Frictional contact problem with wear for electro-viscoelastic materials with long memory

Mohamed Selmani

Abstract

We study a mathematical model for a quasistatic process of contact with normal compliance and friction when the wear of the contact surface due to friction is taken into account. The material is electro-viscoelastic with long memory. We establish a variational formulation for the model and prove the existence and uniqueness of the weak solution. The proof is based on classical results for elliptic variational inequalities and fixed point arguments.

1 Introduction

The piezoelectric effect is characterized by the coupling between the mechanical and electrical behavior of the materials. Indeed, the apparition of electric charges on some crystals submitted to the action of body forces and surfaces tractions was observed and their dependence on the deformation process was underlined. Conversely, it was proved experimentally that the action of electric field on the crystals may generate strain and stress. A deformable material which presents such a behavior is called a piezoelectric material. Piezoelectric materials are used extensively as switches and actuators in many engineering systems, in radioelectronics, electroacoustics and measuring equipments. However, there are very few mathematical results concerning contact problems involving piezoelectric materials and therefore there is a need to extend the results on models for contact with deformable bodies which include coupling between mechanical and electrical properties. General models for elastic materials with piezoelectric effects

Received by the editors in January 2012 - In revised form in June 2012.
Communicated by P. Godin.
2010 *Mathematics Subject Classification* : 74M15, 74M10, 74F15, 49J40.
Key words and phrases : Quasistatic process, electro-viscoelastic materials, normal compliance, friction, wear, existence and uniqueness, fixed point arguments, weak solution.

can be found in [1, 6]. A static frictional contact problem for electric-elastic materials was considered in [2, 8, 12]. A slip-dependent frictional contact problem for electro-elastic materials was studied in [17]. Contact problems with friction or adhesion for electro-viscoelastic materials were studied in [4, 7, 15, 16, 17] and recently in [10, 11] in the case of an electrically conductive foundation.

In the present paper we consider a mathematical model for the process of contact with normal compliance and friction contact conditions when the wear of the contact surface due to friction is taken into account. The foundation is assumed to move steadily and only sliding contact takes places. The material is electro-viscoelastic with long memory, defined by a relaxation operator. We derive the variational formulation and prove the existence and uniqueness of the weak solution of the model.

The paper is organized as follows. In section 2 we present the notation and some preliminaries. In section 3 we present the mechanical problem, we list the assumptions on the data and give the variational formulation of the problem. In section 4 we state our main existence and uniqueness result. It is based on arguments of classical results for elliptic variational inequalities and fixed point arguments.

2 Notation and preliminaries

In this section we present the notation we shall use and some preliminary material. For further details, we refer the reader to [3, 5]. We denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$, while "." and $|\ .\ |$ will represent the inner product and the Euclidean norm on S_d and $\mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ and let $\boldsymbol{\nu}$ denote the unit outer normal on Γ. Everywhere in the sequel the index i and j run from 1 to d, summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the independent spatial variable. We use the standard notation for Lebesgue and Sobolev spaces associated to Ω and Γ and introduce the spaces:

$$H = L^2(\Omega)^d = \Big\{ \boldsymbol{u} = (u_i)\ /\ u_i \in L^2(\Omega)\ \Big\},$$

$$\mathcal{H} = \Big\{ \boldsymbol{\sigma} = (\sigma_{ij})\ /\ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\ \Big\},$$

$$H_1 = \{ \boldsymbol{u} = (u_i)\ /\ \boldsymbol{\varepsilon}(\boldsymbol{u}) \in \mathcal{H}\ \},$$

$$\mathcal{H}_1 = \{ \boldsymbol{\sigma} \in \mathcal{H}\ /\ Div\ \boldsymbol{\sigma} \in H \}.$$

Here $\boldsymbol{\varepsilon}$ and Div are the deformation and divergence operators, respectively, defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})),\quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}),\ Div\, \boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u}, \boldsymbol{v})_H = \int_\Omega u_i v_i\, dx \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in H,$$

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}\, dx \qquad \forall\, \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathcal{H},$$
$$(\boldsymbol{u},\boldsymbol{v})_{H_1} = (\boldsymbol{u},\boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \qquad \forall\, \boldsymbol{u},\boldsymbol{v} \in H_1,$$
$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} + (Div\ \boldsymbol{\sigma}, Div\ \boldsymbol{\tau})_H \ \ \forall\, \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathcal{H}_1.$$

The associated norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $\mid . \mid_H$, $\mid . \mid_{\mathcal{H}}$, $\mid . \mid_{H_1}$ and $\mid . \mid_{\mathcal{H}_1}$, respectively. For every element $\boldsymbol{v} \in H_1$ we also use the notation $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on Γ and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and the tangential components of $\boldsymbol{v}$ on Γ given by

$$v_\nu = \boldsymbol{v}.\boldsymbol{\nu},\ \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}. \tag{2.1}$$

We also denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and the tangential traces of a function $\boldsymbol{\sigma} \in \mathcal{H}_1$, we recall that when $\boldsymbol{\sigma}$ is a regular function then

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}).\boldsymbol{\nu},\ \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}, \tag{2.2}$$

and the following Green's formula holds:

$$(\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (Div\ \boldsymbol{\sigma},\boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}.\boldsymbol{v}\, da \qquad \forall\, \boldsymbol{v} \in H_1. \tag{2.3}$$

Let $T > 0$. For every real Banach space X we use the notation $C(0,T;X)$ and $C^1(0,T;X)$ for the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively; $C(0,T;X)$ is a real Banach space with the norm

$$\mid \boldsymbol{f} \mid_{C(0,T;X)} = \max_{t\in[0,T]} \mid \boldsymbol{f}(t) \mid_X$$

while $C^1(0,T;X)$ is a real Banach space with the norm

$$\mid \boldsymbol{f} \mid_{C^1(0,T;X)} = \max_{t\in[0,T]} \mid \boldsymbol{f}(t) \mid_X + \max_{t\in[0,T]} \mid \dot{\boldsymbol{f}}(t) \mid_X .$$

Finally, for $k \in \mathbb{N}$ and $p \in [1,\infty]$, we use the standard notation for the Lebesgue spaces $L^p(0,T;X)$ and for the Sobolev spaces $W^{k,p}(0,T;X)$. Moreover, if X_1 and X_2 are real Hilbert spaces then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(.,.)_{X_1\times X_2}$.

3 Mechanical and variational formulations

The physical setting is the following. An electro-viscoelastic body with long memory occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ with outer Lipschitz surface Γ. The body is submitted to the action of body forces of density $\boldsymbol{f}_0$ and volume electric charges of density q_0. It is also submitted to mechanical and electric constraint on the boundary. We consider a partition of Γ into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, on one hand, and on two measurable parts Γ_a and Γ_b, on the other hand, such that *meas* $(\Gamma_1) > 0$, *meas* $(\Gamma_a) > 0$ and $\Gamma_3 \subset \Gamma_b$. Let $T > 0$ and let $[0,T]$ be the time interval of interest. The body is clamped on

$\Gamma_1 \times (0,T)$, so the displacement field vanishes there. A surface tractions of density $\boldsymbol{f}_2$ act on $\Gamma_2 \times (0,T)$ and a body force of density $\boldsymbol{f}_0$ acts in $\Omega \times (0,T)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (0,T)$ and a surface electric charge of density q_2 is prescribed on $\Gamma_b \times (0,T)$. The contact is frictional, it is modeled with normal compliance and the wear of the contact surfaces is taken into account. The foundation is assumed to move steadily and only sliding contact takes places. We suppose that the body forces and tractions vary slowly in time, and therefore, the accelerations in the system may be neglected. Neglecting the inertial terms in the equation of motion leads to a quasistatic approach of the process. We use an electro-viscoelastic constitutive law with long memory given by

