# On the approximation of functions by Fourier Stieltjes Series *

Chengbo Lu Dansheng Yu†

**Abstract**

Recently, Leindler [2] introduced the sequences of Head Bounded Variation ($HBVS$) and the sequences of Rest Bounded Variation ($RBVS$), which are nontrivial generalizations of nondecreasing sequences and nonincreasing sequences respectively. In the present paper, we generalize a classical result of Mazhar([1]) on the approximation by means of Fourier Stieltjes series by using the $HBVS$ and $RBVS$.

## 1 Introduction

Let $F(x)$ be a function of bounded variation on $[0, 2\pi]$. Then the Fourier Stieltjes Series of $dF$ and its conjugate series are defined by

$$dF(x) \sim \sum_{v=-\infty}^{+\infty} c_v e^{ivx}, \tag{1.1}$$

and

$$-i \sum_{v=-\infty}^{+\infty} (\text{sign } v) c_v e^{ivx}, \tag{1.2}$$

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†Corresponding author.
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where

$$c_v = \frac{1}{2\pi}\int_0^{2\pi} e^{-ivt}dF(t) \quad (v = 0, \pm 1, \pm 2, \ldots).$$

It is convenient to define $F(x)$ for all values of $x$ by $F(x+2\pi) - F(x) = F(2\pi) - F(0)$. This enables us to integrate, in the formula for $c_v$, over any interval of length $2\pi$.

Write

$$F_x(t) = F(x+t) - F(x-t) - 2tF'(x),$$
$$G_x(t) = F(x+t) + F(x-t) - 2F(x),$$

and denote the total variation of $f(t)$ over the interval $[0,t]$ by $V_0^t(f)$.

Let $\Lambda = (\lambda_{n,k}),\ n = 0,1,2,\ldots; k = 0,1,\cdots,n$ be a triangular matrix, $\{s_k\}$ be a given sequence of numbers. Then the so called $\Lambda-$mean of $\{s_k\}$ is defined as

$$\sigma_n = \sum_{k=0}^{n} \lambda_{n,k}s_k, \ \ n = 1,2,\cdots.$$

In what follows we assume that $C$ is a positive constant not necessarily the same at each occurrence.

There are a lot of papers on the degree of approximation by means of Fourier Stieltjes Series. Among them, Mazhar [1] proved the following.

**Theorem A.** *Let $\{\lambda_{n,k}\}$ satisfy the following conditions*

$$\lambda_{n,k} \geq 0 \qquad and \qquad \sum_{k=0}^{n} \lambda_{n,k} = 1, \tag{1.3}$$

$$\lambda_{n,k} \geq \lambda_{n,k+1}, \qquad (k = 0,1,\ldots,n-1; n = 0,1,\ldots). \tag{1.4}$$

*And let $t_n(x)$ and $\tilde{t}_n(x)$ denote respectively the $\Lambda$-means of series (1.1) and (1.2). Then*

$$|t_n(x) - F'(x)| \leq C\sum_{k=0}^{n} V_0^{\frac{\pi}{k+1}}(F_x)\sum_{\nu=0}^{k}\lambda_{n,n-\nu}; \tag{1.5}$$

$$\left|\tilde{t}_n(x) - \left\{-\frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\pi}\frac{G_x(t)dt}{(2sint/2)^2}\right\}\right| \leq C\sum_{k=0}^{n} V_0^{\frac{\pi}{k+1}}(G_x)\sum_{\nu=0}^{k}\lambda_{n,n-\nu}. \tag{1.6}$$

Recently, Leindler [2] introduced the sequences of Head Bounded Variation ($HBVS$) and the sequences of Rest Bounded Variation ($RBVS$), which are nontrivial generalizations of nondecreasing sequences and nonincreasing sequences respectively.

For a fixed $n$, $\alpha_n := \{\alpha_{n,k}\}_{k=0}^{\infty}$ of nonnegative numbers tending to zero is called of *Head bounded variation*, or briefly $\alpha_n \in HBVS$, if there is a constant $C(\alpha_n)$ only depend on $\alpha_n$ such that

$$\sum_{k=0}^{m-1} |\Delta a_{n,k}| := \sum_{k=0}^{m-1} |a_{n,k} - a_{n,k+1}| \leq C(\alpha_n)a_{n,m} \tag{1.7}$$

for all natural numbers $m$, or only for all $m \leq N$ if the sequence $\lambda_n$ has only finite nonzero terms, and the last nonzero term is $a_{n,N}$.

