A decomposition of orthogonally additive polynomials on archimedean vector lattices

Mohamed Ali Toumi

Abstract

We constructively prove a decomposition theorem for order bounded homogeneous orthogonally-additive polynomials on archimedean vector lattices. In this way, we constructively generalize results of Sundaresan (1991), D. Pérez-Garcia, and I. Villanueva, (2005), Y. Benyamini, S. Lassalle and J. L. G. Llavona (2006), D. Carando, S. Lassalle and I. Zalduendo (2006 and 2012).

1 Introduction

Let A and B be vectors lattices. An order bounded map $P : A \to B$ is called a *homogeneous polynomial of degree n* (or a *n-homogeneous polynomial*) if $P(x) = \Psi(x, ..., x)$, where Ψ is an order bounded n-multilinear map from $A^n = A \times ... \times A$ (n-times) into B. In this paper, we only deal with order bounded polynomials, and we will therefore omit the adjective order bounded.
A subset D of A^n is called *order bounded* if there exist $(a_1, ..., a_n)$ and $(b_1, ..., b_n) \in A^n$ such that

$$(a_1, ..., a_n) \leq (x_1, ..., x_n) \leq (b_1, ..., b_n)$$

for all $(x_1, ..., x_n) \in A^n$. A multilinear (or multimorphism) $\Psi : A^n \to B$ is *order bounded* if Ψ maps order bounded subsets of A^n onto order bounded subsets of B.

Received by the editors in April 2012 - In revised form in September 2012.
Communicated by F. Bastin.
2010 *Mathematics Subject Classification :* Primary 06F25, 47B65; Secondary 46G20, 46E50.
Key words and phrases : Orthosymmetric multimorphisms, Orthogonally additive polynomials.

A homogeneous polynomial (of degree n) $P : A \to B$ is said to be *orthogonally-additive* if $P(x+y) = P(x) + P(y)$ whenever $x, y \in A$ are orthogonally (i.e., $|x| \wedge |y| = 0$). We denote by $\mathcal{P}_0\left({}^nA, B\right)$ the set of n-homogeneous orthogonally-additive polynomials from A to B. Interest in orthogonally-additive polynomials on Banach lattices originates in the work of K. Sundaresan [19], in which it has been characterized the space of n-homogeneous orthogonally-additive polynomials on L^p and ℓ^p. More precisely, K. Sundaresan proved that every n-homogeneous orthogonally-additive polynomial $P : L^p \to \mathbb{R}$ is determined by some $g \in L^{\frac{p}{p-n}}$ via the formula $P(f) = \int f^n g \, d\mu$, for all $f \in L^p$. Very recently, D. Pérez-Garcia, and I. Villanueva in [17], D. Carando, S. Lassalle and I. Zalduendo in [11] proved the following analogous result for $C(X)$ spaces: Let Y be a Banach space, let $P : C(X) \to Y$ be an orthogonally-additive n-homogeneous polynomial and let $\Psi : (C(X))^n \to Y$ be its unique associated symmetric multilinear operator. Then there exists a linear operator $S : C(X) \to Y$ such that $\|S\| = \|\Psi\|$ and there exists a finitely additive measure $\nu : \Sigma \to Y^{**}$ such that for every $f \in C(X)$, we have $P(f) = S(f^n) = \int_X f^n g \, d\nu$. Here, Σ is the Borel σ-algebra on X. Using different techniques, Y. Benyamini, S. Lassalle and J. L. G. Llavona [3] have proven the analogous result of K. Sundaresan for the classes of Banach lattices of functions and Köthe Banach lattices. The disadvantage of the above results is that their proofs rely heavily on the representation of vector lattices as vector spaces of extended continuous functions, and that they are not applicable to general vector lattices. Our main purpose is to prove constructively a decomposition theorem for homogeneous orthogonally-additive polynomials on archimedean vector lattices. In fact, the innovation of this paper consists in making a relationship between orthogonally-additive homogeneous polynomials and orthosymmetric multimorphisms which leads to a constructively generalization of Sundaresan result (1991) and those of D. Pérez-Garcia, and I. Villanueva (2005), Y. Benyamini, S. Lassalle and J. L. G. Llavona (2006), D. Carando, S. Lassalle and I. Zalduendo (2006), .A. Ibort, P. Linares and J. G. Llavona on the representation of orthogonally additive polynomials on ℓ_p and positive orthogonally additive polynomials on vector lattices (2009 and 2012). Moreover, involving the vector lattice $\Pi^n(A) = \left\{ \prod_{i=1}^{n} a_i, a_i \in A \right\}$ (here the multiplication under consideration is the f-algebra multiplication of the universal completion of A), the author proved in [20] the following Theorem: Let A be a σ-Dedekind complete vector lattice, B be an archimedean vector lattice and $P : A \to B$ be an orthogonally-additive n-homogeneous polynomial . If P is continuous, with respect to the relatively uniform topology then there exists a linear map $T : \Pi^n(A) \to B$ such that $P(x) = T(x^n)$, for all $x \in A$.

By using the the notion of component,the calculus in σ-Dedekind complete vector lattice is too easy, see [20]. So a natural question raised: what about order bounded orthogonally-additive n-homogeneous polynomials on a vector lattice A (not necessary σ-Dedekind complete)?

The answer is affirmative. Indeed, in this paper, we give an identification of the space of orthogonally-additive n-homogeneous polynomials on a vector lattice A.

We take it for granted that the reader is familiar with the notions of vector lattices (or Riesz spaces) and operators between them. For terminology, notations and concepts that are not explained in this paper we refer to the standard monographs [1], [13] , [15] and [18].

2 Definitions and notations

In order to avoid unnecessary repetitions, we will assume that all vector lattices and ℓ-algebras under consideration are *Archimedean*.

Let us recall some of the relevant notions. Let A be a (real) vector lattice. A vector subspace I of A is called *order ideal* (or *o-ideal*) whenever $|a| \leq |b|$ and $b \in I$ imply $a \in I$. Every o-ideal is a vector sublattice of A. The principal o-ideal generated by $0 \leq e \in A$ is denoted by A_e. An o-ideal I of A is called *band* if each subset J of I such that $\sup J = x \in A$ implies $x \in I$. For each subset B of A, B^d denotes the set $\{x \in A, |x| \wedge |y| = 0, \forall y \in B\}$ and B^d is called the *orthogonal band* of B. The set B^{dd} denotes $\left(B^d\right)^d$ and called the *band generated* by B. A band B of A is called *order dense* in A if $B^{dd} = A$.

Let A be a vector lattice (or Riesz space). A subset S of the positive cone A^+ is called an *orthogonal system of A* if $0 \notin S$ and if $u \wedge v = 0$ for each pair (u, v) of distinct elements of S. It is clear from Zorn's lemma that every orthogonal system of A is contained in a maximal orthogonal system. An element e of a vector lattice A is called *weak order unit* (resp *strong order unit*) of A whenever $\{e\}$ is a maximal orthogonal system of A (resp $A_e = A$).

Let A be a vector lattice, let $0 \leq v \in A$, the sequence $\{a_{n,}\ n = 1, 2, ...\}$ in A is called (v) *relatively uniformly convergent* to $a \in A$ if for every real number $\varepsilon > 0$, there exists a natural number n_ε such that $|a_n - a| \leq \varepsilon v$ for all $n \geq n_\varepsilon$. This will be denoted by $a_n \to a\ (v)$. If $a_n \to a\ (v)$ for some $0 \leq v \in A$, then the sequence $\{a_{n,}\ n = 1, 2, ...\}$ is called *(relatively) uniformly convergent* to a, which is denoted by $a_n \to a(r.u)$. The notion of (v) *relatively uniformly Cauchy* sequence is defined in the obvious way. A vector lattice is called *relatively uniform complete* if every relatively uniformly Cauchy sequence in A has a unique limit. Relatively uniformly limits are unique in archimedean vector lattices, see [14, Theorem 63.2].