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \int_0^t M(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds - \mathcal{E}^*\boldsymbol{E}(\varphi), \tag{3.1}$$

$$\boldsymbol{D} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) + B\boldsymbol{E}(\varphi), \tag{3.2}$$

where $\boldsymbol{u}$ is the displacement field and $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ are the stress and the linearized strain tensor, respectively. Here $\mathcal{A}$ and $\mathcal{F}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, and M is a relaxation fourth order tensor. $\boldsymbol{E}(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{E} = (e_{ijk})$ represents the third order piezoelectric tensor, $\mathcal{E}^*$ is its transposed and B denotes the electric permittivity tensor. We use dots for derivatives with respect to the time variable t. When $M = 0$ the constitutive law (3.1)-(3.2) reduces to the electro-viscoelastic constitutive law given by (3.2) and

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{E}^*\boldsymbol{E}(\varphi). \tag{3.3}$$

We now briefly describe the boundary conditions on the contact surface Γ_3, based on the model derived in [19,20]. We introduce the wear function $w : \Gamma_3 \times [0,T] \to \mathbb{R}$ which measures the wear accumulated of the surface. The evolution of the wear of the contacting surface is governed by a simplified version of Archard's law (see [19,20]) which we now describe. The rate form of Archard's law is

$$\dot{w} = -k_1\sigma_\nu \mid \dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^* \mid, \tag{3.4}$$

where $k_1 > 0$ is a wear coefficient, $\boldsymbol{v}^*$ is the tangential velocity of the foundation and $\mid \dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^* \mid$ represents the slip speed between the contact surface and the foundation. We see that the rate of wear is assumed to be proportional to the contact stress and the slip speed. For the sake of simplicity we assume in the rest of the section that the motion of the foundation is uniform, i.e. $\boldsymbol{v}^*$ does not vary in time. Denote $v^* =\mid \boldsymbol{v}^* \mid > 0$. We assume that the tangential speed $\boldsymbol{v}^*$ is large so that we can neglect in the sequel $\dot{\boldsymbol{u}}_\tau$ as compared with $\boldsymbol{v}^*$ to obtain the following version of Archard's law

$$\dot{w} = -k_1 v^*\sigma_\nu. \tag{3.5}$$

Use of the simplified law (3.5) for the evolution of the wear avoids some mathematical difficulties in the study of the quasistatic electro-viscoelastic contact problem. Let now p_i , for $i = \nu, \tau$, denote the normal compliance functions satisfying $p_i(r) = 0$ if $r \leq 0$ and other assumptions given in what follows and note that the wear appears in the normal compliance condition as follows:

$$-\sigma_\nu = p_\nu(u_\nu - h - w)\,, \quad \mid \boldsymbol{\sigma}_\tau \mid = p_\tau(u_\nu - h - w),$$

where h represents the initial gap between the body and the foundation. We model the frictional contact between the electro-viscoelastic body and the foundation with a version of Coulomb's law of dry friction. Since there is only sliding contact it follows that

$$\mid \boldsymbol{\sigma}_\tau \mid = p_\tau(u_\nu - h - w),\ \boldsymbol{\sigma}_\tau = -\lambda(\boldsymbol{v}^* - \boldsymbol{u}_\tau),\ \lambda \geq 0\,,$$

these relations set constraints on the evolution of the tangential stress, in particular, the tangential stress is in the direction opposite to the relative sliding velocity $\boldsymbol{v}^* - \boldsymbol{u}_\tau$. An example of the normal compliance function p_ν is

$$p_\nu(r) = c_\nu r_+, \tag{3.6}$$

or, more general,

$$p_\nu(r) = c_\nu(r_+)^m. \tag{3.7}$$

Here c_ν is a positive constant, $r_+ = \max\{0, r\}$ and $m > 0$. The normal compliance contact condition was proposed in [9], in the particular form of (3.7). Then it was used in a large number of papers, see e.g. [14] and the references therein. A related form of normal compliance functions is given by

$$p_\tau = \mu\, p_\nu, \tag{3.8}$$

where $\mu \geq 0$ is the coefficient of friction, and

$$p_\tau = \mu\, p_\nu(1 - \delta p_\nu)_+, \tag{3.9}$$

where δ is a small positive material constant related to the wear and hardness of the surface and $\mu \geq 0$ is the coefficient of friction. This related form of compliance contact condition was used in [19, 20].

To simplify the notation, we do not indicate explicitely the dependence of various functions on the variables $\boldsymbol{x} \in \Omega \cup \Gamma$ and $t \in [0, T]$. Then, the classical formulation of the mechanical problem of sliding frictional contact problem with normal compliance and wear may be stated as follows.

Problem P. Find a displacement field $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^d$*, a stress field* $\boldsymbol{\sigma} : \Omega \times [0, T] \to S_d$*, an electric potential field* $\varphi : \Omega \times [0, T] \to \mathbb{R}$ *and an electric displacement field* $\boldsymbol{D} : \Omega \times [0, T] \to \mathbb{R}^d$ *such that*

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \int_0^t M(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds + \mathcal{E}^*\nabla\varphi \quad \text{in } \Omega \times (0, T), \tag{3.10}$$

$$\boldsymbol{D} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) - B\nabla\varphi \quad \text{in } \Omega \times (0, T), \tag{3.11}$$

$$Div\boldsymbol{\sigma} + \boldsymbol{f}_0 = 0 \quad \text{in } \Omega \times (0, T), \tag{3.12}$$

$$div\boldsymbol{D} = q_0 \quad \text{in } \Omega \times (0, T), \tag{3.13}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1 \times (0, T), \tag{3.14}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{3.15}$$

$$\left\{\begin{array}{l} -\sigma_\nu = p_\nu(u_\nu - h - w)\,,\ \mid \boldsymbol{\sigma}_\tau \mid = p_\tau(u_\nu - h - w) \\ \boldsymbol{\sigma}_\tau = -\lambda(\boldsymbol{v}^* - \boldsymbol{u}_\tau),\ \lambda \geq 0 \\ \qquad \dot{w} = -k_1 v^* \sigma_\nu \end{array}\right. \quad \text{on } \Gamma_3 \times (0, T), \tag{3.16}$$

$$\varphi = 0 \quad \text{on} \quad \Gamma_a \times (0, T), \tag{3.17}$$

$$\boldsymbol{D} \,.\, \boldsymbol{\nu} = q_2 \quad \text{on} \quad \Gamma_b \times (0, T), \tag{3.18}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0,\ w(0) = 0 \quad \text{in} \quad \Omega. \tag{3.19}$$

Here equations (3.10) and (3.11) represent the electro-viscoelastic constitutive law with long memory introduced in the third section. Equations (3.12) and (3.13) represent the equilibrium equations for the stress and electric-displacement fields while equations (3.14) and (3.15) are the displacement and traction boundary condition, respectively. Equation (3.16) represents the condition with normal compliance, friction and wear described above. Equations (3.17) and (3.18) represent the electric boundary conditions. In equation (3.19) $\boldsymbol{u}_0$ is the given initial displacement and $w(0) = 0$ means that at the initial moment the body is not subject to any prior wear. To obtain the variational formulation of the problem (3.10)-(3.19), we introduce the closed subspace of $H^1(\Omega)^d$ defined by

$$V = \left\{ \boldsymbol{v} \in H^1(\Omega)^d \ / \ \boldsymbol{v} = \boldsymbol{0} \quad \text{on}\, \Gamma_1 \right\}.$$

Since *meas* $(\Gamma_1) > 0$, Korn's inequality holds and there exists a constant $C_k > 0$, that depends only on Ω and Γ_1, such that

$$\mid \boldsymbol{\varepsilon}(\boldsymbol{v}) \mid_{\mathcal{H}} \geq C_k \mid \boldsymbol{v} \mid_{H^1(\Omega)^d} \ \ \forall \boldsymbol{v} \in V.$$

A proof of Korn's inequality may be found in [13] p. 79. On the space V we consider the inner product and the associated norm given by

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \ \mid \boldsymbol{v} \mid_V = \mid \boldsymbol{\varepsilon}(\boldsymbol{v}) \mid_{\mathcal{H}} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V. \tag{3.20}$$