For a fixed $n$, $\alpha_n = \{\alpha_{n,k}\}_{k=0}^{\infty}$ of nonnegative numbers tending to zero is called of *Rest bounded variation*, or briefly $\alpha_n \in RBVS$, if there is a constant $C(\alpha_n)$ only depend on $\alpha_n$ such that

$$\sum_{k=m}^{\infty} |\Delta a_{n,k}| \leq C(\alpha_n) a_{n,m} \tag{1.8}$$

for all natural numbers $m$.

It should be noted that, in (1.7) and (1.8), a sequence of monotone sequences $\alpha_n := \{a_{n,k}\}_{k=0}^{\infty}$ are involved. Thus, it is natural to assume that $\{C(\alpha_n)\}_{n=0}^{\infty}$ is bounded, that is an absolute constant $C$ such that

$$0 \leq C(\alpha_n) \leq C$$

holds for all $n$.

It is clear that every monotone increasing sequence is an $HBVS$, but not conversely. Similarly every monotone decreasing null-sequence is an $RBVS$, but not conversely ([2]).

In the present paper, we show that the monotonic condition of $\{\lambda_{n,k}\}$ in Theorem A can be essentially relaxed to $HBVS$. And we further get some new results when the sequence $\{\lambda_{n,k}\}$ belongs to the class $RBVS$. In fact, we have the following:

**Theorem 1.** If $(\lambda_{n,k})$ *satisfies the condition (1.3) and* $(\lambda_{n,k}) \in HBVS$, *then (1.5) and (1.6) still hold.*

**Theorem 2.** If $(\lambda_{n,k})$ *satisfies the condition (1.3) and* $(\lambda_{n,k}) \in RBVS$, *then*

$$|t_n(x) - F'(x)| \leq C \sum_{k=0}^{n} V_0^{\frac{\pi}{k+1}}(F_x) \sum_{\nu=0}^{k} \lambda_{n,\nu}; \tag{1.9}$$

$$\left| \tilde{t}_n(x) - \left\{ -\frac{1}{\pi} \int_{\frac{\pi}{n+1}}^{\pi} \frac{G_x(t)dt}{(2sint/2)^2} \right\} \right| \leq C \sum_{k=0}^{n} V_0^{\frac{\pi}{k+1}}(G_x) \sum_{\nu=0}^{k} \lambda_{n,\nu}. \tag{1.10}$$

## 2 Auxiliary Lemmas

We need some Lemmas.

**Lemma 1. ([2])** *If* $\{\lambda_n\} \in HBVS$, *then*

$$\lambda_n \leq C\lambda_m \tag{2.1}$$

*holds for any* $m \geq n \geq 0$.

**Lemma 2. ([2])** *If* $\{\lambda_n\} \in RBVS$, *then*

$$\lambda_m \leq C\lambda_n \tag{2.2}$$

*holds for any* $m \geq n \geq 0$.

**Lemma 3.** *Let $(\lambda_{n,k})$ satisfy (1.3) and $\{\lambda_{n,k}\}_{k=0}^{\infty} \in HBVS$, then*

$$\frac{1}{n+1}\sum_{\nu=0}^{n}\lambda_{n,n-\nu} \leq \frac{C}{k+1}\sum_{\nu=0}^{k}\lambda_{n,n-\nu}. \tag{2.3}$$

*Proof.* By Lemma 1, for any $\nu \geq k+1$, we have

$$\lambda_{n,n-\nu} \leq C\frac{1}{k+1}\sum_{\nu=0}^{k}\lambda_{n,n-\nu},$$

and thus

$$\sum_{\nu=k+1}^{n}\lambda_{n,n-\nu} \leq C\frac{n-k}{k+1}\sum_{\nu=0}^{k}\lambda_{n,n-\nu}. \tag{2.4}$$

By (2.4), we deduce that

$$\begin{aligned}\frac{1}{n+1}\sum_{\nu=0}^{n}\lambda_{n,n-\nu} &= \frac{1}{n+1}\sum_{\nu=0}^{k}\lambda_{n,n-\nu} + \frac{1}{n+1}\sum_{\nu=k+1}^{n}\lambda_{n,n-\nu} \\ &\leq \frac{1}{n+1}\sum_{\nu=0}^{k}\lambda_{n,n-\nu}\left(1+C\frac{n-k}{k+1}\right) \\ &\leq \frac{C}{n+1}\sum_{\nu=0}^{k}\lambda_{n,n-\nu}\cdot\frac{k+1+n-k}{k+1} \\ &= \frac{C}{k+1}\sum_{\nu=0}^{k}\lambda_{n,n-\nu}.\end{aligned}$$

