Global Existence and Finite Time Blowup for a Nonlocal Parabolic System*

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Abstract

This paper concerns with a parabolic system coupled via nonlocal sources, subjecting to homogeneous Dirichlet boundary condition. The main aim of this paper is to study conditions on the global existence and finite time blowup of solutions. By using the super- and sub-solution techniques, the critical exponent of the system is determined. Furthermore, the related classification for the parameters in the model is optimal and complete.

1 Introduction and main results

In this paper, we investigate the global existence and finite time blowup of nonnegative solutions for the following parabolic system with nonlocal sources

$$\begin{cases} u_t = \Delta u + \|uv\|_\alpha^p, & (x,t) \in \Omega \times (0,T), \\ v_t = \Delta v + \|uv\|_\beta^q, & (x,t) \in \Omega \times (0,T) \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \\ u(x,t) = 0, \quad v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{1.1}$$

where Ω be a bounded domain in $\mathbb{R}^N (N \geq 1)$ with smooth boundary $\partial\Omega$, and $u_0(x), v_0(x)$ are nonnegative bounded functions in Ω, constants $\alpha, \beta \geq 1$, $p, q > 0$, where $\|\cdot\|_\alpha^\alpha = \int_\Omega |\cdot|^\alpha \mathrm{d}x$.

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Equations (1.1) constitute a simple example of a reaction diffusion system exhibiting a nontrivial coupling on the unknowns $u(x,t), v(x,t)$. Such as heat propagations in a two-component combustible mixture [1], chemical processes [2], interaction of two biological groups without self-limiting [3], etc.

In the past several decades, a number of works have been contributed to the study of the following weakly coupled reaction-diffusion system

$$u_t = \Delta u + u^{p_1} v^{q_1}, \quad v_t = \Delta v + u^{p_2} v^{q_2}, \quad (x,t) \in \Omega \times (0,T), \tag{1.2}$$

(see [1],[4]-[6] and references therein), especially its special cases $p_1 = q_2 = 0$ (variational) or $q_1 = p_2 = 0$ (uncoupled single equation). For the case $p_1 = q_2 = 0$, Escobedo analyzed the boundedness and blow-up of solutions[7]; Caristi obtained the blow-up estimates of solutions[4]. Wu Yuan studied the uniqueness of generalized solutions with degenerate diffusion[6]. It is well known that there have been much more results for the uncoupled single equation case $q_1 = p_2 = 0$, including necessary and sufficient conditions for finite blow-up[8], estimates of blow-up time[9], blow-up rates[10] and blow-up behavior[11].

The general form of (1.2) was systematically studied by Escobedo[5]. They gave a complete analysis on the critical blow-up and the global existence numbers for the Cauchy problem of (1.2), where the introduced parameters α and β satisfying

$$\begin{pmatrix} p_1 - 1 & q_1 \\ p_2 & q_2 - 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

played important roles in their framework.

Meanwhile, the system (1.2) was also studied by Wang in [12] and Zheng in [13] with different methods. Some interesting results concerning the global existence and blow-up conditions of the solutions are established.

Lately, Li et al. in[14] and Zhang et al. in[15] studied the following system

$$u_t = \Delta u + \int_\Omega u^m(x,t) v^n(x,t) \mathrm{d}x, \; v_t = \Delta v + \int_\Omega u^p(x,t) v^q(x,t) \mathrm{d}x, \; x \in \Omega, t > 0,$$

$$u_t = \Delta u + a(x) u^{p_1}(x,t) v^{q_1}(0,t), \quad v_t = \Delta v + b(x) v^{p_2}(x,t) u^{q_2}(0,t), \quad x \in B, t > 0,$$

respectively. They obtained some results on the global solutions, the blow-up solutions and the blow-up rates.

Our present work is motivated by [5] and [12]-[15] mentioned above. The main purpose is to extend Escobedo's method in [5] to system (1.1) and establish the critical exponent which concern with the global existence and finite time blowup of solutions.

For a solution $(u(x,t), v(x,t))$ of (1.1), we define

$$T^* = T^*(u,v) = \sup\{T > 0 : (u,v) \text{ is bounded and satisfies (1.1)}\}.$$

Note that if $T^* < +\infty$, then we have

$$\limsup_{t \to T^*} \|u(x,t)\|_{L^\infty} = +\infty \quad \text{or} \quad \limsup_{t \to T^*} \|v(x,t)\|_{L^\infty} = +\infty,$$

in this case, we say that the solution (u,v) blows up in finite time.