It follows that $\mid . \mid_{H^1(\Omega)^d}$ and $\mid . \mid_V$ are equivalent norms on V and therefore $(V, \mid . \mid_V)$ is a real Hilbert space. Moreover, by the Sobolev trace Theorem and (3.20), there exists a constant $C_0 > 0$, depending only on Ω, Γ_1 and Γ_3 such that

$$\mid \boldsymbol{v} \mid_{L^2(\Gamma_3)^d} \leq C_0 \mid \boldsymbol{v} \mid_V \qquad \forall \boldsymbol{v} \in V. \tag{3.21}$$

We also introduce the spaces

$$W = \left\{ \phi \in H^1(\Omega) \ / \ \phi = 0 \text{ on } \Gamma_a \right\},$$

$$\mathcal{W} = \left\{ \boldsymbol{D} = (D_i) \ / \ D_i \in L^2(\Omega),\ div\boldsymbol{D} \in L^2(\Omega) \right\},$$

where $div\boldsymbol{D} = (D_{i,i})$. The spaces W and $\mathcal{W}$ are real Hilbert spaces with the inner products given by

$$(\varphi, \phi)_W = \int_\Omega \nabla\varphi \,.\nabla\phi \, dx,$$

$$(\boldsymbol{D}, \boldsymbol{E})_{\mathcal{W}} = \int_\Omega \boldsymbol{D} \,.\, \boldsymbol{E} \, dx + \int_\Omega div\boldsymbol{D} \,.\, div\boldsymbol{E} \, dx.$$

The associated norms will be denoted by $\mid . \mid_W$ and $\mid . \mid_{\mathcal{W}}$, respectively. Moreover, when $\boldsymbol{D} \in \mathcal{W}$ is a regular function, the following Green's type formula holds:

$$(\boldsymbol{D}, \nabla\phi)_H + (div\boldsymbol{D}, \phi)_{L^2(\Omega)} = \int_\Gamma \boldsymbol{D} \,.\, \boldsymbol{\nu} \, \phi \, da \qquad \forall \phi \in H^1(\Omega).$$

Notice also that, since $meas(\Gamma_a) > 0$, the following Friedrichs-Poincaré inequality holds:

$$| \nabla\phi |_H \geq C_F | \phi |_{H^1(\Omega)} \quad \forall\phi \in W, \tag{3.22}$$

where $C_F > 0$ is a constant which depends only on Ω and Γ_a. In the study of the mechanical problem (3.10)-(3.19), we make the following assumptions.

The viscosity operator $\mathcal{A} : \Omega \times S_d \to S_d$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_{\mathcal{A}} > 0 \text{ such that} \\ | \mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2) | \leq L_{\mathcal{A}} | \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 | \ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ (b) \text{ There exists a constant } m_{\mathcal{A}} > 0 \text{ such that} \\ (\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)).(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}} | \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 |^2 \ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ (c) \text{ The mapping } \boldsymbol{x} \to \mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega \text{ for any } \boldsymbol{\varepsilon} \in S_d. \\ (d) \text{ The mapping } \boldsymbol{x} \to \mathcal{A}(\boldsymbol{x},\boldsymbol{0}) \text{ belongs to } \mathcal{H}. \end{cases} \tag{3.23}$$

The elasticity Operator $\mathcal{F} : \Omega \times S_d \to S_d$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_{\mathcal{F}} > 0 \text{ such that} \\ | \mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_2) | \leq L_{\mathcal{F}} | \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 | \\ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ (b) \text{ The mapping } \boldsymbol{x} \to \mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega \\ \text{for any } \boldsymbol{\varepsilon} \in S_d. \\ (c) \text{ The mapping } \boldsymbol{x} \to \mathcal{F}(\boldsymbol{x},\boldsymbol{0}) \text{ belongs to } \mathcal{H}. \end{cases} \tag{3.24}$$

The relaxation operator M satisfies

$$M \in C(0,T;\mathcal{H}_\infty), \tag{3.25}$$

where $\mathcal{H}_\infty$ is the space of fourth order tensor field given by

$$\mathcal{H}_\infty = \left\{ E = (E_{ijkl}) / E_{ijkl} = E_{klij} \in L^\infty(\Omega), 1 \leq i,j,k,l \leq d \right\},$$

which is a real Banach space with the norm

$$| E |_{\mathcal{H}_\infty} = \max_{1 \leq i,j,k,l \leq d} | E_{ijkl} |_{L^\infty(\Omega)} .$$

The normal compliance functions $p_r : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ $(r = \nu, \tau)$ satisfy

$$\begin{cases} (a) \text{ There exists a constant } L_r > 0 \text{ such that} \\ | p_r(\boldsymbol{x},u_1) - p_r(\boldsymbol{x},u_2) | \leq L_r | u_1 - u_2 | \ \forall u_1, u_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ (b) \text{ For any } u \in \mathbb{R},\ \boldsymbol{x} \to p_r(\boldsymbol{x},u) \text{ is measurable.} \\ (c) \text{ The mapping } \boldsymbol{x} \to p_r(\boldsymbol{x},0) \text{ blongs to } L^2(\Gamma_3). \end{cases} \tag{3.26}$$

We observe that the assumption (3.26) on the functions p_ν and p_τ are pretty general except the assumption (3.26)(a) which, roughly speaking, requires the functions to grow at most linearly. It is easily seen that the functions defined in (3.6) and (3.7) satisfy the condition (3.26)(a). We also observe that if the functions p_ν and p_τ are related by (3.8) or (3.9) and p_ν satisfies (3.26)(a), then p_τ also satisfies

(3.26)(a) with $L_\tau = \mu\, L_\nu$. So our results below are valid for the boundary value problems associated with these choices of the normal compliance functions.

The electric permittivity operator $B = (b_{ij})$: $\Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} (a)\ B(\boldsymbol{x})\boldsymbol{E} = (b_{ij}(\boldsymbol{x})E_j)\ \forall \boldsymbol{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ (b)\ b_{ij} = b_{ji},\ \ b_{ij} \in L^\infty(\Omega). \\ (c)\ \text{There exists a constant } m_B > 0 \text{ such that} \\ B\boldsymbol{E}.\boldsymbol{E} \geq m_B \mid \boldsymbol{E} \mid^2 \quad \forall\, \boldsymbol{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \end{cases} \tag{3.27}$$

The piezoelectric operator $\mathcal{E} : \Omega \times S_d \to \mathbb{R}^d$ satisfies

$$\begin{cases} (a)\ \mathcal{E}(\boldsymbol{x})\boldsymbol{\tau} = (e_{ijk}(\boldsymbol{x})\tau_{jk}) \quad \forall \boldsymbol{\tau} = (\tau_{ij}) \in S_d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ (b)\ e_{ijk} = e_{ikj} \in L^\infty(\Omega). \end{cases} \tag{3.28}$$

We also suppose that the body forces and surface tractions have the regularity

$$\boldsymbol{f}_0 \in C(0,T;H), \quad \boldsymbol{f}_2 \in C(0,T;L^2(\Gamma_2)^d), \tag{3.29}$$

$$q_0 \in C(0,T;L^2(\Omega)), \quad q_2 \in C(0,T;L^2(\Gamma_b)), \tag{3.30}$$

$$q_2(t) = 0 \text{ on } \Gamma_3\ \forall t \in [0,T]\,. \tag{3.31}$$

Note that we need to impose assumption (3.31) for physical reasons, indeed the foundation is assumed to be insulator and therefore the electric charges (which are prescribed on $\Gamma_b \supset \Gamma_3$) have to vanish on the potential contact surface. The initial displacement field satisfies

$$\boldsymbol{u}_0 \in V. \tag{3.32}$$

Next, we denote by $\boldsymbol{f} : [0,T] \to V$ the function defined by

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t)\,.\,\boldsymbol{v}\,dx + \int_{\Gamma_2} \boldsymbol{f}_2(t)\,.\boldsymbol{v}\,da \quad \forall \boldsymbol{v} \in V, t \in [0,T]\,, \tag{3.33}$$

and we denote by $q : [0,T] \to W$ the function defined by

$$(q(t), \phi)_W = \int_\Omega q_0(t)\,.\phi\,dx - \int_{\Gamma_b} q_2(t)\,.\phi\,da\ \forall \phi \in W,\ t \in [0,T]\,. \tag{3.34}$$