Similarly, we can get

**Lemma 4.** *Let $(\lambda_{n,k})$ satisfy (1.3) and $\{\lambda_{n,k}\}_{k=0}^{\infty} \in RBVS$, then*

$$\frac{1}{n+1} \leq \frac{C}{k+1}\sum_{\nu=0}^{k}\lambda_{n,\nu}. \tag{2.5}$$

Let $\gamma_n(t)$ be a linear function on $[k, k+1]$ such that $\gamma_n(k) = \lambda_{n,n-k}$, $k = 0, 1, 2, \ldots, n$, and let

$$\Gamma_n(t) = \int_0^t \gamma_n(u)du, \quad t \geq 0. \tag{2.6}$$

By (1.3) and Lemma 1, one can prove the following.

**Lemma 5.** *Let $\{\lambda_{n,k}\}_{k=0}^{\infty} \in HBVS$, then*

$$\Gamma_n(k) \sim \Gamma_n(k+1) \sim \sum_{\nu=0}^{k}\lambda_{n,n-\nu}, \tag{2.7}$$

where $A \sim B$ means that there exists positive constants $K_1$ and $K_2$ such that:

$$K_1 B \leq A \leq K_2 B.$$

**Lemma 6.** *Let* $\{\lambda_{n,k}\}_{k=0}^{\infty} \in HBVS$. *Then*

$$\left|\sum_{k=0}^{n} \lambda_{n,n-k} sin(n-k)t\right| \leq C\left(\Gamma_n(\pi/t)\right).$$

*Proof.* By Lemma 5, it is element to deduce that

$$\begin{aligned}
&\left|\sum_{k=0}^{n} \lambda_{n,n-k} sin(n-k)t\right| \\
&\leq \sum_{k=0}^{\tau} \lambda_{n,n-k} + O\left(\frac{1}{t}\right)\cdot\left(\lambda_{n,n-\tau} + \sum_{k=\tau}^{n-1} |\Delta\lambda_{n,n-k}| + \lambda_{n,0}\right) \qquad \left(\tau := \left[\frac{\pi}{t}\right]\right) \\
&\leq \sum_{k=0}^{\tau} \lambda_{n,n-k} + O\left(\frac{1}{t}\right)\cdot\left(\lambda_{n,n-\tau} + \lambda_{n,0}\right) \\
&= O\left(\sum_{k=0}^{\tau} \lambda_{n,n-k}\right) \qquad (\lambda_{n,n-\tau} = O(\lambda_{n,n-k}), k = 0,1,2,\ldots,\tau) \\
&= O(\Gamma_n(\tau)) \qquad (\text{By}(2.7)) \\
&= O\left(\Gamma_n(\pi/t)\right). && (2.8)
\end{aligned}$$

Let $\phi_n(t)$ be a linear function on $[k, k+1]$ such that $\phi_n(k) = \lambda_{n,k}, k = 0,1,2,\ldots,n$, and let

$$\Phi_n(t) = \int_0^t \phi_n(u)du, \quad t \geq 0. \tag{2.9}$$

Similar to Lemma 5 and Lemma 6, we have
**Lemma 7.** If $\{\lambda_{n,k}\}_{k=0}^{\infty} \in RBVS$, then

$$\Phi_n(k) \sim \Phi_n(k+1) \sim \sum_{v=0}^{k} \lambda_{n,v}.$$

**Lemma 8**. If $\{\lambda_{n,k}\}_{k=0}^{\infty} \in RBVS$, then

$$\left|\sum_{k=0}^{n} \lambda_{n,k} sinkt\right| = O\left(\Phi_n\left(\frac{\pi}{t}\right)\right). \tag{2.10}$$

## 3 Proofs of Results

*Proof of (1.5).* We will follow the ideas of Mazhar [1].