Throughout the remainder of this paper, we denote

$$A = \begin{pmatrix} 1-p & -p \\ -q & 1-q \end{pmatrix}, \qquad L = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}.$$

Then, let us state our main results, the two theorems concern the global existence and blow-up conditions of the solutions to system (1.1).

Theorem 1. *If one of the following conditions holds:*

(1) $p,q<1$ *and* $pq<(1-p)(1-q)$;

(2) $p,q<1$, $pq>(1-p)(1-q)$ *and the initial data* $u_0(x),v_0(x)$ *are sufficiently small;*

(3) $p,q<1$, $pq=(1-p)(1-q)$ *and the domain* $(|\Omega|)$ *is sufficiently small.*

Then every nonnegative solution of system (1.1) *exists globally.*

Theorem 2. *If one of the following conditions holds:*

(1) $p,q<1$, $pq>(1-p)(1-q)$, *and the initial data* $u_0(x),v_0(x)$ *are sufficiently large;*

(2) $p,q<1$, $pq=(1-p)(1-q)$, *the domain contains a sufficiently large ball, and initial data* $u_0(x),v_0(x)$ *are sufficiently large.*

Then the nonnegative solution of system (1.1) *blows up in a finite time.*

This paper is organized as follows. In the next Section, we establish the local existence and give some auxiliary lemmas. In Section 3, which concerns global existence, we prove Theorem 1. Theorem 2 which deals with the blow-up phenomenon is proved in Section 4.

2 Local existence and comparison principle

At first, we give the maximum principle and the comparison principle for the nonlocal parabolic system. Let $0<T<+\infty$, we set $Q_T=\Omega\times(0,T)$, $\bar{Q}_T=\bar{\Omega}\times[0,T]$, $S_T=\partial\Omega\times(0,T)$ and define the following class of test functions:

$$\Psi\equiv\{\psi(x,t)\in C(\bar{Q}_T);\psi_t,\Delta\psi\in C(Q_T)\cap L^2(Q_T);\psi\geqslant 0;\psi(x,t)|_{x\in\partial\Omega}=0\}.$$

Definition 1. *A pair of vector function* $(\tilde{u}(x,t),\tilde{v}(x,t))$ *defined on* $\bar{Q}_T$, *for some* $T>0$, *is called a sub-solution of* (1.1), *if* $\tilde{u},\tilde{v}\in C^{2,1}(Q_T)\cap C(\bar{Q}_T)$ *and all the following hold:*

(1) $\tilde{u}(x,t),\tilde{v}(x,t)\in L^\infty(Q_T)$;

(2) $\tilde{u}(x,t),\tilde{v}(x,t)\le 0$ *for* $(x,t)\in S_T$, *and for all* $x\in\Omega$, $\tilde{u}(x,0)\le u_0(x),\tilde{v}(x,0)\le v_0(x)$;

(3) *For every* $t\in[0,T]$ *and any* $\psi_1,\psi_2\in\Psi$,

$$\begin{cases} \displaystyle\int_\Omega \big(\tilde{u}(x,t)\psi_1(x,t)-u_0(x)\psi_1(x,0)\big)\mathrm{d}x \leqslant \int_0^t\int_\Omega \big(\tilde{u}\psi_{1s}+\tilde{u}\Delta\psi_1+\|\tilde{u}\tilde{v}\|_\alpha^p\psi_1\big)\mathrm{d}x\mathrm{d}s, \\ \displaystyle\int_\Omega \big(\tilde{v}(x,t)\psi_2(x,t)-v_0(x)\psi_2(x,0)\big)\mathrm{d}x \leqslant \int_0^t\int_\Omega \big(\tilde{v}\psi_{2s}+\tilde{v}\Delta\psi_2+\|\tilde{u}\tilde{v}\|_\beta^q\psi_2\big)\mathrm{d}x\mathrm{d}s. \end{cases} \tag{2.1}$$

A super-solution $(\bar{u}(x,t), \bar{v}(x,t))$ can be defined in a similar way.

A weak solution (u, v) of (1.1) is a vector function which is both a sub-solution and a super-solution of (1.1). For every $T < +\infty$, if (u, v) is a weak solution of (1.1), we say the (u, v) is global.