A linear mapping T defined on a vector lattice A with values in a vector lattice B is called *positive* if $T(A^+) \subset B^+$ (notation $T \in \mathcal{L}^+(A,B)$ or $T \in \mathcal{L}^+(A)$ if $A = B$).

A positive operator π on a vector lattice A is called *positive orthomorphism* if it follows from $x \wedge y = 0$ that $\pi(x) \wedge y = 0$. The difference of two positive orthomorphisms is called an *orthomorphism*. The collection of all orthomorphisms on A is denoted by *Orth(A)*.

In the following lines, we recall definitions and some basic facts about *f*-algebras. For more information about this field, we refer the reader to [1,16]. A (real) algebra A which is simultaneously a vector lattice such that the partial ordering and the multiplication in A are compatible, that is $a, b \in A^+$ implies $ab \in A^+$, is called *lattice-ordered algebra* (briefly *ℓ-algebra*). In an ℓ-algebra A, we denote the collection of all nilpotent elements of A by $N(A)$. An ℓ-algebra A is referred to be *semiprime* if $N(A) = \{0\}$. An ℓ-algebra A is called an *f-algebra* if A verifies the property that $a \wedge b = 0$ and $c \geq 0$ imply $ac \wedge b = ca \wedge b = 0$. Any *f*-algebra is automatically commutative and has positive squares. Every unital *f*-algebra (i.e., an *f*-algebra with a unit element) is semiprime.

The next paragraph of this section deals with some facts about Dedekind complete and universally complete vector lattices. A vector lattice A is called *Dedekind complete* if for each non-void majorized set $B \subset A$, sup B exists in A. Every vector lattice A has a *Dedekind completion* $A^{\mathfrak{d}}$, this means that there exists a Dedekind complete vector lattice $A^{\mathfrak{d}}$ containing A as a vector sublattice and such that

$$x' = \sup\left\{x \in A, x \leq x'\right\} = \inf\left\{x \in A, x \geq x'\right\}$$

holds for each $x' \in A^{\mathfrak{d}}$. For more about this concept, see [14, chap IV].

A vector lattice A is called *laterally complete* if every orthogonal system in A has a supremum in A and if A is Dedekind complete and laterally complete, A is said to be *universally complete*. Every vector lattice A has a *universal completion* A^u, this means that there exists a unique (up to a lattice isomorphism) universally complete vector lattice A^u such that A can be identified as an order dense sublattice of A^u. For more properties about universal completion, see [1, Chap II].
We finish this section with some definitions about multilinear maps on vector lattices. Let A and B be vector lattices. A multilinear map Ψ from A^n into B is said to be *positive* whenever $(a_1, ..., a_n) \in (A^+)^n$ imply $\Psi(a_1, ..., a_n) \in B^+$. A bilinear map Ψ from A^n into B is said to be *orthosymmetric* if for all $(a_1, ..., a_n) \in A^n$ such that $a_i \wedge a_j = 0$ for some $1 \leq i, j \leq n$ implies $\Psi(a_1, ..., a_n) = 0$, see [5,7].

3 A decomposition theorem

Our main purpose is to prove constructively a decomposition theorem for homogeneous orthogonally-additive polynomials on archimedean vector lattices. To reach our goal, we need some prerequisites. The following proposition, which is important for the context of this work, is already proven in [17, Proposition 2.2] for the special case $B = \mathbb{R}$. In order to make this paper self contained, we reproduce the same proof as in [17, Proposition 2.2]

Proposition 1. *Let A and B be vector lattices, let P: $A \to B$ be a homogeneous polynomial of degree n and let $\Psi : A^n \to B$ be its associated symmetric n-multilinear operator. Then P is orthogonally-additive if and only if for every $1 < s \leq n$ and $1 \leq n_1, ..., n_s \leq n-1$ such that $n_1 + ... + n_s = n$ and for every mutually orthogonal $f_1, ..., f_s \in A$, we have that*

$$\Psi\left(f_1, \overset{n_1}{\ldots}, f_1, ..., f_s, \overset{n_S}{\ldots}, f_s\right) = 0. \qquad (FFF)$$

Proof. One of the implications is clear. For the other, we fix $1 \leq n_1, ..., n_s \leq n-1$ such that $n_1 + ... + n_s = n$ and mutually orthogonal $f_1, ..., f_s \in A$. Let us take scalars $\lambda_1, ..., \lambda_s$. The orthogonal additivity of the polynomial gives us that

$$P(\lambda_1 f_1 + ... + \lambda_s f_s) = \lambda_1^n P(f_1) + ... + \lambda_s^n P(f_s)$$

Moreover, we have that

$$\Psi(\lambda_1 f_1 + ... + \lambda_s f_s, .., \lambda_1 f_1 + ... + \lambda_s f_s) = \sum_{i=1}^{s} \lambda_i^n \Psi(f_i, .., f_i).$$

Using the symmetry of Ψ, we get

$$\sum_{1\leq\gamma_1+..+\gamma_s=n}\lambda_1^{\gamma_1}..\lambda_s^{\gamma_s}\Psi(f_1,\overset{\gamma_1}{..},f_1,...,f_s,\overset{\gamma_s}{..},f_s)=0.$$

Thus we have the polynomial Q in $\lambda_1,..,\lambda_s$, with coefficients in B, given by

$$Q\left(\lambda_1,..,\lambda_s\right)=\sum_{1\leq\gamma_1+..+\gamma_s=n}\lambda_1^{\gamma_1}..\lambda_s^{\gamma_s}\Psi(f_1,\overset{\gamma_1}{..},f_1,...,f_s,\overset{\gamma_s}{..},f_s)=0.$$

We get then

$$\Psi\left(f_1,\overset{n_1}{...},f_1,...,f_s,\overset{n_S}{...},f_s\right)=0$$

which gives the required result. ■

Lemma 1. *Let A and B be vector lattices, let $P\colon A\to B$ be a homogeneous polynomial of degree n, let $\Psi:A^n\to B$ be its associated symmetric n-multilinear operator and let $1\leq n_1,n_2,n_3\leq n-1$ such that $n_1+n_2+n_3=n$. Then*

$$\Psi\left(f_1,\overset{n_1}{...},f_1,f_2,\overset{n_2}{...},f_2,g,\overset{n_3}{...},g\right)=0.$$

for all $f_1,f_2,g\in A$ such that $f_1\wedge f_2=f_1\wedge g=0$.