We define the functional $\quad j : V \times V \times L^2(\Gamma_3) \to \mathbb{R}$ by

$$\begin{aligned} j(\boldsymbol{u}, \boldsymbol{v}, w) = &\int_{\Gamma_3} p_\nu(u_\nu - h - w)v_\nu\,da \\ &+ \int_{\Gamma_3} p_\tau(u_\nu - h - w) \mid \boldsymbol{v}_\tau - \boldsymbol{v}^* \mid da \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V, w \in L^2(\Gamma_3), \end{aligned} \tag{3.35}$$

the functional j satisfies

for all $\boldsymbol{g} \in V$ and $w \in L^2(\Gamma_3)$

$$\boldsymbol{v} \to j(\boldsymbol{g}, \boldsymbol{v}, w) \text{ is proper, convex and lower semicontinuous on } V. \tag{3.36}$$

We note that conditions (3.29) and (3.30) imply

$$\boldsymbol{f} \in C(0,T;V), \;\; q \in C(0,T;W). \tag{3.37}$$

Using standard arguments we obtain the variational formulation of the mechanical problem (3.10)-(3.19).

Problem PV. Find a displacement field $\boldsymbol{u} : [0,T] \to V$, *a stress field* $\boldsymbol{\sigma} : [0,T] \to \mathcal{H}_1$, *an electric potential field* $\varphi : [0,T] \to W$, an electric displacement field $\boldsymbol{D} : [0,T] \to H$ *and* a wear function $w : [0,T] \to L^2(\Gamma_3)$ *such that for all* $t \in [0,T]$,

$$\boldsymbol{\sigma}(t) \;\; = \;\; \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t M(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds + \mathcal{E}^*\nabla\varphi(t), \tag{3.38}$$

$$\dot{w} = -k_1 v^* \sigma_\nu, \tag{3.39}$$

$$\begin{aligned}(\boldsymbol{\sigma}(t)\,),\boldsymbol{\varepsilon}(\boldsymbol{v}-\dot{\boldsymbol{u}}(t)))_{\mathcal{H}} + j(\boldsymbol{u}(t),\boldsymbol{v},w(t)) - j(\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t),w(t))&\\ \geq (\boldsymbol{f}(t),\boldsymbol{v}-\dot{\boldsymbol{u}}(t))_V \quad \forall \boldsymbol{v} \in V,&\end{aligned} \tag{3.40}$$

$$\boldsymbol{D}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - B\nabla\varphi(t), \tag{3.41}$$

$$(\boldsymbol{D}(t),\nabla\phi)_H = -(q(t),\phi)_W \quad \forall \phi \in W, \tag{3.42}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \; w(0) = 0. \tag{3.43}$$

We notice that the variational problem *PV* is formulated in terms of displacement field, stress field, electrical potential field, electric displacement field *and* a wear function. The functions $\boldsymbol{u}$, $\boldsymbol{\sigma}$, φ, $\boldsymbol{D}$ and w which satisfy (3.38)-(3.43) are called weak solution to contact problem P. The existence of the unique solution to problem *PV* is stated and proved in the next section.

4 An existence and uniqueness result

The main result in this section is the following existence and uniqueness result.

Theorem 4.1. *Let the assumptions (3.23)-(3.32) hold. Then there exists a unique solution* $\{\boldsymbol{u},\boldsymbol{\sigma},\varphi,\boldsymbol{D},w\}$ *to problem PV. Moreover, the solution satisfies*

$$\boldsymbol{u} \in C^1(0,T;V), \tag{4.1}$$

$$\boldsymbol{\sigma} \in C(0,T;\mathcal{H}_1), \tag{4.2}$$

$$\varphi \in C(0,T;W), \tag{4.3}$$

$$\boldsymbol{D} \in C(0,T;\mathcal{W}), \tag{4.4}$$

$$w \in C^1(0,T;L^2(\Gamma_3)). \tag{4.5}$$

The functions $\boldsymbol{u}$, $\boldsymbol{\sigma}$, φ, $\boldsymbol{D}$ and w which satisfy (3.38)-(3.43) are called weak solution to contact problem P. We conclude that, under the assumptions (3.23)-(3.32), the mechanical problem (3.10)-(3.19) has a unique weak solution satisfying (4.1)-(4.5). The proof of Theorem 4.1 is carried out in several steps that we prove in what follows. Everywhere in this section we suppose that assumptions of Theorem 4.1 hold. Below, C denotes a generic positive constant which may depend on $\Omega, \Gamma_1, \Gamma_2, \Gamma_3, \mathcal{A}, \mathcal{E}, \mathcal{F}, p_\nu, p_\tau$ and T but does not depend on t nor of the rest of input data, and whose value may change from place to place.

In the first step let $w \in C(0,T;L^2(\Gamma_3))$, $\boldsymbol{\eta} \in C(0,T;\mathcal{H})$ and $\boldsymbol{g} \in C(0,T;V)$ be given and consider the following variational problem.

Problem $PV_{w\eta g}$: Find a displacement field $\boldsymbol{v}_{w\eta g} : [0,T] \to V$ and a stress field $\boldsymbol{\sigma}_{w\eta g} : [0,T] \to \mathcal{H}$ such that for all $t \in [0,T]$,

$$\boldsymbol{\sigma}_{w\eta g}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_{w\eta g}(t)) + \boldsymbol{\eta}(t), \tag{4.6}$$

$$\begin{aligned}(\boldsymbol{\sigma}_{w\eta g}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{v}_{w\eta g}(t)))_{\mathcal{H}} \; + \; j(\boldsymbol{g}(t), \boldsymbol{v}, w(t)) - j(\boldsymbol{g}(t), \boldsymbol{v}_{w\eta g}(t), w(t)) \\ \geqslant (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{v}_{w\eta g}(t))_V \quad \forall \boldsymbol{v} \in V.\end{aligned} \tag{4.7}$$

In the study of problem $PV_{w\eta g}$ we have the following result.

Lemma 4.2. *$PV_{w\eta g}$ has a unique weak solution such that*

$$\boldsymbol{v}_{w\eta g} \in C(0,T;V), \quad \boldsymbol{\sigma}_{w\eta g} \in C(0,T;\mathcal{H}_1). \tag{4.8}$$

Proof. We define the operator $A : V \to V$ such that

$$(A\boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V. \tag{4.9}$$

It follows from (4.9) and (3.23)(a) that

$$| A\boldsymbol{u} - A\boldsymbol{v} |_V \leq L_{\mathcal{A}} | \boldsymbol{u} - \boldsymbol{v} |_V \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V, \tag{4.10}$$

which shows that $A : V \to V$ is Lipschitz continuous. Now, by (4.9) and (3.23)(b) we find

$$(A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_V \geq m_{\mathcal{A}} | \boldsymbol{u} - \boldsymbol{v} |_V^2 \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V, \tag{4.11}$$

i.e., that $A : V \to V$ is a strongly monotone operator on V. Moreover using Riesz Representation Theorem we may define an element $\boldsymbol{F} \in C(0,T;V)$ by

$$(\boldsymbol{F}(t), \boldsymbol{v})_V = (\boldsymbol{f}(t), \boldsymbol{v})_V - (\boldsymbol{\eta}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}.$$