Lemma 1. *(Maximum principle) Suppose that $\omega_1(x,t), \omega_2(x,t) \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ and satisfy*

$$\begin{cases} M_1\omega = \omega_{1t} - \Delta\omega_1 - f_1 \displaystyle\int_\Omega (c_{11}\omega_1 + c_{12}\omega_2)\mathrm{d}x \geq 0, & (x,t) \in Q_T, \\ M_2\omega = \omega_{2t} - \Delta\omega_2 - f_2 \displaystyle\int_\Omega (c_{21}\omega_1 + c_{22}\omega_2)\mathrm{d}x \geq 0, & (x,t) \in Q_T, \\ \omega_1(x,t) \geq 0, \quad \omega_2(x,t) \geq 0, & (x,t) \in S_T, \\ \omega_1(x,0) \geq 0, \quad \omega_2(x,0) \geq 0, & x \in \Omega, \end{cases} \tag{2.2}$$

where $f_i(x,t)$ and $c_{ij}(x,t)$ are the nonnegative functions in Q_T. Then $\omega_i(x,t) \geq 0$ on $\bar{Q}_T$.

Proof. The technique for proving the maximum principle for parabolic equation is quite standard. Here we shall sketch the argument for the convenience of the reader.

Suppose the strict inequality of (2.2) hold, then we assert that $\omega_i(x,t) > 0$ $(i = 1,2)$ on $\bar{Q}_T$. According to $\omega_i(x,0) > 0 (i = 1,2)$, by continuity, there exist $\delta > 0$ such that $\omega_i(x,t) > 0$ for all $x \in \Omega, 0 \leqslant t \leqslant \delta$. Let

$$A = \{\delta < T : \omega_i(x,t) > 0, (x,t) \in \Omega \times [0,\delta], i = 1,2\}$$

and $\bar{t} = \sup A$, then $0 < \bar{t} \leqslant T$.

If $\bar{t} < T$, there holds $\omega_i(x,t) \geqslant 0$ in $\Omega \times (0,\bar{t}]$, and at least one of ω_1, ω_2 vanishes at $(\bar{x},\bar{t})$ for some $\bar{x} \in \bar{\Omega}$. Furthermore, by the boundary conditions we know that $\bar{x} \in \Omega$. Without loss of generality, we suppose $\omega_1(\bar{x},\bar{t}) = 0$. In view of the boundary conditions, we know $\omega_1 > 0$ on $\partial\Omega \times (0,\bar{t}]$. So ω_1 takes the nonnegative minimum on $\bar{Q}_{\bar{t}}$ at $(\bar{x},\bar{t})$. Then, at $(\bar{x},\bar{t})$ we find that

$$M_1\omega = \omega_{1t} - \Delta\omega_1 - f_1 \int_\Omega (c_{11}\omega_1 + c_{12}\omega_2)\mathrm{d}x \leqslant 0.$$

This is a contradiction from (2.2). Hence $\bar{t} = T$, that is $\omega_i(x,t) > 0 (i = 1,2)$ on $\bar{Q}_T$.

Now, we consider the general case. Take constant γ satisfies

$$\gamma > \max_{(x,t)\in\bar{Q}_T} \left\{ f_1 \int_\Omega (c_{11} + c_{12})\mathrm{d}x,\ f_2 \int_\Omega (c_{21} + c_{22})\mathrm{d}x \right\}$$

and set $\tilde{\omega}_i(x,t) = \omega_i(x,t) + \varepsilon e^{\gamma t}$, $(i = 1,2)$, where ε is any fixed positive constant. In view of (2.2), we can get

$$\begin{cases} M_1\tilde{\omega} = M_1\omega + \varepsilon e^{\gamma t}\big(\gamma - f_1 \displaystyle\int_\Omega (c_{11} + c_{12})\mathrm{d}x\big) > 0, & (x,t) \in Q_T, \\ M_2\tilde{\omega} = M_2\omega + \varepsilon e^{\gamma t}\big(\gamma - f_2 \displaystyle\int_\Omega (c_{21} + c_{22})\mathrm{d}x\big) > 0, & (x,t) \in Q_T, \\ \tilde{\omega}_1(x,t) \geqslant \varepsilon e^{\gamma t} > 0, \quad \tilde{\omega}_2(x,t) \geqslant \varepsilon e^{\gamma t} > 0, & (x,t) \in S_T, \\ \tilde{\omega}_1(x,0) = \omega_1(x,0) + \varepsilon > 0, \quad \tilde{\omega}_2(x,0) = \omega_2(x,0) + \varepsilon > 0, & x \in \Omega. \end{cases}$$