Proof. Let $f_1,f_2,g\in A$ such that $f_1\wedge f_2=f_1\wedge g=0$. Since Ψ satisfies the property (FFF), it follows that

$$\Psi\left(f_1,\overset{n_1}{...},f_1,f_2+g,\overset{n_2+n_3}{...},f_2+g\right)=0.$$

Hence

$$\sum_{\substack{0\leq i,j\leq n_2+n_3\\ i+j=n_2+n_3}}\frac{(n_2+n_3)!}{i!j!}\Psi\left(f_1,\overset{n_1}{...},f_1,f_2,\overset{i}{...},f_2,g,\overset{j}{...},g\right)=0. \quad (1)$$

Again, since Ψsatisfies the property (FFF), it follows that

$$\Psi\left(f_1,\overset{n_1}{...},f_1,f_2,\overset{n_2+n_3}{...},f_2\right)=\Psi\left(f_1,\overset{n_1}{...},f_1,g,\overset{n_2+n_3}{...},g\right)=0.$$

Then the equality (1) becomes

$$\sum_{\substack{1\leq i,j\leq n_2+n_3\\ i+j=n_2+n_3}}\frac{(n_2+n_3)!}{i!j!}\Psi\left(f_1,\overset{n_1}{...},f_1,f_2,\overset{i}{...},f_2,g,\overset{j}{...},g\right)=0.$$

So

$$\sum_{\substack{1\leq i,j\leq n_2+n_3\\ i+j=n_2+n_3\\ i\neq 1}}\frac{(n_2+n_3)!}{i!j!}\Psi\left(f_1,\overset{n_1}{...},f_1,f_2,\overset{i}{...},f_2,g,\overset{j}{...},g\right)$$

$$=-\frac{(n_2+n_3)!}{1!\left(n_2+n_3-1\right)!}\Psi\left(f_1,\overset{n_1}{...},f_1,f_2,g,\overset{n_2+n_3-1}{...},g\right).$$

Therefore, repeating the same argument with pg in place of g $(0 < p \in \mathbb{N})$, one finds from the last equality that

$$\Psi\left(f_1, \overset{n_1}{\ldots}, f_1, f_2, \overset{n_2+n_3-1}{g, \ldots, g}\right) = 0.$$

Using the iterated process, we conclude that

$$\Psi\left(f_1, \overset{n_1}{\ldots}, f_1, , f_2, \overset{i}{\ldots}, f_2, g, \overset{j}{\ldots}, g\right) = 0$$

for all $0 \leq i, j \leq n_2 + n_3$ such that $i + j = n_2 + n_3$, as required. ∎

Lemma 2. *Let A and B be vector lattices, let $P\colon A \to B$ be a homogeneous polynomial of degree n, let $\Psi : A^n \to B$ be its associated symmetric n-multilinear operator and let $1 \leq n_1, n_2, n_3 \leq n-1$ such that $n_1 + n_2 + n_3 = n$. Then*

$$\Psi\left(f_1, \overset{n_1}{\ldots}, f_1, , f_2, \overset{n_2}{\ldots}, f_2, g, \overset{n_3}{\ldots}, g\right) = 0.$$

for all $f_1, f_2, g \in A$ such that $f_1 \wedge f_2 = 0$ and $g \in \{f_1\}^{dd}$.

Proof. Let $f_1, f_2, g \in A$ such that $f_1 \wedge f_2 = 0$ and $g \in \{f_1\}^{dd}$. Since Ψ satisfies the property (*FFF*), it follows that

$$\Psi\left(f_1 + g, \overset{n_1+n_3}{\ldots}, f_1 + g, f_2, \overset{n_2}{\ldots}, f_2\right) = 0.$$

Hence

$$\sum_{\substack{0 \leq i,j \leq n_1+n_3 \\ i+j=n_1+n_3}} \frac{(n_1+n_3)!}{i!j!} \Psi\left(f_1, \overset{i}{\ldots}, f_1, f_2, \overset{n_2}{\ldots}, f_2, g, \overset{j}{\ldots}, g\right) = 0. \tag{2}$$

Again, since Ψ satisfies the property (*FFF*), it follows that

$$\Psi\left(f_1, \overset{n_1+n_3}{\ldots}, f_1, f_2, \overset{n_2}{\ldots}, f_2\right) = \Psi\left(f_2, \overset{n_2}{\ldots}, f_2, g, \overset{n_1+n_3}{\ldots}, g\right) = 0.$$

Then the equality (2) becomes

$$\sum_{\substack{1 \leq i,j \leq n_1+n_3 \\ i+j=n_1+n_3}} \frac{(n_1+n_3)!}{i!j!} \Psi\left(f_1, \overset{i}{\ldots}, f_1, f_2, \overset{n_2}{\ldots}, f_2, g, \overset{j}{\ldots}, g\right) = 0.$$

So

$$\sum_{\substack{1 \leq i,j \leq n_2+n_3 \\ i+j=n_2+n_3 \\ i \neq 1}} \frac{(n_1+n_3)!}{i!j!} \Psi\left(f_1, \overset{i}{\ldots}, f_1, f_2, \overset{n_2}{\ldots}, f_2, g, \overset{j}{\ldots}, g\right)$$
$$= -\frac{(n_1+n_3)!}{1!\,(n_2+n_3-1)!} \Psi\left(f_1, f_2, \overset{n_2}{\ldots}, f_2, \overset{n_2+n_3-1}{g, \ldots, g}\right).$$

Therefore, repeating the same argument with pg in place of g ($0 < p \in \mathbb{N}$), one finds from the last equality that

$$\Psi\left(f_1, f_2, \overset{n_2}{\dots}, f_2, \overset{n_2+n_3-1}{g, \dots, g}\right) = 0.$$

Using the iterated process, we conclude that

$$\Psi\left(f_1, \overset{i}{\dots}, f_1, f_2, \overset{n_2}{\dots}, f_2, g, \overset{j}{\dots}, g\right) = 0$$

for all $0 \leq i, j \leq n_2 + n_3$ such that $i + j = n_2 + n_3$, as required. ∎

Lemma 3. *Let A and B be vector lattices, let $P\colon A \to B$ be a homogeneous polynomial of degree n, let $\Psi : A^n \to B$ be its associated symmetric n-multilinear operator and let $1 \leq n_1, n_2, n_3, n_4 \leq n-1$ such that $n_1 + n_2 + n_3 + n_4 = n$. Then*

$$\Psi\left(f_1, \overset{n_1}{\dots}, f_1, f_2, \overset{n_2}{\dots}, f_2, g_1, \overset{n_3}{\dots}, g_1, g_2, \overset{n_3}{\dots}, g_2\right) = 0.$$

for all $f_1, f_2, g_1, g_2 \in A$ such that $f_1 \wedge f_2 = 0$ and $g_1, g_2 \in \{f_1\}^{dd}$.

Proof. Let $f_1, f_2, g \in A$ such that $f_1 \wedge f_2 = 0$ and $g_1, g_2 \in \{f_1\}^{dd}$. Since Ψ satisfies the property (*FFF*), it follows that

$$\Psi\left(f_1 + g_1 + \overset{n_1+n_3+n_4}{g_2, \dots, f_1} + g_1 + g_2, f_2, \overset{n_2}{\dots}, f_2\right) = 0.$$

Hence

$$\sum_{\substack{0 \leq i,j \leq n_1+n_3+n_4 \\ i+j = n_1+n_3+n_4}} \frac{(n_1+n_3+n_4)!}{i!j!} \Psi\left(f_1, \overset{i}{\dots}, f_1, f_2, \overset{n_2}{\dots}, f_2, g_1 + g_2, \overset{j}{\dots}, g_1 + g_2\right) = 0. \quad (3)$$

Again, since Ψ satisfies the property (*FFF*), it follows that

$$\Psi\left(\overset{n_1+n_3+n_4}{f_1, \dots, f_1}, f_2, \overset{n_2}{\dots}, f_2\right) = \Psi\left(f_2, \overset{n_2}{\dots}, f_2, g_1 + \overset{n_1+n_3}{g_2, \dots, g_1} + g_2\right) = 0$$

Then the equality (3) becomes

$$\sum_{\substack{1 \leq i,j \leq n_1+n_3+n_4 \\ i+j = n_1+n_3+n_4}} \frac{(n_1+n_3+n_4)!}{i!j!} \Psi\left(f_1, \overset{i}{\dots}, f_1, f_2, \overset{n_2}{\dots}, f_2, g_1 + g_2, \overset{j}{\dots}, g_1 + g_2\right) = 0.$$