Since A is a strongly monotone and Lipschitz continuous operator on V and since $\boldsymbol{v} \to j(\boldsymbol{g}(t), \boldsymbol{v}, w(t))$ is a proper convex lower semicontinuous functional, it follows from classical result on elliptic inequalities (see for example [5]) that there exists a unique function $\boldsymbol{v}_{w\eta g}(t) \in V$ which satisfies

$$\begin{aligned}(A\boldsymbol{v}_{w\eta g}(t), \boldsymbol{v} - \boldsymbol{v}_{w\eta g}(t))_V + j(\boldsymbol{g}(t), \boldsymbol{v}, w(t)) - j(\boldsymbol{g}(t), \boldsymbol{v}_{w\eta g}(t), w(t)) \\ \geqslant (\boldsymbol{F}(t), \boldsymbol{v} - \boldsymbol{v}_{w\eta g}(t))_V \quad \forall \boldsymbol{v} \in V.\end{aligned} \tag{4.12}$$

We use the relation (4.6), the assumption (3.23) and the properties of the deformation tensor to obtain that $\sigma_{w\eta g}(t) \in \mathcal{H}$. Since $v = v_{w\eta g}(t) \pm \psi$ satisfies (4.7), where $\psi \in \mathcal{D}(\Omega)^d$ is arbitrary, using the definition (3.33) we find

$$Div\sigma_{w\eta g}(t) + f_0(t) = 0. \tag{4.13}$$

With the regularity assumption (3.29) on f_0 we see that $Div\sigma_{w\eta g}(t) \in H$. Therefore $\sigma_{w\eta g}(t) \in \mathcal{H}_1$. Let $t_1, t_2 \in [0, T]$ and denote $\eta(t_i) = \eta_i$, $f(t_i) = f_i$, $g(t_i) = g_i$, $v_{w\eta g}(t_i) = v_i$, $\sigma_{w\eta g}(t_i) = \sigma_i$ for $i = 1, 2$. Using the relation (4.12) we find that

$$\begin{aligned} &(Av_1 - Av_2, v_1 - v_2)_V \\ &\qquad \leq (f_1 - f_2, v_1 - v_2)_V + (\eta_2 - \eta_1, \varepsilon(v_1 - v_2))_{\mathcal{H}} \\ &\qquad + j(g_1, v_2, w) - j(g_1, v_1, w) + j(g_2, v_1, w) - j(g_2, v_2, w). \end{aligned} \tag{4.14}$$

From the definition of the functional j given by (3.35) we have

$$\begin{aligned} &j(g_1, v_2, w) - j(g_1, v_1, w) + j(g_2, v_1, w) - j(g_2, v_2, w) \\ &\qquad = \int_{\Gamma_3} \{(p_\nu(g_{1\nu} - w - h) - p_\nu(g_{2\nu} - w - h)\} (v_{2\nu} - v_{1\nu})\, da \\ &\qquad + \int_{\Gamma_3} \{(p_\tau(g_{1\nu} - w - h) - p_\tau(g_{2\nu} - w - h)\} (|v_{2\tau} - v^*| - |v_{1\tau} - v^*|)\, da \end{aligned}$$

We use (3.21) and (3.26) to deduce that

$$\begin{aligned} &j(g_1, v_2, w) - j(g_1, v_1, w) + j(g_2, v_1, w) - j(g_2, v_2, w) \\ &\qquad \leq C \mid g_1 - g_2 \mid_V \mid v_1 - v_2 \mid_V . \end{aligned} \tag{4.15}$$

The relation (3.20), the estimate (4.11) and the inequality (4.15) combined with (4.14) give us

$$m_{\mathcal{A}} \mid v_1 - v_2 \mid_V \leq C(\mid f_1 - f_2 \mid_V + \mid \eta_1 - \eta_2 \mid_{\mathcal{H}} + \mid g_1 - g_2 \mid_V). \tag{4.16}$$

The inequality (4.16) and the regularity of the functions f, g and η show that

$$v_{w\eta g} \in C(0, T; V).$$

From assumption (3.23) and the relation (4.6) we have

$$\mid \sigma_1 - \sigma_2 \mid_{\mathcal{H}} \leq C(\mid v_1 - v_2 \mid_V + \mid \eta_1 - \eta_2 \mid_{\mathcal{H}}), \tag{4.17}$$

and from (4.13) we have

$$Div\sigma(t_i) + f_0(t_i) = 0,\ i = 1, 2. \tag{4.18}$$

The regularity of the function η, v, f_0 and the relations (4.17)-(4.18) show that

$$\sigma_{w\eta g} \in C(0, T; \mathcal{H}_1).$$

■

Let $w \in C(0,T;L^2(\Gamma_3))$, $\boldsymbol{g} \in C(0,T;V)$ and let $\boldsymbol{\eta} \in C(0,T;\mathcal{H})$ be given. We consider the following operator

$$\Lambda_{w\eta} : C(0,T;V) \to C(0,T;V)$$

defined by

$$\Lambda_{w\eta}\boldsymbol{g} = \boldsymbol{u}_0 + \int_0^t \boldsymbol{v}_{w\eta g}\,(s)\,ds \quad \forall \boldsymbol{g} \in C(0,T;V). \tag{4.19}$$

Lemma 4.3. *Let the assumptions (3.23)-(3.32) hold. Then the operator $\Lambda_{w\eta}$ has a unique fixed point $\boldsymbol{g}_{w\eta} \in C(0,T;V)$.*

Proof. Let $\boldsymbol{g}_1, \boldsymbol{g}_2 \in C(0,T;V)$ and let $\boldsymbol{\eta} \in C(0,T;\mathcal{H})$. We use the notation $\boldsymbol{v}_i = \boldsymbol{v}_{w\eta g_i}$ and $\boldsymbol{\sigma}_i = \boldsymbol{\sigma}_{w\eta g_i}$ for $i = 1,2$. Using similar arguments as those used in (4.16) we find

$$m_{\mathcal{A}} \mid \boldsymbol{v}_1(t) - \boldsymbol{v}_2(t) \mid_V \leq C \mid \boldsymbol{g}_1(t) - \boldsymbol{g}_2(t) \mid_V \ \forall t \in [0,T]. \tag{4.20}$$

From (4.19) and (4.20) we find that

$$\mid \Lambda_{w\eta}\boldsymbol{g}_1(t) - \Lambda_{w\eta}\boldsymbol{g}_2(t) \mid_V \leq C \int_0^t \mid \boldsymbol{g}_1(s) - \boldsymbol{g}_2(s) \mid_V ds \ \forall t \in [0,T]. \tag{4.21}$$

Reiterating this inequality m times, we obtain

$$\mid \Lambda_{w\eta}^m \boldsymbol{g}_1 - \Lambda_{w\eta}^m \boldsymbol{g}_2 \mid_{C(0,T;V)} \leq \frac{C^m T^m}{m!} \mid \boldsymbol{g}_1 - \boldsymbol{g}_2 \mid_{C(0,T;V)} . \tag{4.22}$$

This shows that for m large enough the operator $\Lambda_{w\eta}^m$ is a contraction in the Banach space $C(0,T;V)$. Thus, from Banach's fixed point theorem the operator $\Lambda_{w\eta}$ has a unique fixed point $\boldsymbol{g}_{w\eta} \in C(0,T;V)$. ∎

Now we consider the following problem.

Problem $PV_{w\eta}$. Find a displacement field $\boldsymbol{u}_{w\eta} : [0,T] \to V$ such that for all $t \in [0,T]$,

$$(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_{w\eta}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v} - \dot{\boldsymbol{u}}_{w\eta}(t)))_{\mathcal{H}} + j(\boldsymbol{u}_{w\eta}(t), \boldsymbol{v}, w(t)) - j(\boldsymbol{u}_{w\eta}(t), \dot{\boldsymbol{u}}_{w\eta}(t), w(t))$$
$$+ (\boldsymbol{\eta}(t), \boldsymbol{\varepsilon}(\boldsymbol{v} - \dot{\boldsymbol{u}}_{w\eta}(t))_{\mathcal{H}} \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_{w\eta}(t))_V \ \forall \boldsymbol{v} \in V, \tag{4.23}$$

$$\boldsymbol{u}_{w\eta}(0) = \boldsymbol{u}_0. \tag{4.24}$$

In the study of the problem $PV_{w\eta}$ we have the following result.