Write $K_n(t) = \sum\limits_{k=0}^{n} \lambda_{n,k} D_k(t)$, with $D_k(t) = \frac{sin(k+\frac{1}{2}t)}{2sin\frac{t}{2}}$ and denote by $s_n(x)$ the $n$-th partial sum of (1.1), we have

$$t_n(x) = \sum_{k=0}^{n} \lambda_{n,k} s_k(x) = \sum_{k=0}^{n} \lambda_{n,k} \frac{1}{\pi} \int_{-\pi}^{\pi} D_k(x-t) dF(t)$$
$$= \frac{1}{\pi} \int_0^{\pi} \sum_{k=0}^{n} \lambda_{n,k} D_k(t) d[F(x+t) - F(x-t)]$$
$$= \frac{1}{\pi} \int_0^{\pi} K_n(t) d[F(x+t) - F(x-t)].$$

and hence

$$t_n(x) - F'(x) = \frac{1}{\pi} \int_0^{\pi} K_n(t) d[F(x+t) - F(x-t) - 2tF'(x)]$$
$$= \frac{1}{\pi} \int_0^{\pi} K_n(t) dF_x(t) = \frac{1}{\pi} \left( \int_0^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^{\pi} \right) K_n(t) dF_x(t) := I_1 + I_2.$$

Since $|K_n(t)| \leq 2n$ uniformly in $t$, we have

$$|I_1| \leq \frac{1}{\pi} \int_0^{\frac{\pi}{n+1}} 2n |dF_x(t)| \leq \frac{2n}{\pi} V_0^{\frac{\pi}{n+1}}(F_x)$$
$$\leq C \frac{1}{n+1} V_0^{\frac{\pi}{n+1}} F_x \sum_{k=0}^{n} (k+1) = C V_0^{\frac{\pi}{n+1}} F_x \sum_{k=0}^{n} (k+1) \frac{1}{n+1} \sum_{\nu=0}^{n} \lambda_{n,n-\nu}$$
$$\leq C V_0^{\frac{\pi}{n+1}} F_x \sum_{k=0}^{n} (k+1) \frac{1}{k+1} \sum_{\nu=0}^{k} \lambda_{n,n-\nu} \qquad \text{(By Lemma 3)}$$
$$\leq C \sum_{k=0}^{n} V_0^{\frac{\pi}{n+1}} F_x \sum_{\nu=0}^{k} \lambda_{n,n-\nu}. \quad (3.1)$$

$$|I_2| \leq \frac{1}{\pi} \int_{\frac{\pi}{n+1}}^{\pi} |K_n(t)||dF_x(t)| \leq C \int_{\frac{\pi}{n+1}}^{\pi} |dF_x(t)| \frac{\Gamma_n(\pi/t)}{t} \qquad \text{(By (2.8))}$$
$$= C \int_{\frac{\pi}{n+1}}^{\pi} \frac{\Gamma_n(\pi/t)}{t} dV_0^t(F_x)$$
$$= C \left\{ \left[ \frac{\Gamma(\pi/t)}{t} V_0^t(F_x) \right]_{\frac{\pi}{n+1}}^{\pi} + \int_{\frac{\pi}{n+1}}^{\pi} V_0^t(F_x) \frac{\Gamma(\pi/t)}{t^2} dt + \int_{\frac{\pi}{n+1}}^{\pi} \pi V_0^t(F_x) \frac{\gamma_n(\pi/t)}{t^3} dt \right\}$$
$$= \frac{C}{\pi} \Gamma_n(1) V_0^{\pi}(F_x) - \frac{(n+1)C}{\pi} \Gamma_n(n+1) V_0^{\frac{\pi}{n+1}}(F_x)$$
$$+ \frac{C}{\pi} \int_1^{n+1} V_0^{\pi/t}(F_x) \Gamma_n(t) dt + \frac{C}{\pi} \int_1^{n+1} t V_0^{\pi/t}(F_x) \gamma_n(t) dt$$
$$\leq C a_{n,n} V_0^{\pi}(F_x) + C(n+1) V_0^{\frac{\pi}{n+1}}(F_x) + C \sum_{k=1}^{n} \int_k^{k+1} V_0^{\pi/t}(F_x) \Gamma_n(t) dt$$
$$+ C \sum_{k=1}^{n} \int_k^{k+1} V_0^{\pi/t}(F_x) t \gamma_n(t) dt$$

$$\leq C\sum_{k=0}^{n} V_0^{\frac{\pi}{k+1}}(F_x)\sum_{\nu=0}^{k}\lambda_{n,n-\nu} \qquad \text{(By Lemma 3 again)}$$

$$+ C\sum_{k=1}^{n} V_0^{\frac{\pi}{k}}(F_x)\Gamma_n(k+1) + C\sum_{k=1}^{n} V_0^{\frac{\pi}{k}}(F_x)(k+1)\left(\frac{\gamma_n(k)+\gamma_n(k+1)}{2}\right)$$

$$\leq C\sum_{k=0}^{n} V_0^{\frac{\pi}{k+1}}(F_x)\lambda_{n,n-\nu} + C\sum_{k=1}^{n} V_0^{\frac{\pi}{k}}(F_x)(k+1)\lambda_{n,n-k}$$

$$\leq C\sum_{k=0}^{n} V_0^{\frac{\pi}{k+1}}(F_x)\sum_{\nu=0}^{k}\lambda_{n,n-\nu}. \quad (3.2)$$

We obtain (1.5) by combining (3.1) and (3.2).