By using the previous lemma, we deduce that

$$\Psi\left(f_1, \overset{i}{\dots}, f_1, f_2, \overset{n_2}{\dots}, f_2, g_1 + g_2, \overset{j}{\dots}, g_1 + g_2\right) = 0$$

for all $1 \leq i, j \leq n_1 + n_3$ such that $i + j = n_1 + n_3 + n_4$. It follows that

$$\Psi\left(f_1, \overset{n_1}{\ldots}, f_1, , f_2, \overset{n_2}{\ldots}, f_2, g_1 + g_2, \overset{n_3+n_4}{\ldots}, g_1 + g_2\right) = 0.$$

So

$$\sum_{\substack{0 \leq i,j \leq n_2+n_4 \\ i+j=n_3+n_4}} \frac{(n_3+n_4)!}{i!j!} \Psi\left(f_1, \overset{n_1}{\ldots}, f_1, f_2, \overset{n_2}{\ldots}, f_2, g_1, \overset{i}{\ldots}, g_1, g_2, \overset{j}{\ldots}, g_2\right) = 0.$$

Since

$$\Psi\left(f_1, \overset{n_1}{\ldots}, f_1, f_2, \overset{n_2}{\ldots}, f_2, g_1, \overset{n_3+n_4}{\ldots}, g_1\right) = \Psi\left(f_1, \overset{n_1}{\ldots}, f_1, f_2, \overset{n_2}{\ldots}, f_2, g_2, \overset{n_4+n_3}{\ldots}, g_2\right) = 0,$$

the last equality becomes

$$\sum_{\substack{1 \leq i,j \leq n_3+n_4 \\ i+j=n_3+n_4}} \frac{(n_3+n_4)!}{i!j!} \Psi\left(f_1, \overset{n_1}{\ldots}, f_1, , f_2, \overset{n_2}{\ldots}, f_2, g_1, \overset{i}{\ldots}, g_1, g_2, \overset{j}{\ldots}, g_2\right) = 0.$$

Thus

$$\sum_{\substack{1 \leq i,j \leq n_3+n_4 \\ i+j=n_3+n_4 \\ i \neq 1}} \frac{(n_3+n_4)!}{i!j!} \Psi\left(f_1, \overset{n_1}{\ldots}, f_1, f_2, \overset{n_2}{\ldots}, f_2, g_1, \overset{i}{\ldots}, g_1, g_2, \overset{j}{\ldots}, g_2\right)$$
$$= -\frac{(n_3+n_4)!}{(n_3+n_4-1)!} \Psi\left(f_1, \overset{n_1}{\ldots}, f_1, f_2, \overset{n_2}{\ldots}, f_2, g_1, g_2, \overset{n_3+n_4-1}{\ldots}, g_2\right)$$

Therefore, repeating the same argument with pg_2 in place of g_2 $(0 < p \in \mathbb{N})$, one finds from the last equality that

$$\Psi\left(f_1, \overset{n_1}{\ldots}, f_1, f_2, \overset{n_2}{\ldots}, f_2, g_1, g_2, \overset{n_3+n_4-1}{\ldots}, g_2\right) = 0.$$

Using the iterated process, we conclude that

$$\Psi\left(f_1, \overset{n_1}{\ldots}, f_1, , f_2, \overset{n_2}{\ldots}, f_2, g_1, \overset{i}{\ldots}, g_1, g_2, \overset{j}{\ldots}, g_2\right) = 0$$

for all $0 \leq i, j \leq n_3 + n_4$ such that $i + j = n_3 + n_4$ and the proof is complete. ∎

The following result is crucial for our main result.

Proposition 2. *Let A and B be vector lattices and let Ψ be a symmetric n-multilinear operator from A^n into B satisfying the property (FFF), then Ψ is inevitably orthosymmetric.*

Proof. Let Ψ be a symmetric positive n-multilinear mapping satisfying the property (*FFF*), let $1 \leq n_1, n_2 \leq n - 2$ such that $n_1 + n_2 = n - 1$, let $f_1, f_2 \in A_+$ such that $f_1 \wedge f_2 = 0$ and let $0 \leq g \in A$. Let $g_1 = f_1 - g \wedge f_1$ and $g_2 = g - g \wedge f_1$. It follows that $g = g_2 + g \wedge f_1, f_1 = g_1 + g \wedge f_1$ and $g_1 \wedge g_2 = 0$. Therefore

$$\begin{aligned}
\Psi\left(f_1, \overset{n_1}{\ldots}, f_1, f_2, \overset{n_2}{\ldots} f_2, g\right) &= \Psi\left(g_1 + g \wedge f_1, \overset{n_1}{\ldots}, g_1 + g \wedge f_1, f_2, \overset{n_2}{\ldots} f_2, g_2 + g \wedge f_1\right) \\
&= \Psi\left(g_1, f_1, \overset{n_1-1}{\ldots}, f_1, f_2, \overset{n_2}{\ldots}, f_2, g_2 + g \wedge f_1\right) \\
&\quad + \Psi\left(g \wedge f_1, f_1, \ldots, f_1, f_2, \overset{n_2}{\ldots} f_2, g_2 + g \wedge f_1\right) \\
&= \Psi\left(g_1, f_1, \ldots, f_1, f_2, \overset{n_2}{\ldots}, f_2, g_2\right) \\
&\quad + \Psi\left(g_1, f_1, \overset{n_1-1}{\ldots}, f_1, f_2, \overset{n_2}{\ldots}, f_2, g \wedge f_1\right) \\
&\quad + \Psi\left(g \wedge f_1, f_1, \overset{n_1-1}{\ldots}, f_1, f_2, \overset{n_2}{\ldots}, f_2, g_2\right) \\
&\quad + \Psi\left(g \wedge f_1, f_1, \overset{n_1-1}{.}, f_1, f_2, \overset{n_2}{\ldots}, f_2, g \wedge f_1\right).
\end{aligned}$$

From the fact that $g_1, g \wedge f_1 \leq f_1$, it follows, by using the same idea as in Lemma 1, that

$$\Psi\left(g_1, f_1, \overset{n_1-1}{\ldots}, f_1, f_2, \overset{n_2}{\ldots}, f_2, g \wedge f_1\right) = 0$$

and

$$\Psi\left(g \wedge f_1, f_1, \overset{n_1-1}{\ldots}, f_1, f_2, \overset{n_2}{\ldots}, f_2, g \wedge f_1\right) = 0$$

Moreover, since $f_1 \wedge f_2 = 0$ for all $i \neq 1$ and $g_1 \leq f_1$ then $g_1 \wedge f_2 = 0$, it follows, by using Lemma 3, that

$$\Psi\left(g_1, \overset{n_1}{\ldots}, g_1, f_2, \overset{n_2}{\ldots}, f_2, g_2\right) = 0.$$

Accordingly

$$\Psi\left(g_1, f_1, \overset{n_1-1}{\ldots}, f_1, f_2, \overset{n_2}{\ldots}, f_2, g_2\right) = \sum_{\substack{k+l=n_1 \\ l \geq 1}} \frac{n_1!}{l!k!} \Psi\left(g_1, \overset{k}{\ldots}, g_1, g \wedge f_1, \overset{l}{\ldots}, g \wedge f_1, f_2, \overset{n_2}{\ldots}, f_2, g_2\right).$$