Lemma 4.4. *$PV_{w\eta}$ has a unique solution satisfying the regularity (4.1).*

Proof. For each $w \in C(0,T;L^2(\Gamma_3))$ and $\boldsymbol{\eta} \in C(0,T;\mathcal{H})$, we denote by $\boldsymbol{g}_{w\eta} \in C(0,T;V)$ be the fixed point obtained in lemma 4.3 and let $\boldsymbol{u}_{w\eta}$ be the function defined by

$$\boldsymbol{u}_{w\eta}(t) = \boldsymbol{u}_0 + \int_0^t \boldsymbol{v}_{w\eta g_{w\eta}}(s)\,ds \qquad \forall t \in [0,T]. \tag{4.25}$$

We have $\Lambda_{w\eta}\boldsymbol{g}_{w\eta} = \boldsymbol{g}_{w\eta}$. From (4.19) and (4.25) it follows that

$$\boldsymbol{u}_{w\eta} = \boldsymbol{g}_{w\eta}. \tag{4.26}$$

Therefore, taking $\boldsymbol{g} = \boldsymbol{g}_{w\eta}$ in (4.7) and using (4.6), (4.25) and (4.26) we see that $\boldsymbol{u}_{w\eta}$ is the unique solution to problem $PV_{w\eta}$ satisfiying the regularity expressed in (4.1). ∎

In the second step, let $w \in C(0,T;L^2(\Gamma_3))$ and $\boldsymbol{\eta} \in C(0,T;\mathcal{H})$, we use the displacement field $\boldsymbol{u}_{w\eta}$ obtained in (4.25) and consider the following variational problem.

Problem $QV_{w\eta}$. Find the electric potential field $\varphi_{w\eta} : [0,T] \to W$ such that

$$(B\nabla\varphi_{w\eta}(t), \nabla\phi)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{w\eta}(t)), \nabla\phi)_H = (q(t), \phi)_W \quad \forall\phi \in W, \tag{4.27}$$

we have the following result.

Lemma 4.5. *$QV_{w\eta}$ has a unique solution $\varphi_{w\eta}$ which satisfies the regularity (4.3).*

Proof. We define a bilinear form: $b(.,.) : W \times W \to \mathbb{R}$ such that

$$b(\varphi, \phi) = (B\nabla\varphi, \nabla\phi)_H \quad \forall\varphi, \phi \in W. \tag{4.28}$$

We use (3.27) to show that the bilinear form b is continuous, symmetric and coercive on W, moreover using Riesz Representation Theorem we may define an element $q_{w\eta} : [0,T] \to W$ such that

$$(q_{w\eta}(t), \phi)_W = (q(t), \phi)_W + (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{w\eta}(t)), \nabla\phi)_H \quad \forall\phi \in W.$$

We apply Lax-Milgram Theorem to deduce that there exists a unique element $\varphi_{w\eta}(t) \in W$ such that

$$b(\varphi_{w\eta}(t), \phi) = (q_{w\eta}(t), \phi)_W \quad \forall\phi \in W. \tag{4.29}$$

We conclude that $\varphi_{w\eta}(t)$ is a solution to $QV_{w\eta}$. Let $t_1, t_2 \in [0,T]$, it follows from (3.22), (3.27), (3.28), (4.28) and (4.29) that

$$|\ \varphi_{w\eta}(t_1) - \varphi_{w\eta}(t_2)\ |_W \le C(|\ \boldsymbol{u}_{w\eta}(t_1) - \boldsymbol{u}_{w\eta}(t_2)\ |_V + |\ q(t_1) - q(t_2)\ |_W),$$

the previous inequality and the regularity of $\boldsymbol{u}_{w\eta}$ and q imply that $\varphi_{w\eta} \in C(0,T;W)$. ∎

Finally as a consequence of these results and using the properties of the operator $\mathcal{F}$ and the operator $\mathcal{E}$, for $t \in [0,T]$, we consider the operator $\Lambda : C(0,T;\mathcal{H}) \to C(0,T;\mathcal{H})$ defined by

$$\Lambda\boldsymbol{\eta}(t) = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_{w\eta}(t)) + \int_0^t M(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_{w\eta}(s))\, ds + \mathcal{E}^*\nabla\varphi_{w\eta}(t), \tag{4.30}$$

We have the following result.

Lemma 4.6. *The operator Λ hase a unique fixed point $\boldsymbol{\eta}^* \in C(0,T;\mathcal{H})$ such that $\Lambda\boldsymbol{\eta}^* = \boldsymbol{\eta}^*$.*

Proof. Let $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in C(0,T;\mathcal{H})$. We use the notation $\boldsymbol{u}_{w\eta_i} = \boldsymbol{u}_i$, $\dot{\boldsymbol{u}}_{w\eta_i} = \boldsymbol{v}_{w\eta_i} = \boldsymbol{v}_i$, $\boldsymbol{\sigma}_{w\eta_i} = \boldsymbol{\sigma}_i$ and $\varphi_{w\eta_i} = \varphi_i$ for $i = 1,2$. Using (4.30), (3.20), (3.24), (3.25) and (3.28) to obtain

$$|\ \Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)\ |^2_{\mathcal{H}} \le C(|\ \boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\ |^2_V + \int_0^t |\ \boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\ |^2_V\, ds + |\ \varphi_1(t) - \varphi_2(t)\ |^2_W). \tag{4.31}$$

Since

$$\boldsymbol{u}_i(t) = \int_0^t \boldsymbol{v}_i(s)ds + \boldsymbol{u}_0, \quad t \in [0,T],$$

we have

$$\mid \boldsymbol{u}_1(t) - \boldsymbol{u}_2(t) \mid_V^2 \leq C \int_0^t \mid \boldsymbol{v}_1(s) - \boldsymbol{v}_2(s) \mid_V^2 ds \quad \forall t \in [0,T]. \tag{4.32}$$

For the electric potential field, we use (4.27), (3.22), (3.27) and (3.34) to obtain

$$\mid \varphi_1(t) - \varphi_2(t) \mid_W^2 \leq C \mid \boldsymbol{u}_1(t) - \boldsymbol{u}_2(t) \mid_V^2. \tag{4.33}$$

We substitute (4.33) in (4.31) and use (4.32) to obtain

$$\mid \Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t) \mid_{\mathcal{H}}^2 \leq C \int_0^t \mid \boldsymbol{v}_1(s) - \boldsymbol{v}_2(s) \mid_V^2 ds. \tag{4.34}$$

Moreover, from (4.23) we obtain

$$\begin{aligned}(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_1) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_2), \boldsymbol{\varepsilon}(\boldsymbol{v}_1 - \boldsymbol{v}_2))_{\mathcal{H}} &\leq j(\boldsymbol{u}_1, \boldsymbol{v}_2, w) - j(\boldsymbol{u}_1, \boldsymbol{v}_1, w) \\ &+ j(\boldsymbol{u}_2, \boldsymbol{v}_1, w) - j(\boldsymbol{u}_2, \boldsymbol{v}_2, w) - (\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2, \boldsymbol{\varepsilon}(\boldsymbol{v}_1 - \boldsymbol{v}_2))_{\mathcal{H}}.\end{aligned} \tag{4.35}$$

Using similar arguments as those used in (4.15) we find

$$\begin{aligned}&j(\boldsymbol{u}_1, \boldsymbol{v}_2, w) - j(\boldsymbol{u}_1, \boldsymbol{v}_1, w) + j(\boldsymbol{u}_2, \boldsymbol{v}_1, w) - j(\boldsymbol{u}_2, \boldsymbol{v}_2, w). \\ &\qquad \leq C \mid \boldsymbol{u}_1 - \boldsymbol{u}_2 \mid_V \mid \boldsymbol{v}_1 - \boldsymbol{v}_2 \mid_V.\end{aligned} \tag{4.36}$$