*Proof of (1.6).* It is obvious that

$$\tilde{t}_n(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi}\tilde{K}_n(t)dF(x+t),$$

where

$$\tilde{K}_n(t) = \sum_{\nu=0}^{n}\lambda_{n,k}\tilde{D}_k(t),$$

and

$$\tilde{D}_k(t) = \sum_{\nu=1}^{k}\sin\nu t = \frac{\cos t/2 - \cos(k+\frac{1}{2})t}{2\sin t/2}.$$

Since

$$\tilde{t}_n(x) = -\frac{1}{\pi}\int_0^{\pi}\tilde{K}_n(t)d[F(x+t)+F(x-t)] = -\frac{1}{\pi}\int_0^{\pi}\tilde{K}_n(t)dG_x(t),$$

then

$$\tilde{t}_n(x) - \left(-\frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\pi}\frac{G_x(t)}{(2\sin t/2)^2}\right) = -\frac{1}{\pi}\int_0^{\frac{\pi}{n+1}}\tilde{K}_n(t)dG_x(t)$$
$$+\frac{1}{\pi}\left[-\frac{G_x(t)}{2\tan t/2}\right]_{\frac{\pi}{n+1}}^{\pi} + \frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\pi}\left\{\frac{1}{2\tan t/2} - \tilde{K}_n(t)dt\right\}dG_x(t)$$
$$:= L_1 + L_2 + L_3.$$

Since $|\tilde{K}_n(t)| \leq n$, as shown in (3.1) we have

$$|L_1| \leq \frac{n}{\pi}\int_0^{\frac{\pi}{n+1}}|dG_x(t)| \leq C\sum_{k=0}^{n} V_0^{\frac{\pi}{k+1}}(G_x)\sum_{\nu=0}^{k}\lambda_{n,n-\nu}. \quad (3.3)$$

Also

$$|L_2| = \frac{1}{\pi}\left|G_x\left(\frac{\pi}{n+1}\right) - G_x(0)\right|\frac{1}{2\tan\frac{\pi}{2(n+1)}}$$
$$\leq \frac{(n+1)}{\pi^2}V_0^{\frac{\pi}{n+1}}(G_x) \leq C\sum_{k=0}^{n} V_0^{\frac{\pi}{k+1}}(G_x)\sum_{\nu=0}^{k}\lambda_{n,n-\nu}. \quad (3.4)$$

By (2.8), we have

$$\left|\frac{1}{2\tan t/2}-\tilde{K}_n(t)\right| = \left|\sum_{k=0}^{n}\lambda_{n,k}\left\{\frac{1}{2\tan t/2}-\frac{\cos t/2-\cos(k+\frac{1}{2})t}{2\sin t/2}\right\}\right|$$
$$= \left|\sum_{k=0}^{n}\lambda_{n,n-k}\frac{\cos\left(n-k+\frac{1}{2}\right)t}{2\sin t/2}\right| \le \frac{C}{t}\Gamma_n\left(\frac{\pi}{t}\right),$$

thus

$$|L_3| \le C\int_{\frac{\pi}{n+1}}^{\pi}\frac{1}{t}\Gamma_n\left(\frac{\pi}{t}\right)|dG_x(t)| \le C\sum_{k=0}^{n}V_0^{\frac{\pi}{k+1}}(G_x)\sum_{\nu=0}^{k}\lambda_{n,n-\nu}. \tag{3.5}$$

as shown in $I_2$.

We get (1.6) by combining (3.3)−(3.5).

*Proof of Theorem 2.* By using Lemma 4 and Lemma 8 instead of Lemma 3 and Lemma 6, in a similar way to the proof of Theorem 1, one can prove Theorem 2, we omit the details here.

## References

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Department of Mathematics, Lishui University,
Lishui, 323000, P.R.China
email: lu.chengbo@yahoo.com.cn

Department of Mathematics, Hangzhou Normal University, Xiasha Economic Development Area, Hangzhou, 310036, P.R.China
email: danshengyu@yahoo.com.cn