Therefore, we have $\tilde{\omega}_i(x,t) > 0$, that is $\omega_i(x,t) + \varepsilon e^{\gamma t} > 0$ on $\bar{Q}_T$. Let $\varepsilon \to 0^+$, it follows that $\omega_i(x,t) \geqslant 0 (i = 1,2)$ on $\bar{Q}_T$. Thus the proof is completed. ■

Based on the above lemma, we obtain the following comparison principle.

Lemma 2. (*Comparison principle*) *Let* $(\tilde{u},\tilde{v})$ *and* $(\bar{u},\bar{v})$ *be a nonnegative sub-solution and a nonnegative super-solution of system* (1.1), *respectively. Then* $(\tilde{u},\tilde{v}) \leq (\bar{u},\bar{v})$ *on* Q_T *if*
$(\tilde{u}(x,0),\tilde{v}(x,0)) \leq (\bar{u}(x,0),\bar{v}(x,0))$ *and either*

$$\int_\Omega (\bar{u}\bar{v})^\alpha \mathrm{d}x \geq \rho > 0, \int_\Omega (\bar{u}\bar{v})^\beta \mathrm{d}x \geq \rho > 0 \quad \textit{or}$$
$$\int_\Omega (\tilde{u}\tilde{v})^\alpha \mathrm{d}x \geq \rho > 0, \int_\Omega (\tilde{u}\tilde{v})^\beta \mathrm{d}x \geq \rho > 0 \quad (2.3)$$

hold.

Proof. Subtracting the integral inequalities of (2.1) for $(\bar{u},\bar{v})$ and $(\tilde{u},\tilde{v})$, and using the mean value theorem, we have

$$\begin{aligned}\int_\Omega \big(\tilde{u}(x,t) - \bar{u}(x,t)\big)\psi_1(x,t)\mathrm{d}x \leqslant & \int_\Omega \big(\tilde{u}(x,0) - \bar{u}(x,0)\big)\psi_1(x,0)\mathrm{d}x \\ & + \int_0^t \int_\Omega (\tilde{u} - \bar{u})(\psi_{1s} + \Delta\psi_1)\mathrm{d}x\mathrm{d}s \\ & + \int_0^t D_1(s)\Big(\int_\Omega (\tilde{v} - \bar{v})\tilde{u}H_1(x,s)\mathrm{d}x\Big)\Big(\int_\Omega \psi_1 \mathrm{d}x\Big)\mathrm{d}s \\ & + \int_0^t D_1(s)\Big(\int_\Omega (\tilde{u} - \bar{u})\bar{v}H_1(x,s)\mathrm{d}x\Big)\Big(\int_\Omega \psi_1 \mathrm{d}x\Big)\mathrm{d}s.\end{aligned}$$

where

$$D_1(s) = \int_0^1 \frac{p}{\alpha}\Big(\theta \int_\Omega (\tilde{u}\tilde{v})^\alpha \mathrm{d}x + (1-\theta)\int_\Omega (\bar{u}\bar{v})^\alpha \mathrm{d}x\Big)^{p/\alpha - 1}\mathrm{d}\theta,$$
$$H_1(x,s) = \int_0^1 \alpha\big(\theta\tilde{u}\tilde{v} + (1-\theta)\bar{u}\bar{v}\big)^{\alpha-1}\mathrm{d}\theta.$$