The previous inequalities enable us to get

$$\begin{aligned}
\Psi\left(f_1,\overset{n_1}{\ldots},f_1,f_2,\overset{n_2}{\ldots},f_2,g\right) &= \Psi\left(g_1,f_1,\overset{n_1}{\ldots},f_1,f_2,\overset{n_2}{\ldots},f_2,g_2\right) \\
&\quad + \Psi\left(g\wedge f_1,f_1,\overset{n_1-1}{\ldots},,f_1,f_2,\overset{n_2}{\ldots},f_2,g_2\right) \\
&= \sum_{\substack{k+l=n_1\\ l\geq 1}} \frac{n_1!}{l!k!}\Psi\left(g_1,\overset{k}{\ldots},g_1,g\wedge f_1,\overset{l}{\ldots},g\wedge f_1,f_2,\overset{n_2}{\ldots},f_2,g_2\right) \\
&\quad + \Psi\left(g\wedge f_1,f_1,\overset{n_1-1}{\ldots},f_1,f_2,\overset{n_2}{\ldots},f_2,g_2\right).
\end{aligned} \tag{4}$$

Since

$$\begin{aligned}
&\Psi\left(g_1,\overset{k}{\ldots},g_1,g\wedge f_1,\overset{l}{\ldots},g\wedge f_1,f_2,\overset{n_2}{\ldots},f_2,g_2\right) = \\
&\qquad \Psi\left(g_1,\overset{k}{\ldots},g_1,g\wedge f_1,\overset{l}{\ldots},g\wedge f_1,f_2,\overset{n_2}{\ldots},f_2,g\right) - \\
&\qquad\qquad \Psi\left(g_1,\overset{k}{\ldots},g_1,g\wedge f_1,\overset{l}{\ldots},g\wedge f_1,f_2,\overset{n_2}{\ldots},f_2,g\wedge f_1\right)
\end{aligned}$$

and

$$\begin{aligned}
\Psi\left(g\wedge f_1,f_1,\overset{n_1-1}{\ldots},f_1,f_2,\overset{n_2}{\ldots},f_2,g_2\right) &= \Psi\left(g\wedge f_1,f_1,\overset{n_1-1}{\ldots},f_1,f_2,\overset{n_2}{\ldots},f_2,g\right) \\
&\quad + \Psi\left(g\wedge f_1,f_1,\overset{n_1-1}{\ldots},f_1,f_2,\overset{n_2}{\ldots},f_2,g\wedge f_1\right)
\end{aligned}$$

it follows, by using the same argument as in Lemma 1 and Lemma 3, that

$$\begin{aligned}
&\Psi\left(g_1,\overset{k}{\ldots},g_1,g\wedge f_1,\overset{l}{\ldots},g\wedge f_1,f_2,\overset{n_2}{\ldots},f_2,g\wedge f_1\right) \\
&\qquad = \Psi\left(g\wedge f_1,f_1,\overset{n_1-1}{\ldots},f_1,f_2,\overset{n_2}{\ldots},f_2,g\wedge f_1\right) = 0
\end{aligned}$$

Therefore Equality (4) becomes

$$\begin{aligned}
\Psi\left(f_1,\overset{n_1}{\ldots},f_1,f_2,\overset{n_2}{\ldots},f_2,g\right) &= \sum_{\substack{k+l=n_1\\ l\geq 1}} \frac{n_1!}{l!k!}\Psi\left(g_1,\overset{k}{\ldots},g_1,g\wedge f_1,\overset{l}{\ldots},g\wedge f_1,f_2,\overset{n_2}{\ldots},f_2,g\right) \\
&\quad + \Psi\left(g\wedge f_1,f_1,\overset{n_1-1}{\ldots},f_1,f_2,\overset{n_2}{\ldots},f_2,g\right).
\end{aligned}$$

Repeating the argument with $\frac{g}{p}$ in place of g redefined ($0 < p \in \mathbb{N}$), one finds that the latter equality still hold. So we deduce that

$$\frac{1}{p}\Psi\left(f_1,\overset{n_1}{\ldots},f_1,f_2,\overset{n_2}{\ldots},f_2,g\right)=\sum_{\substack{k+l=n_1\\ l\geq 1}}\frac{n_1!}{l!k!}\Psi\left(g_1,\overset{k}{\ldots},g_1,\frac{g}{p}\wedge f_1,\overset{l}{\ldots},\frac{g}{p}\wedge f_1,f_2,\overset{n_2}{\ldots},f_2,\frac{g}{p}\right)$$
$$+\Psi\left(\frac{g}{p}\wedge f_1,f_1,\overset{n_1-1}{\ldots},f_1,f_2,\overset{n_2}{\ldots},f_2,\frac{g}{p}\right).$$

Then

$$\Psi\left(f_1,\overset{n_1}{\ldots},f_1,f_2,\overset{n_2}{\ldots},f_2,g\right)=\sum_{\substack{k+l=n_1\\ l\geq 1}}\frac{n_1!}{l!k!}\Psi\left(g_1,\overset{k}{\ldots},g_1,\frac{g}{p}\wedge f_1,\overset{l}{\ldots},\frac{g}{p}\wedge f_1,f_2,\overset{n_2}{\ldots},f_2,g\right)$$
$$+\Psi\left(\frac{g}{p}\wedge f_1,f_1,\overset{n_1-1}{\ldots},f_1,f_2,\overset{n_2}{\ldots},f_2,g\right).$$

If $p\to\infty$, it follows that $\frac{g}{p}\wedge f_1\to 0\ (r.u)$ and $g_1=f_1-\frac{g}{p}\wedge f_1\to f_1\ (r.u)$. Hence

$$\Psi\left(g_1,\overset{k}{\ldots},g_1,\frac{g}{p}\wedge f_1,\overset{l}{\ldots},\frac{g}{p}\wedge f_1,f_2,\overset{n_2}{\ldots},f_2,g\right)\to 0\,(r.u)$$

and

$$\Psi\left(\frac{g}{p}\wedge f_1,f_1,\overset{n_1-1}{\ldots},f_1,f_2,\overset{n_2}{\ldots},f_2,g\right)\to 0\,(r.u)$$

Therefore

$$\Psi\left(f_1,\overset{n_1}{\ldots},f_1,f_2,\overset{n_2}{\ldots},f_2,g\right)=0. \tag{5}$$

By using the same argument, we find that

$$\Psi\left(f_1,f_2,g,\ldots,g\right)=0 \tag{6}$$

for all $g\in A$. Now let $f_1\wedge f_2=0$, let $g_1,g_2\in A_+$ and let $n_1,n_2\in\mathbb{N}^*$ such that $n_1+n_2=n-2$. It follows that

$$\Psi\left(f_1,f_2,g_1+g_2,\overset{n_1+n_2}{\ldots},g_1+g_2\right)=0. \tag{7}$$

Since

$$\Psi\left(f_1,f_2,g_1,\overset{n_1+n_2}{\ldots},g_1\right)=\Psi\left(f_1,f_2,g_2,\overset{n_1+n_2}{\ldots},g_2\right)=0$$

it follows that

$$\Psi\left(f_1,f_2,g_1+g_2,\overset{n_1+n_2}{\ldots},g_1+g_2\right)=$$
$$\sum_{\substack{1\leq 1i,j\leq n_1+n_2\\ i+j=n_1+n_2}}\frac{(n_1+n_2)!}{i!j!}\Psi\left(f_1,f_2,g_1,\overset{i}{\ldots},g_1,g_2,\overset{j}{\ldots},g_2\right)=0$$

Thus

$$\sum_{\substack{1\leq i,j\leq n_1+n_2\\ i+j=n_1+n_2\\ i\neq 1}} \frac{(n_1+n_2)!}{i!j!}\Psi\left(f_1,f_2,g_1,\overset{i}{\ldots},g_1,g_1,\overset{j}{\ldots},g_1\right) =$$

$$\frac{(n_1+n_2)!}{(n_1+n_2-1)!}\Psi\left(f_1,f_2,g_1,\overset{n_1+n_2-1}{g_2,\ldots,g_2}\right).$$