From (3.23)(b), (4.35) and (4.36) it follows that

$$\mid \boldsymbol{v}_1 - \boldsymbol{v}_2 \mid_V^2 \leq C(\mid \boldsymbol{u}_1 - \boldsymbol{u}_2 \mid_V^2 + \mid \boldsymbol{\eta}_1 - \boldsymbol{\eta}_2 \mid_{\mathcal{H}}^2) \tag{4.37}$$

Integrating this equality with respect to time, we find

$$\begin{aligned}&\int_0^t \mid \boldsymbol{v}_1(s) - \boldsymbol{v}_2(s) \mid_V^2 ds \\ &\qquad \leq C \int_0^t (\mid \boldsymbol{u}_1(s) - \boldsymbol{u}_2(s) \mid_V^2 + \mid \boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s) \mid_{\mathcal{H}}^2) ds \ \ \forall t \in [0,T].\end{aligned} \tag{4.38}$$

From (4.32) we have

$$\begin{aligned}&\mid \boldsymbol{u}_1(t) - \boldsymbol{u}_2(t) \mid_V^2 \\ &\leq C \int_0^t \mid \boldsymbol{u}_1(s) - \boldsymbol{u}_2(s) \mid_V^2 ds + C \int_0^t \mid \boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s) \mid_{\mathcal{H}}^2 ds \quad \forall t \in [0,T].\end{aligned} \tag{4.39}$$

This inequality, combined with Gronwall's inequality, leads to

$$\mid \boldsymbol{u}_1(t) - \boldsymbol{u}_2(t) \mid_V^2 \leq C \int_0^t \mid \boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s) \mid_{\mathcal{H}}^2 ds \quad \forall t \in [0,T]. \tag{4.40}$$

It follows now from (4.38) that

$$\int_0^t \mid \boldsymbol{v}_1(s) - \boldsymbol{v}_2(s) \mid_V^2 ds \leq C \int_0^t \mid \boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s) \mid_{\mathcal{H}}^2 ds \quad \forall t \in [0,T]. \tag{4.41}$$

The previous inequality and estimate (4.34) imply

$$\mid \Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t) \mid^2_{\mathcal{H}}$$

$$\leq C \int_0^t \mid \boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s) \mid^2_{\mathcal{H}} ds.$$

Reiterating this inequality m times leads to

$$\mid \Lambda^m\boldsymbol{\eta}_1 - \Lambda^m\boldsymbol{\eta}_2 \mid^2_{C(0,T;\mathcal{H})} \leq \frac{C^m T^m}{m!} \mid \boldsymbol{\eta}_1 - \boldsymbol{\eta}_2, \mid^2_{C(0,T;\mathcal{H})} .$$

Thus, for m sufficiently large, Λ^m is a contraction on the Banach space $C(0,T;\mathcal{H})$, and so Λ has a unique fixed point. ■

Let $w \in C(0,T;L^2(\Gamma_3))$. In the third step we consider the following variational problem.

Problem PV_w. Find a displacement field $\boldsymbol{u}_w : [0,T] \to V$, a stress field $\boldsymbol{\sigma}_w : [0,T] \to \mathcal{H}$, an electric potential field $\varphi_w : [0,T] \to W$ and an electric displacement field $\boldsymbol{D}_w : [0,T] \to H$ such that for all $t \in [0,T]$,

$$\boldsymbol{\sigma}_w(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_w(t)) + \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_w(t)) + \int_0^t M(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_w(s))\, ds + \mathcal{E}^*\nabla\varphi_w(t), \quad (4.42)$$

$$(\boldsymbol{\sigma}_w(t)\,, \boldsymbol{\varepsilon}(\boldsymbol{v}-\dot{\boldsymbol{u}}_w(t)))_{\mathcal{H}} + j(\boldsymbol{u}_w(t), \boldsymbol{v}, w(t)) - j(\boldsymbol{u}_w(t), \dot{\boldsymbol{u}}_w(t), w(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v}-\dot{\boldsymbol{u}}_w(t))_V \quad \forall \boldsymbol{v} \in V, \quad (4.43)$$

$$\boldsymbol{D}_w(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_w(t)) - B\nabla\varphi_w(t), \quad (4.44)$$

$$(\boldsymbol{D}_w(t), \nabla\phi)_H = -(q(t), \phi)_W \quad \forall \phi \in W, \quad (4.45)$$

$$\boldsymbol{u}_w(0) = \boldsymbol{u}_0. \quad (4.46)$$

Lemma 4.7. *Problem PV_w has a unique solution $(\boldsymbol{u}_w, \sigma_w, \varphi_w, D_w)$ satisfying (4.1)-(4.4).*

Proof. Let $\boldsymbol{\eta}^* \in C(0,T;\mathcal{H})$ be the fixed point of Λ defined by (4.30) and denote $\boldsymbol{u}_w = \boldsymbol{u}_{w\eta^*}, \varphi_w = \varphi_{w\eta^*}$ be the solutions to problems $PV_{w\eta}$ and $QV_{w\eta}$ obtained in lemmas 4.4 and 4.5 for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$. Let

$$\boldsymbol{\sigma}_w(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_w(t)) + \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_w(t)) + \int_0^t M(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_w(s))\, ds + \mathcal{E}^*\nabla\varphi_w(t).$$

Equation $\Lambda\boldsymbol{\eta}^* = \boldsymbol{\eta}^*$, combined with (4.30), shows that $(\boldsymbol{u}_w, \boldsymbol{\sigma}_w, \varphi_w, D_w)$ satisfies (4.42)-(4.45). Next, (4.46) and the regularities (4.1)-(4.4) follow from Lemmas 4.4, 4.5 and assumptions on $\mathcal{A}$, $\mathcal{F}$, M and $\mathcal{E}$ which concludes the existence part of the lemma 4.7.

The uniqueness part of lemma 4.7 is a consequence of the uniqueness of the fixed point of the operator Λ defined by (4.30) and the unique solvability of problems $PV_{w\eta^*}$ and $QV_{w\eta^*}$. ■

Let us now consider the operator $\mathcal{L} : C(0,T;L^2(\Gamma_3)) \to C(0,T;L^2(\Gamma_3))$ defined by

$$\mathcal{L}w(t) = -k_1 v^* \int_0^t (\sigma_w)_\nu(s)ds \quad \forall t \in [0,T]\,. \tag{4.47}$$

The last step in the proof of Theorem 4.1 is the next result.

Lemma 4.8. *The operator $\mathcal{L}$ has a unique fixed point*

$$w^* \in C(0,T;L^2(\Gamma_3)).$$

Proof. Let $w_1, w_2 \in C(0,T;L^2(\Gamma_3))$ and denote by $(\boldsymbol{u}_i, \boldsymbol{\sigma}_i, \varphi_i, \boldsymbol{D}_i)$, $i = 1,2$, the solutions to problem PV_w for $w = w_i$, i.e. $\boldsymbol{u}_i = \boldsymbol{u}_{w_i}$, $\boldsymbol{v}_i = \dot{\boldsymbol{u}}_i = \dot{\boldsymbol{u}}_{w_i}$, $\boldsymbol{\sigma}_i = \boldsymbol{\sigma}_{w_i}$, $\varphi_i = \varphi_{w_i}$ and $\boldsymbol{D}_i = \boldsymbol{D}_{w_i}$. Moreover, we denote in the sequel by C various positive constants which may depend on k_1 and $\boldsymbol{v}^*$. We use similar arguments that those used in the proof of the relation (4.41) to find that

$$\begin{aligned}\int_0^t &\mid \boldsymbol{v}_1(s) - \boldsymbol{v}_2(s) \mid_V^2 ds \\ &\leq C\Big(\int_0^t \mid \boldsymbol{u}_1(s) - \boldsymbol{u}_2(s) \mid_V^2 ds + \int_0^t \mid w_1(s) - w_2(s) \mid_{L^2(\Gamma_3)}^2 ds\Big).\end{aligned} \tag{4.48}$$