Since $(\tilde{u},\tilde{v})$ and $(\bar{u},\bar{v})$ on Q_T are bounded, it follows from $\alpha \geqslant 1$ that $H_1(x,s)$ is a bounded, nonnegative function. Similarly, $D_1(s)$ is bounded if $p/\alpha \geqslant 1$. Now if $p/\alpha < 1$, we have $D_1(s) \leqslant \rho^{p/\alpha - 1}$ by the assumptions. Thus, appropriate test function ψ_1 may be chosen exactly as in [16] to obtain

$$\begin{aligned}\int_\Omega [\tilde{u}(x,t) - \bar{u}(x,t)]_+ \mathrm{d}x \leqslant \|\psi_1\|_\infty & \int_\Omega [\tilde{u}(x,0) - \bar{u}(x,0)]_+ \mathrm{d}x \\ & + k_1 \int_0^t \int_\Omega \big([\tilde{u} - \bar{u}]_+ + [\tilde{v} - \bar{v}]_+\big)\mathrm{d}x\mathrm{d}s, \end{aligned} \quad (2.4)$$

where $\omega_+ = \max\{\omega, 0\}$ and $k_1 > 0$ is a bounded constant. Similarly, we can prove

$$\begin{aligned}\int_\Omega [\tilde{v}(x,t) - \bar{v}(x,t)]_+ \mathrm{d}x \leqslant \|\psi_2\|_\infty & \int_\Omega [\tilde{v}(x,0) - \bar{v}(x,0)]_+ \mathrm{d}x \\ & + k_2 \int_0^t \int_\Omega \big([\tilde{u} - \bar{u}]_+ + [\tilde{v} - \bar{v}]_+\big)\mathrm{d}x\mathrm{d}s \end{aligned} \quad (2.5)$$

for some bounded constant $k_2 > 0$. Now, (2.4)-(2.5) combined with the Gronwall's lemma show $(\tilde{u},\tilde{v}) \leqslant (\bar{u},\bar{v})$ since $(\tilde{u}(x,0),\tilde{v}(x,0)) \leqslant (\bar{u}(x,0),\bar{v}(x,0))$. Thus the proof is completed. ■

In order to prove the local existence of solution, for $k = 1, 2, \cdots$, we consider the following corresponding regularized system

$$\begin{cases} (u_k)_t = \Delta f_k(u_k) + \|f_k(u_k)g_k(v_k)\|_\alpha^p, & (x,t) \in Q_T, \\ (v_k)_t = \Delta g_k(v_k) + \|f_k(u_k)g_k(v_k)\|_\beta^q, & (x,t) \in Q_T, \\ u_k(x,t) = v_k(x,t) = 1/k, & (x,t) \in S_T, \\ u_k(x,0) = u_{0i}(x) + 1/k, \quad v_k(x,0) = v_{0i}(x) + 1/k, & x \in \Omega, \end{cases} \tag{2.6}$$

where

$$f_k(u_k) = \begin{cases} u_k, & u_k \geqslant 1/k, \\ 1/k, & u_k < 1/k, \end{cases} \quad g_k(v_k) = \begin{cases} v_k, & v_k \geqslant 1/k, \\ 1/k, & v_k < 1/k \end{cases}$$

and $u_{0i}(x), v_{0i}(x)$ are smooth approximation of $u_0(x), v_0(x)$ with $\mathrm{supp} u_{0i} \subset \Omega$ and $\mathrm{supp} v_{0i} \subset \Omega$, respectively. It is known that the system (2.6) has a unique classical solution $(u_k^i, v_k^i) \in C(\bar{\Omega} \times [0, T_i(k))) \cap C^{2,1}(\Omega \times (0, T_i(k)))$ for $0 < T_i(k) \leqslant \infty$ by the classical theory for parabolic equation, where $T_i(k)$ is the maximal existence time. By a direct computation and the classical maximum principle, we have $u_k^i, v_k^i \geqslant 1/k$. Hence (u_k^i, v_k^i) satisfies

$$(u_k^i)_t = \Delta(u_k^i) + \|u_k^i v_k^i\|_\alpha^p, \quad (v_k^i)_t = \Delta(v_k^i) + \|u_k^i v_k^i\|_\beta^q, \quad (x,t) \in Q_{T_i(k)} \tag{2.7}$$

with the corresponding initial and boundary conditions. At the same time, passing to the limit $i \to \infty$, it follows that

$$u_k(x,t) = \lim_{i\to\infty} u_k^i(x,t), \quad v_k(x,t) = \lim_{i\to\infty} v_k^i(x,t)$$

and (u_k, v_k) is a weak solution of

$$(u_k)_t = \Delta(u_k) + \|u_k v_k\|_\alpha^p, \quad (v_k)_t = \Delta(v_k) + \|u_k v_k\|_\beta^q, \quad (x,t) \in Q_{T(k)} \tag{2.8}$$