Repeating the argument with pg_2 in place of g_2 redefined ($0 < p \in \mathbb{N}$), one finds that the latter equality still hold. If $p \to \infty$, we have

$$\Psi\left(f_1,f_2,g_1,\overset{n_1+n_2-1}{g_2,\ldots,g_2}\right) = 0.$$

By using the same argument, we deduce that

$$\Psi\left(f_1,f_2,g_1,\overset{i}{\ldots},g_1,g_2,\overset{j}{\ldots},g_2\right) = 0$$

for all $1 \leq i,j \leq n_1+n_2$ such that $i+j = n_1+n_2$ and then

$$\Psi\left(f_1,f_2,g_1,\overset{n_1}{\ldots},g_1,g_2,\overset{n_2}{\ldots},g_2\right) = 0.$$

Using the iterated process, we conclude that if $f_1 \wedge f_2 = 0$ and $g_1,\ldots,g_{n-2} \in A_+$ then

$$\Psi\left(f_1,f_2,g_1,\ldots,g_{n-2}\right) = 0,$$

which gives the desired result. ∎

In order to reach our aim, we need the following definition.

Definition 1. *Let A and B be vector lattices. An orthogonally-additive homogeneous polynomial P: $A \to B$ is called positive orthosymmetric polynomial if its associated symmetric n-multilinear operator $\Psi : A^n \to B$ is positive orthosymmetric.*

Now, we present our decomposition theorem.

Theorem 1. *Let A and B be vector lattices, let P: $A \to B$ be a homogeneous polynomial of degree n and let $\Psi : A^n \to B$ be its associated symmetric n-multilinear operator. Then P is an orthogonally-additive polynomial if and only if there exist two positive orthosymmetric polynomials P_1, P_2: $A \to B^{\delta}$(the Dedekind completion of B) such that*

$$P = P_1 - P_2.$$

Proof. Let P: $A \to B$ be an orthogonally-additive homogeneous polynomial of degree n and let $\Psi : A^n \to B$ be its associated symmetric n-multilinear operator. Since Ψ is an orthosymmetric order bounded n-multilinear map, it follows that Ψ is an order bounded variation, see ([8, Theorem 9] or [10, Theorem 3.4]). By using ([9, Theorem 3.1], we deduce that $|\Psi| : A^n \to B^{\delta}$ is defined by

$$|\Psi|\left(f_1,\ldots,f_n\right) = \sup\left\{\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}\ldots\sum_{i_2=1}^{n_n}\left|\Psi\left(\left(x_{1,i_1},\ldots,x_{n,i_n}\right)\right)\right|\right\}$$

where supremum is taken over all $n_1, n_2, ..., n_n \in \mathbb{N}$, and all finite collections $x_{1,i_1}, ..., x_{n,i_n} \in A_+$ with $\sum_{i_1=1}^{n_1} x_{1,i_1} = f_1, ..., \sum_{i_1=1}^{n_n} x_{n,i_n} = f_n$. It is an easy task to show that $|\Psi|(f_1, ..., f_n) \geq 0$, for all $0 \leq f_1, ..., f_n \in A$ and that $|\Psi|$ is orthosymmetric. Let $\Psi_1 = \frac{1}{2}(|\Psi| + \Psi)$ and $\Psi_2 = \frac{1}{2}(|\Psi| - \Psi)$. It is clear that Ψ_1 and Ψ_2 are positive orthosymmetric n-multilinear maps, as required. ■

Let us denote $\Delta(^n A, B)$ by the set all n-multilinear orthosymmetric maps from A^n into B. When examining Theorem 1, we shall prove that $\Delta(^n A, B)$ and $\mathcal{P}_0(^n A, B)$ are mutually equivalent. Hence in what follows, we will study the order aspects of the set $\Delta(^n A, B)$.

Actually, when examining the proof of Theorem 1, we have proved the following corollary.

Corollary 1. *Let A be a vector lattice and let B be a Dedekind complete vector lattice. Then under the following ordering*

$$\Psi_1 \leq \Psi_2 \textit{ if and only if } \Psi_1(f_1, ..., f_n) \leq \Psi_2(f_1, ..., f_n)$$

for all $f_1, .., f_n \in A_+$ and for all $\Psi_1, \Psi_2 \in \Delta(^n A, B)$, $\Delta(^n A, B)$ is a vector lattice.

It is well known that for any uniformly complete vector lattice A with strong order unit e, there exists a unique multiplication in A such that A is an f-algebra with a unit element e, see |16, Remark 19.5]. Next, we give a short proof of the corresponding result for any universally complete vector lattice.

Lemma 4. *Let A be a universally complete vector lattice and let e be a weak order unit of A. Then there exists a unique multiplication in A such that A is an f-algebra with a unit element e.*

Proof. Since there exists a unique multiplication (denoted by juxtaposition) on A_e in such a manner A_e becomes an f-algebra with a unit element e (see [16, Remark 19.5]). Let $x \in A_e$. Since every $\pi_x \in Orth(A_e)$ defined by $\pi_x(y) = xy$ is an order continuous, it is not hard to prove that π_x has a unique extension $\pi^* \in Orth(A)$, then A becomes an f-algebra with a unit element e, which completes the proof. ■

Remark 1. *We remark that any universally complete vector lattice can be seen as a universally complete unital f-algebra. So in the sequel we denote its f-algebra multiplication by juxtaposition.*

Recall that if A is a vector lattice, if A^u is its universal completion and if $F(X_1, .., X_n) \in \mathbb{R}_+[X_1, .., X_n]$ is a homogeneous polynomial of degree p ($0 < p \in \mathbb{N}$) then the element $(F(u_1, .., u_n))^{\frac{1}{p}}$ exists in $(A^u)_+$ for all $u_1, .., u_n \in (A^u)_+$, see ([6], Corollary 4.11). Next we will show that any uniformly complete vector sublattice of A^u is closed under this property. The corresponding proof is omitted because it is almost similar to ([6], Theorem 3.7 and Corollary 4.11).

Proposition 3. *Let A be a universally complete vector lattice, let B be a relatively uniformly complete vector sublattice of A and let $F(X_1, .., X_n) \in \mathbb{R}_+[X_1, .., X_n]$ be a homogeneous polynomial of degree p ($0 < p \in \mathbb{N}$). Then $(F(u_1, .., u_n))^{\frac{1}{p}} \in B_+$ for all $u_1, .., u_n \in B_+$.*

Theorem 2. *([4], Theorem 3.2) Let A be a uniformly complete vector lattice and let p be a natural number such that $p \geq 1$. Then the set $\prod_p(A) = \{u_1...u_p, u_i \in A, 1 \leq i \leq p\}$ is a vector lattice under the ordering inherited from A with $\prod_p(A_+)$ as a positive cone.*

Proposition 4 (20, Proposition 3.11)**.** *Let A be a relatively uniformly complete vector lattice, let A^u be its universal completion, let $S = \{e_i, i \in I\}$ be a maximal orthogonal system of A and let $e_{i_0} \in S$. Then A^u can be equipped with an f-algebra multiplication (denoted by $*$) such that $a * b \in A$ and $a * e_{i_0} = a$ for all $a, b \in I_{e_{i_0}}$, where $I_{e_{i_0}}$ is the order ideal generated by e_{i_0} in A.*

Proof. Let $S = \{e_i, i \in I\}$ be a maximal orthogonal system of A and let $e_{i_0} \in S$. By [16, Remark 19.5], the order ideal $I_{e_{i_0}}$ generated by e_{i_0} in A can be equipped with a unital f-algebra multiplication denoted by $*$. This f-algebra multiplication has a similar extension to the band $\{e_{i_0}\}^{dd}$ generated by e_{i_0} in A^u (denoted also by $*$). This latter multiplication has also a similar extension to A^u (denoted also by $*$), by the following way

$$a * b = a_1 * b_1, \text{ for all } a, b \in A^u$$

where a_1 (resp b_1) is the projection component of a (resp of b) in the band $\{e_{i_0}\}^{dd}$ generated by e_{i_0} in A^u, which gives the desired result. ■

The proof of the following result is almost identical to [4, Theorem 3.4].

