v \in \Lambda$, we have

$$\begin{aligned}
&\left|\int_D G_D(x,y)f(y,v(y))\,dy - \int_D G_D(x',y)f(y,v(y))\,dy\right| \\
&\le \int_D \left|G_D(x,y) - G_D(x',y)\right| f(y,\|\omega\|_\infty)\,dy \\
&\le 2\sup_{z\in D} \int_{B(x_0,2r)\cap D} G_D(z,y)f(y,\|\omega\|_\infty)\,dy \\
&\quad + 2\sup_{z\in D} \int_{(|x_0-y|\ge 2r)\cap(|y|\ge M)\cap D} G_D(z,y)f(y,\|\omega\|_\infty)\,dy \\
&\quad + \int_{(|x_0-y|\ge 2r)\cap(|y|\le M)\cap D} \left|G_D(x,y) - G_D(x',y)\right| f(y,\|\omega\|_\infty)\,dy \\
&\le \varepsilon + \int_{(|x_0-y|\ge 2r)\cap(|y|\le M)\cap D} \left|G_D(x,y) - G_D(x',y)\right| f(y,\|\omega\|_\infty)\,dy.
\end{aligned}$$

On the other hand, for every $y \in B^c(x_0,2r)\cap B(0,M)\cap D$ and $x, x' \in B(x_0,r)\cap D$, we have by using (1.9),

$$\begin{aligned}
|G_D(x,y) - G_D(x',y)| &\le G_D(x,y) + G_D(x',y) \\
&\le C\left[\frac{\rho_D(x)\rho_D(y)}{|x-y|^{n-2}} + \frac{\rho_D(x')\rho_D(y)}{|x'-y|^{n-2}}\right] \\
&\le C\left[\frac{1}{|x-y|^{n-2}} + \frac{1}{|x'-y|^{n-2}}\right]\rho_D(y) \\
&\le C.\delta(y) \\
&\le C\frac{\delta(y)}{(|y|^{n-1}+1)}
\end{aligned}$$

Now since G_D is continuous outside the diagonal,we deduce by the dominated convergence theorem, (H_2) and Proposition 1.5 (iii), that

$$\int_{(|x_0-y|\ge 2r)\cap(|y|\le M)\cap D} \left|G_D(x,y) - G_D(x',y)\right| f(y,\|\omega\|_\infty)\,dy \to 0 \text{ as } \left|x - x'\right| \to 0.$$

Hence $\{\int_D G_D(.,y)f(y,v(y))\,dy,\ v\in\Lambda\}$, is equicontinuous on $\overline{D}$.
Next, we need to prove that $\{\int_D G_D(.,y)f(y,v(y))\,dy,\ v\in\Lambda\}$, is equicontinuous at ∞, whenever D is unbounded.
Let $x\in D$ such that $|x|\geq M+1$. Then for each $v\in\Lambda$, we have

$$\begin{aligned}\left|\int_D G_D(x,y)f(y,v(y))\,dy\right| &\leq \int_{(|y|\geq M)\cap D} G_D(x,y)f(y,\|\omega\|_\infty)\,dy\\ &\quad+\int_{(|y|\leq M)\cap D} G_D(x,y)f(y,\|\omega\|_\infty)\,dy\\ &\leq \frac{\varepsilon}{4}+\int_{(|y|\leq M)\cap D} G_D(x,y)f(y,\|\omega\|_\infty)\,dy.\end{aligned}$$

For $y\in B(0,M)\cap D$, we have $|x-y|\geq 1$. Hence by (1.9), we get

$$\begin{aligned}\left|\int_D G_D(x,y)f(y,v(y))\,dy\right| &\leq \frac{\varepsilon}{4}+C\int_{(|y|\leq M)\cap D}\frac{\rho_D(y)}{|x-y|^{n-2}}f(y,\|\omega\|_\infty)\,dy\\ &\leq \frac{\varepsilon}{4}+\frac{C}{(|x|-M)^{n-2}}\int_{(|y|\leq M)\cap D}\delta(y)f(y,\|\omega\|_\infty)\,dy\\ &\leq \frac{\varepsilon}{4}+\frac{C}{(|x|-M)^{n-2}}\int_{(|y|\leq M)\cap D}\frac{\delta(y)f(y,\|\omega\|_\infty)}{(|y|^{n-1}+1)}dy.\end{aligned}$$