with the corresponding initial and boundary conditions on $Q_{T(k)}$, where $T(k) = \lim_{i\to\infty} T_i(k)$ is the maximal existence time. Here a weak solution of (2.8) is defined in a manner similar to that for (1.1), only the integral equalities for u and v, (2.1) may be replaced with

$$\begin{aligned} &\int_\Omega \big(u_k(x,t)\psi_1(x,t) - (u_0(x) + 1/k)\psi_1(x,0)\big)\mathrm{d}x \\ &= \int_0^t \int_\Omega \big(u_k\psi_{1s} + u_k\Delta\psi_1 + \|u_k v_k\|_\alpha^p \psi_1\big)\mathrm{d}x\mathrm{d}s + \frac{1}{k}\int_0^t \int_{\partial\Omega} (\partial\psi_1/\partial\nu)\mathrm{d}\sigma\mathrm{d}s, \end{aligned} \tag{2.9}$$

$$\begin{aligned} &\int_\Omega \big(v_k(x,t)\psi_2(x,t) - (v_0(x) + 1/k)\psi_2(x,0)\big)\mathrm{d}x \\ &= \int_0^t \int_\Omega \big(v_k\psi_{2s} + v_k\Delta\psi_2 + \|u_k v_k\|_\beta^q \psi_2\big)\mathrm{d}x\mathrm{d}s + \frac{1}{k}\int_0^t \int_{\partial\Omega} (\partial\psi_2/\partial\nu)\mathrm{d}\sigma\mathrm{d}s, \end{aligned} \tag{2.10}$$

respectively.

Since $u_k^i, v_k^i \geqslant 1/k$, applying Lemma 1, we have the following lemma.

Lemma 3. *Assume that $\omega(x,t), s(x,t) \in C(\bar{\Omega}\times[0,T_i(k)))\cap C^{2,1}(\Omega\times(0,T_i(k)))$ is a sub- (or super-) solution of* (2.7). *Then $(\omega,s)\leqslant(\geqslant)(u_k^i,v_k^i)$ on $\bar{\Omega}\times[0,T_i(k))$.*

According to Lemma 3, we have

Lemma 4. *If $k_1 > k_2$, then we have $(u_{k_1}^i, v_{k_1}^i) \leqslant (u_{k_2}^i, v_{k_2}^i)$ on $\bar{\Omega}\times[0,T_i(k_2))$ and $T_i(k_1)\geqslant T_i(k_2)$.*

Then, from Lemma 4, passing to the limit $i\to\infty$, it happens that $(u_{k_1},v_{k_1})\leqslant(u_{k_2},v_{k_2})$ and $T(k_1)\geqslant T(k_2)$ if $k_1>k_2$.

Therefore, the limit $T^* = \lim_{k\to\infty} T(k)$ exists and, as well, the point-wise limit

$$u(x,t)=\lim_{k\to\infty}u_k(x,t),\quad v(x,t)=\lim_{k\to\infty}v_k(x,t)$$

exist for any $(x,t)\in\bar{\Omega}\times[0,T^*)$. Furthermore, as the convergence of the sequence is monotone, passing to the limit $k\to\infty$ in the identities (2.9) and (2.10) is justified by monotone and dominated convergence theorems for any $\psi_1,\psi_2\in\Psi$ and $t\in[0,T^*)$. Thus, the following theorem is established.

Theorem 3. (Local existence and continuation) *Assume $u_0,v_0\geqslant 0, u_0,v_0\in L^\infty(\Omega)$, there is a $T^*=T^*(u_0,v_0)>0$ such that there exists a nonnegative weak solution $(u(x,t),v(x,t))$ of* (1.1) *for each $T<T^*$. Furthermore, either $T^*=+\infty$ or*

$$\limsup_{t\to T^*}(\|u(\cdot,t)\|_\infty+\|v(\cdot,t)\|_\infty)=\infty.$$

3 Global existence

In this section, we will prove Theorem 1. According to Lemma 2, we only need to construct bounded, positive super-solutions for any $T>0$.