Theorem 3 (20, Theorem 3.12)**.** *Let A be a relatively uniformly complete vector lattice, let B be a vector lattice and let Ψ be an order bounded orthosymmetric n-multilinear mapping from A^n into B. Then there exists an order bounded operator $T : \prod_n(A) \to B$ such that $\Psi(u_1, .., u_n) = T(u_1...u_n)$.*

As a corollary we generalize the result of Y. Benyamini, S. Lassalle and J. L. G. Llavona ([3], Theorem 2.3).

Corollary 2. *(Y. Benyamini, S. Lassalle and J. L. G. Llavona) Let A be a relatively uniformly complete vector lattice, let B be a vector lattice and let $P \in \mathcal{P}_0(^nA, B)$. Then there exists an order bounded operator $T : \prod_n(A) \to B^\delta$ (the Dedekind completion of B) such that*

$$P(u) = T(u^n) \qquad \text{for all } u \in A.$$

Proof. Let $P \in \mathcal{P}_0(^nA, B)$ and let $\Psi : A^n \to B$ be its associated orthosymmetric n-multilinear operator. According to Theorem 1, there exist two positive orthosymmetric n-multilinear operators $\Psi_1, \Psi_2 : A^n \to B^\delta$ such that $\Psi = \Psi_1 - \Psi_2$. In view of Theorem 3, there exists two order bounded operators T_1, $T_2 : \prod_n(A) \to B^\delta$ such that

$$\Psi_1(u, ..., u) = T_1(u^n), \Psi_2(u, ..., u) = T_2(u^n) \qquad \text{for all } u \in A.$$

Therefore

$$P(u) = (T_1 - T_2)(u^n) \qquad \text{for all } u \in A$$

and the proof is complete. ■

Also we deduce the result of K. Sundaresan [19].

Corollary 3. *(K. Sundaresan) Let* $p \in \mathbb{N}$*, such that* $1 \leq n < p$ *, then for every* $P \in \mathcal{P}_0\left({}^nL_p(\mu), \mathbb{R}\right)$ *there exists* $g \in L_{\frac{p}{p-n}}$ *such that*

$$P(f) = \int_K f^n g d\mu \qquad \text{for all } f \in L_p(\mu).$$

Proof. Let us denote the classical multiplication in $L_p(\mu)$ by $*$. We remark that the following map

$$\begin{array}{cccc} \phi : \prod_n \left(L_p(\mu)\right) & \to & \prod_n^* \left(L_p(\mu)\right) \\ f_1 ... f_n & \mapsto & f_1 * ... * f_n \end{array}$$

is linear and bijective. Hence $\prod_n^* \left(L_p(\mu)\right)$ can be equipped with the following lattice order

$$f_1 * ... * f_n \geq 0 \text{ in } \prod_n^* \left(L_p(\mu)\right) \text{ if } f_1 ... f_n \geq 0 \text{ in } \prod_n \left(L_p(\mu)\right),$$

which means that ϕ becomes a lattice homomorphism. Moreover it is well known that $\prod_n^* \left(L_p(\mu)\right)$ is naturally identified with $L_{\frac{p}{n}}(\mu)$. Therefore, by the previous corollary, there exists an order bounded operator $T : \prod_n^* \left(L_p(\mu)\right) = L_{\frac{p}{n}}(\mu) \to \mathbb{R}$ such that $P(f) = T(f^n)$ for all $f \in L_p(\mu)$. It is well-known that the dual of $L_{\frac{p}{n}}(\mu)$ is given by integrals. More precisely there exists $g \in L_{\frac{p}{p-n}}$ such that

$$P(f) = \int_K f^n g d\mu \qquad \text{for all } f \in L_p(\mu)$$

and we are done. ■

Similarly we deduce the result of D. Carando, S. Lassalle and I. Zalduendo [11, Theorem 1.4] and D. Pérez-Garcia, and I. Villanueva [17, Theorem 2.1].

Corollary 4. *(D. Carando, S. Lassalle and I. Zalduendo, D. Pérez-Garcia, and I. Villanueva) For any* $P \in \mathcal{P}_0\left({}^nC(K), \mathbb{R}\right)$*, there is a regular Borel measure* μ *over* K *such that*

$$P(f) = \int_K f^n d\mu \qquad \text{for all } f \in C(K).$$

Proof. Let $e = \chi_K$. Since that there exists a unique multiplication (denoted by juxtaposition) on $C(K)$ in such a manner $C(K)$ becomes an f-algebra with a unit element e, then the f-algebra multiplication on $C(K)$ can be extended in a unique to the universal completion $(C(K))^u$ of $C(K)$. It follows that $(C(K))^u$ becomes an f-algebra with a unit element e. Consequently $\prod_n (C(K)) = C(K)$. In view of Corollary 2, there exists an order bounded operator $T : C(K) \to \mathbb{R}$ such that $P(f) = T(f^n)$ for all $f \in C(K)$. By the well-known F. Riesz representation Theorem [2, Theorem 1.1.7], there is a regular Borel measure μ over K such that $P(f) = \int_K f^n d\mu$, for all $f \in C(K)$, as required. ■

In the sequel, we need the following notations. Let us denote $\mathcal{L}\left(\prod_n (A), B\right)$ by the vector lattice of all order bounded linear operators from $\prod_n (A)$ into B.

Theorem 4. *Let A be a relatively uniformly complete vector lattice and let B be a Dedekind complete vector lattice. Then the map*

$$\begin{array}{lll}\Omega: & \Delta\left({}^{n}A,B\right) & \to \mathcal{L}\left(\prod_{n}\left(A\right),B\right) \\ & \Psi & \mapsto T_{\Psi}\end{array}$$

defined by $T_{\Psi}\left(a^{n}\right)=\Psi\left(a,..,a\right)$ for all $0\leq a\in A$, is a lattice isomorphism.