Since $\boldsymbol{u}_1(0) = \boldsymbol{u}_2(0) = \boldsymbol{u}_0$ and using (4.48) we obtain

$$\begin{aligned}\mid \boldsymbol{u}_1(t) - \boldsymbol{u}_2(t) \mid_V^2 &\leq C \int_0^t \mid \boldsymbol{v}_1(s) - \boldsymbol{v}_2(s) \mid_V^2 ds \\ &\leq C \int_0^t \mid \boldsymbol{u}_1(s) - \boldsymbol{u}_2(s) \mid_V^2 ds + C \int_0^t \mid w_1(s) - w_2(s) \mid_{L^2(\Gamma_3)}^2 ds.\end{aligned} \tag{4.49}$$

Applying Gronwall inequality, we deduce that

$$\mid \boldsymbol{u}_1(t) - \boldsymbol{u}_2(t) \mid_V^2 \leq C \int_0^t \mid w_1(s) - w_2(s) \mid_{L^2(\Gamma_3)}^2 ds. \tag{4.50}$$

It follows now from (4.48) and (4.50) that

$$\int_0^t \mid \boldsymbol{v}_1(s) - \boldsymbol{v}_2(s) \mid_V^2 ds \leq C \int_0^t \mid w_1(s) - w_2(s) \mid_{L^2(\Gamma_3)}^2 ds. \tag{4.51}$$

On the other hand, since

$$\boldsymbol{\sigma}_i = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_i) + \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_i) + \int_0^t M(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}_i(s))\, ds + \mathcal{E}^*\nabla\varphi_i,$$

for $i = 1,2$, we use the assumption (3.23)(b), (3.24), (3.25), (3.28) and the relation (4.33) to obtain that for $s \in [0,T]$

$$\mid \boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s) \mid_{\mathcal{H}}^2 \leq C(\mid \boldsymbol{v}_1(s) - \boldsymbol{v}_2(s) \mid_V^2 + \mid \boldsymbol{u}_1(s) - \boldsymbol{u}_2(s) \mid_V^2).$$

We integrate the previous inequality with respect to time to deduce that

$$\begin{aligned}\int_0^t &\mid \boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s) \mid_{\mathcal{H}}^2 ds \\ &\leq C\Big(\int_0^t \mid \boldsymbol{v}_1(s) - \boldsymbol{v}_2(s) \mid_V^2 ds + \int_0^t \mid \boldsymbol{u}_1(s) - \boldsymbol{u}_2(s) \mid_V^2 ds\Big).\end{aligned}$$

We substitute (4.50) and (4.51) in the previous inequality to find

$$\int_0^t \mid \boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s) \mid_{\mathcal{H}}^2 ds \leq C \int_0^t \mid w_1(s) - w_2(s) \mid_{L^2(\Gamma_3)}^2 ds. \tag{4.52}$$

The definition of the operator $\mathcal{L}$ given by (4.47) and estimate (4.52) give us

$$\mid \mathcal{L}w_1(t) - \mathcal{L}w_2(t) \mid_{L^2(\Gamma_3)}^2 \leq C \int_0^t \mid w_1(s) - w_2(s) \mid_{L^2(\Gamma_3)}^2 ds. \tag{4.53}$$

Reiterating this inequality n times leads to

$$\mid \mathcal{L}^n w_1 - \mathcal{L}^n w_2 \mid_{C(0,T;L^2(\Gamma_3))}^2 \leq \frac{C^n T^n}{n!} \mid w_1 - w_2 \mid_{C(0,T;L^2(\Gamma_3))}^2 .$$

Therefore, for n large enough, $\mathcal{L}^n$ is a contractive operator on the Banach space $C(0,T;L^2(\Gamma_3))$. The operator $\mathcal{L}$ has a unique fixed point $w^* \in C(0,T;L^2(\Gamma_3))$. ■

Now we have all the ingredients to prove Theorem 4.1.

Proof. Let w^* be the fixed point of the operator $\mathcal{L}$ given by (4.47). With (4.42)-(4.47) it is easy to verify that $(\boldsymbol{u}_{w^*}, \boldsymbol{\sigma}_{w^*}, \varphi_{w^*}, \boldsymbol{D}_{w^*}, w^*)$ is the unique solution to problem PV satisfying the regularities (4.1)-(4.5). ■

References

[1] R. C. Batra and J. S. Yang, Saint Venant's principle in linear piezoelectricity, Journal of Elasticity 38 (1995), 209-218.

[2] P. Bisenga, F. Lebon and F. Maceri, The unilateral frictional contact of a piezoelectric body with a rigid support, in Contact Mechanics , J.A.C. Martins and Manuel D. P. Monteiro Marques (Eds), Kluwer, Dordrecht, (2002), 347-354.

[3] H. Brézis, Equations et Inequations non linéaires dans les espaces vectoriels en dualité, Ann. Inst. Fourier, 18 (1968), 115-175.

[4] M. Dalah and M. Sofonea Antiplane frictional contact of electro-viscoelasticity cylinders, EDJE, Vol. 2007 (2007), No. 161, 1-14.

[5] G. Duvaut and J.L. Lions, Les Inéquations en Mécanique et en Physique, Springer-Verlag, Berlin (1976).

[6] T. Ikeda, Fundamentals of piezoelectricity, Oxford University Press, Oxford (1990).

[7] Z. Lerguet, M. Shillor and M. Sofonea, A frictional contact problem for an electro-viscoelastic body, EJDE, Vol. 2007 (2007), No. 170, 1-16.

[8] F. Maceri and P. Bisenga, The Unilateral frictionless contact of piezoelectric body with a rigid support, Math. Comp. Modelling, 28, (1998), 19-28.

[9] J. A. C Martins and J. T. Oden, Existence and uniqueness results for dynamic contact problems with nonlinear normal and friction interface laws, Nonlinear Analysis TMA 11 (1987), 407-428.

[10] S. Migórski, A. Ochal and M. Sofonea, Analysis of a quasistatic contact problem for piezoelectric materials, J. Math. Anal. Appl. 382 (2011), 701-713.

[11] S. Migórski, A. Ochal and M. Sofonea, A dynamic frictional contact problem for piezoelectric materials, J. Math. Anal. Appl. 361 (2010), 161-176.

[12] S. Migórski, Hemivariational inequality for a frictional contact problem in elasto-piezoelectricity, Discrete Contin. Dyn. Syst. Ser. B 6 (2006), 1339-1356.

[13] J. Nečas and I. Hlavaček, "Mathematical theory of elastic and elastoplastic bodies: An Introduction", Elsevier, Amsterdam, (1981).

[14] M. Rochdi, M. Shillor and M. Sofonea, Quasistatic viscoelastic contact with normal compliance and friction, Journal of Elasticity 51 (1998),105-126.

[15] M. Selmani and L. Selmani, A frictional contact problem with wear and damage for electro-viscoelastic materials, Applications of Mathematics Vol. 55, No. 2, pp. 89-109, 2010.

[16] M. Sofonea and R. Arhab, An electro-viscoelastic contact problem with adhesion, Dynamics of Continuous Discret and Impulsive Systems, 14, (2007), 577-991.

[17] M. Sofonea and El H. Essoufi, Quasistatic frictional contact of viscoelastic piezoelectric body, Ad . Math. Sci. Appl., 14, (2004), 613-631.

[18] M. Sofonea and El H. Essoufi, A piezoelectric contact problem with slip dependent coefficient of friction, Mathematical Modelling and Analysis, 9, (2004), 229-242.

[19] N. Strömberg, Continuum thermodynamics of contact, friction and wear, Thesis No. 491, Department of Mechanical Engineering, Linköping Institute of Technology, Linköping, Sweden, (1995).

[20] N. Strömberg, L. Johansson and A. Klarbring, Derivation and analysis of a generalized standard model for contact friction and wear Int. J. Solids Structures 33 (1996), 1817-1836.

Department of Mathematics,
University of Setif,
19000 Setif Algeria
email:s_elmanih@yahoo.fr