Using again Proposition 1.5(*iii*), we obtain $\int_D G_D(x,y)f(y,v(y))\,dy \to 0$ as $|x|\to\infty$, uniformly in $v\in\Lambda$. Therefore by Ascoli's theorem, the family $\{\int_D G_D(x,y)f(y,v(y))\,dy,\ v\in\Lambda\}$ becomes relatively compact in $C_0(D)$.
Since $\omega\in C(\overline{D}\cup\{\infty\})$, then we deduce that the set $T\Lambda$ is relatively compact in $C(\overline{D}\cup\{\infty\})$.
On the other hand, since f is a nonnegative function, it is clear from (2.1) and (1.5) that $T\Lambda\subset\Lambda$.
Next, we prove the continuity of the operator T in Λ in the supremum norm. Let $(v_k)_k$ be a sequence in Λ which converges uniformly to a function v in Λ. Then we have

$$|Tv_k(x)-Tv(x)|\leq\lambda\int_D G_D(x,y)\,|f(y,v_k(y))-f(y,v(y))|\,dy.$$

From the monotonicity of f, we have

$$|f(y,v_k(y))-f(y,v(y))|\leq 2f(y,\|\omega\|_\infty),$$

Since by (H_2), (1.10) and Proposition 1.5 (*i*), $Vf(y,\|\omega\|)$ is bounded, we conclude by the continuity of f with respect to the second variable and by the dominated convergence theorem, that

$$\forall x\in\overline{D}\cup\{\infty\},\ Tv_k(x)\to Tv(x)\text{ as }k\to\infty.$$

Using the fact that $T\Lambda$ is relatively compact in $C(\overline{D}\cup\{\infty\})$, we obtain the uniform convergence, namely

$$\|Tv_k-Tv\|_\infty\to 0\ \text{ as }k\to\infty.$$

Thus we have proved that T is a compact mapping from Λ to itself. Hence by the Schauder's fixed point theorem, there exists $u \in \Lambda$ such that

$$u(x) = \omega(x) - \lambda \int_D G_D(x,y) f(y, u(y))\, dy. \tag{2.3}$$

In addition, since for each $x \in D$, $f(y, u(y)) \leq f(y, \|\omega\|_\infty)$, we deduce by the hypothesis (H_2) and Proposition 1.5 (iii) that the map $y \to f(y, u(y)) \in L^1_{loc}(D)$ and by (2.3), that $x \to \int_D G_D(x,y) f(y, u(y))\, dy \in L^1_{loc}(D)$. Thus using these facts, (2.3) and (2.2), we deduce that u is the required solution. ■

Example 2.1. *Assume that $g : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function satisfying for each $c > 0$, there exists $\eta > 0$ such that*

$$g(t) \leq \eta t, \quad \forall t \in [0, c]. \tag{2.4}$$

Let p be a positive measurable function satisfying

$$p(x) \leq \frac{C}{(\delta(x))^\sigma} \text{ with } \sigma < 2, \text{ (if } D \text{ is bounded)}$$

or

$$p(x) \leq \frac{C}{(\delta(x))^\sigma |x|^{\mu-\sigma}} \text{ with } \sigma < 2 < \mu, \text{ (whenever } D \text{ is unbounded)}.$$

Then form [2] *and* [10], $p \in K(D)$.
Let φ is a positive continuous function on ∂D and $a \geq 0$, $\alpha \geq 0$ such that $a + \alpha > 0$.
Put $\omega(x) = aH_D\varphi(x) + \alpha h(x)$. Then by (2.4) *and* (1.10), *we have*

$$\frac{\omega(x)}{V(p(.)g(\omega))(x)} \geq \frac{\omega(x)}{\eta V(p\omega)(x)} \geq \frac{1}{\eta C_0 \|p\|_D} > 0.$$

This implies that $\lambda_0 \geq \frac{1}{\eta C_0 \|p\|_D} > 0$.
Therefore by Theorem 1.2, for each $\lambda \in [0, \lambda_0)$, the problem

$$\begin{cases} \Delta u = \lambda p(x) g(u) & \text{in } D, \text{ (in the sense of distributions)} \\ u > 0 & \text{in } D, \\ u = a\varphi & \text{on } \partial D, \\ \lim_{|x| \to \infty} u(x) = \alpha & \text{(whenever } D \text{ is unbounded)}, \end{cases}$$

has at least one continuous bounded solution u such that

$$(1 - \frac{\lambda}{\lambda_0})\omega \leq u \leq \omega \text{ in } D.$$

Acknowledgement 1. *The authors want to thank the referee for a careful reading of the paper. The research of Imed Bachar is supported by NPST Program of King Saud University; project number 11-MAT1716-02.*

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