Proof. Let $\Psi\in\Delta\left({}^{n}A,B\right)$ such that $\Omega\left(\Psi\right)=0_{\mathcal{L}\left(\prod_{n}(A),B\right)}$. It follows that $\Psi\left(a,..,a\right)=0$ for all $0\leq a\in A$. Since $\prod_{n}\left(A_{+}\right)$ is a positive cone of $\prod_{n}\left(A\right)$, it follows that $\Psi=0$. Moreover, it is an easy task to show that Ω is a surjective map. Hence Ω is a bijective map. It remains to show that Ω is lattice homomorphism. To this end, let $\Psi\in\Delta\left({}^{n}A,B\right)$ and let $0\leq a\in A$, then

$$\Omega\left(\Psi^{+}\right)=T_{\Psi^{+}}.$$

Since

$$\begin{aligned}\Omega\left(\Psi^{+}\right)\left(a^{n}\right)&=T_{\Psi^{+}}\left(a^{n}\right)\\&=\Psi^{+}\left(a,..,a\right)\\&=\sup\left\{\Psi\left(b_{1},..,b_{n}\right),0\leq b_{i}\leq a,1\leq i\leq n\right\}\end{aligned}$$

and since A is a relatively uniformly complete vector lattice, then by [16, Remark 19.5] the order ideal I_{a} generated by a in A can be equipped with a multiplication (denoted by juxtaposition) such that I_{a} is an f-algebra with a unit element a. Therefore, by the symmetry of Ψ and by using the same argument as in the proof of Theorem 3,

$$\begin{aligned}\Psi\left(b_{1},..,b_{n}\right)&=\Psi\left(b_{1}a,..,b_{n}a\right)\\&=\Psi\left(b_{1}...b_{n},a,..,a\right)\end{aligned}$$

for all $0\leq b_{i}\leq a,1\leq i\leq n$. In view of Proposition 4, there exists a unique $0\leq b\leq a\in I_{a}$ such that $b_{1}...b_{n}=b^{n}$. Hence, also by using the same argument as in the proof of Theorem 3, we have that

$$\begin{aligned}\Psi\left(b_{1},...,b_{n}\right)&=\Psi\left(b^{n},a,...,a\right)\\&=\Psi\left(b,...,b\right).\end{aligned}$$

As a conclusion,

$$\Omega\left(\Psi^{+}\right)\left(a^{n}\right)=\sup\left\{\Psi\left(b,...,b\right),0\leq b\leq a\right\}.$$

Moreover,

$$\begin{aligned}\left(\Omega\left(\Psi\right)\right)^{+}\left(a^{n}\right)&=\left(T_{\Psi}\right)^{+}\left(a^{n}\right)\\&=\sup\left\{T_{\Psi}\left(b^{n}\right),0\leq b^{n}\leq a^{n}\right\}\\&=\sup\left\{\Psi\left(b,...,b\right),0\leq b\leq a\right\}\end{aligned}$$

Hence

$$\left(\Omega\left(\Psi\right)\right)^{+}\left(a^{n}\right)=\Omega\left(\Psi^{+}\right)\left(a^{n}\right)$$

for all $0\leq a\in A$. As $\prod_{n}\left(A_{+}\right)$ is the positive cone of the vector lattice $\prod_{n}\left(A\right)$, we deduce that $\left(\Omega\left(\Psi\right)\right)^{+}=\Omega\left(\Psi^{+}\right)$ and we are done. ∎

Since Any orthogonally-additive homogeneous polynomial P of degree n is associated to its associated n-orthosymmetric multimorphism Ψ, we deduce the following.

Theorem 5. *Let A be a relatively uniformly complete vector lattice and let B be a Dedekind complete vector lattice. Then under the following ordering*

$$P_1 \leq P_2 \text{ if and only if } \Psi_1\left(f_1, ..., f_n\right) \leq \Psi_2\left(f_1, ..., f_n\right)$$

where $\Psi_1, \Psi_2 \in \Delta\left({}^n A, B\right)$ are respectively the associated orthosymmetric multimorphisms of P_1 and P_2, for all $f_1, .., f_n \in A_+$ and for all $P_1, P_2 \in \mathcal{P}_0\left({}^n A, B\right)$, $\mathcal{P}_0\left({}^n A, B\right)$ is a vector lattice.

Finally, as simple combination between Theorem 1, Theorem 5 and Theorem 4, we deduce the following corollaries.

Corollary 5. *Let A be a relatively uniformly complete vector lattice and let B be a Dedekind complete vector lattice. Then $\mathcal{P}_0\left({}^n A, B\right)$ is lattice isomorphic to $\mathcal{L}\left(\prod_n (A), B\right)$.*

Corollary 6 (13, Theorem 3.4)**.** *Let E be a uniformly complete Archimedean Riesz subspace of a semiprime f-algebra A and F uniformly complete Archimedean. Then, for every positive orthogonally additive n-homogeneous polynomial $P \in \mathcal{P}_0\left({}^n E, F\right)$ there exists a unique positive linear application $L \in \mathcal{L}\left(\prod_n (E), F\right)$ such that $P(x) = L(x^n)$ for every $x \in E$.*

References

[1] C. D. Aliprantis and O. Burkinshaw, *Positive Operators,* Academic Press. Orlando, 1985.

[2] B. Aupetit, *A prime on Spectral theory*, Springer-verlag, New York 1991.

[3] Y. Benyamini, S. Lassalle and J. L. G. Llavona, *Homogeneous orthogonally-additive polynomials on Banach lattices*, Bull. London Math. Soc. 38(3) (2006), 459-469.

[4] K. Boulabiar, *On products in lattice-ordered algebras*, J. Aust. Math. Soc. 75, No.1, 23-40 (2003).

[5] K. Boulabiar, G. Buskes, Vector lattice Powers: f-Algebras and Functional Calculus, Comm. Alg. 34 (2006), 1435-0442.

[6] G. Buskes, B. de Pagter and A. van Rooij "Functional calculus on Riesz spaces", Indag. Math. (N.S.) 2 (1991), no. 4, 423–436.

[7] G. Buskes and A. van Rooij, *Almost f-algebras, Commutativity and the Cauchy-Schwarz inequality*, Positivity 4 (2000), 227-331.

[8] G. Buskes and A. van Rooij, *Almost f-algebras, Squares of Riesz spaces*, Rocky Mountain J. Math. V31, (3), (2001), 45-56.

[9] G. Buskes and A. van Rooij, *Bounded variation and tensor products of Banach lattices*, Positivity 7(1{2) (2003), 47-59.

[10] G. Buskes, and A. G. Kusraev, *Representation and extension of orthoregular bilinear operators*, Vladikavkaz Math. J. V, 9 (2007), 16-29.

[11] D. Carando, S. Lassalle and I. Zalduendo, *Orthogonally additive polynomials over C(K) are measures –a short proof*, Int. Eq. and Oper. Theory, 56, No. 4, (2006), 597-602.

[12] A. Ibort, P. Linares and J. G. Llavona, *On the Representation of Orthogonally Additive Polynomials in* ℓ_p, Publ. RIMS, Kyoto Univ, 45 (2009), 519–524.

[13] A. Ibort, P. Linares and J. G. Llavona, A representation theorem for orthogonally additive polynomials on Riesz spaces, Revista Matemática Complutense V 25, 1, (2012), 21-30.

[14] W. A. J. Luxemburg and A. C. Zaanen, *Riesz spaces I*, North-Holland. Amsterdam, 1971.

[15] P. Meyer-Neiberg, *Banach lattices,* Springer Verlag, Berlin-Heidelberg-New York, 1974.

[16] B. De Pagter, *f-algebras and Orthomorphisms*, Thesis, Leiden, 1981.

[17] D. Pérez-Garcıa. and I. Villanueva, *Orthogonally additive polynomials on spaces of continuous functions*, J. Math. Anal. Appl. 306(1) (2005), 97-105.

[18] H. H. Schaefer, *Banach lattices and positive operators*, Springer Verlag, New York, 1974.

[19] K. Sundaresan, *Geometry of Spaces of Homogeneous Polynomials on Banach Lattices*, Applied geometry and discrete mathematics, 571-586, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 4, Amer. Math. Soc., Prov., RI, 1991.

[20] M.A. Toumi, Orthogonally additive polynomials on σ-Dedekind complete vector lattices, Mathematical Proceedings of the Royal Irish Academy 110A (2010), 83-94.

Département de Mathématiques,
Faculté des Sciences de Bizerte,
7021, Zarzouna, Bizerte,
TUNISIA
email:MohamedAli.Toumi@fsb.rnu.tn