Let $\varphi(x)$ be the unique positive solution of the following linear elliptic problem

$$-\Delta\varphi(x)=1,\quad x\in\Omega;\quad \varphi(x)=0,\quad x\in\partial\Omega.$$

Denote $C=\max_{x\in\Omega}\varphi(x)$, then $0\le\varphi(x)\le C$. Now, we define the functions $\bar{u},\bar{v}$ as

$$\bar{u}(x,t)=\big(k(\varphi(x)+1)\big)^{l_1},\quad \bar{v}(x,t)=\big(k(\varphi(x)+1)\big)^{l_2},\tag{3.1}$$

where constants $l_1,l_2<1$, and $k>0$ will be fixed later. Clearly, for any $T>0$, $(\bar{u},\bar{v})$ is a bounded function and $\bar{u}\ge k^{l_1}>0, \bar{v}\ge k^{l_2}>0$. Then, a series of computations yields

$$\begin{aligned}\bar{u}_t-\Delta\bar{u}=&-k^{l_1}l_1(l_1-1)(\varphi+1)^{l_1-2}|\nabla\varphi|^2+k^{l_1}l_1(\varphi+1)^{l_1-1}\\ \ge& k^{l_1}l_1(\varphi+1)^{l_1-1}\ge k^{l_1}l_1(C+1)^{l_1-1},\end{aligned}\tag{3.2}$$

$$\|\bar{u}\bar{v}\|_\alpha^p=k^{p(l_1+l_2)}\|(\varphi+1)^{l_1+l_2}\|_\alpha^p\le k^{p(l_1+l_2)}(C+1)^{p(l_1+l_2)}|\Omega|^{p/\alpha}.\tag{3.3}$$

Similarly,

$$\bar{v}_t-\Delta\bar{v}\ge k^{l_2}l_2(C+1)^{l_2-1},\quad \|\bar{u}\bar{v}\|_\beta^q\le k^{q(l_1+l_2)}(C+1)^{q(l_1+l_2)}|\Omega|^{q/\beta}.\tag{3.4}$$

(1) If $p,q < 1$ and $pq < (1-p)(1-q)$, then there exist constant $0 < \varepsilon_1 < 1$ such that

$$(1-\varepsilon_1)(1-p) > (1+\varepsilon_1)q, \quad (1-\varepsilon_1)(1-q) > (1+\varepsilon_1)p. \tag{3.5}$$

Hence, $AL = (\varepsilon_1, \varepsilon_1)^T$ yields

$$(1-p)l_1 - pl_2 = \varepsilon_1, \quad -ql_1 + (1-q)l_2 = \varepsilon_1. \tag{3.6}$$

That is,

$$l_1 = \frac{\varepsilon_1(1-q)+\varepsilon_1 p}{(1-p)(1-q)-pq}, \quad l_2 = \frac{\varepsilon_1(1-p)+\varepsilon_1 q}{(1-p)(1-q)-pq}. \tag{3.7}$$

Moreover, $0 < l_1, l_2 < 1$. Therefore, we can choose k sufficiently large such that

$$k > \max\Big\{\Big(\frac{|\Omega|^{p/\alpha}}{l_1}(1+C)^{1-\varepsilon_1}\Big)^{1/\varepsilon_1}, \Big(\frac{|\Omega|^{q/\beta}}{l_2}(1+C)^{1-\varepsilon_1}\Big)^{1/\varepsilon_1}\Big\} \tag{3.8}$$

and

$$(k(\varphi+1))^{l_1} \geq u_0(x), (k(\varphi+1))^{l_2} \geq v_0(x). \tag{3.9}$$

Now, it follows from (3.2)-(3.9) that $(\bar{u}, \bar{v})$ is a positive super-solution of (1.1).

(2) Next, if $p,q < 1$ and $pq > (1-p)(1-q)$, then there exist constant $0 < \varepsilon_2 < 1$ such that

$$(1+\varepsilon_2)(1-q) < (1-\varepsilon_2)p, \quad (1+\varepsilon_2)(1-p) < (1-\varepsilon_2)q. \tag{3.10}$$

Hence, $AL = (-\varepsilon_2, -\varepsilon_2)^T$ yields

$$(1-p)l_1 - pl_2 = -\varepsilon_2, \quad -ql_1 + (1-q)l_2 = -\varepsilon_2. \tag{3.11}$$

Namely,

$$l_1 = -\frac{\varepsilon_2(1-q)+\varepsilon_2 p}{(1-p)(1-q)-pq}, \quad l_2 = -\frac{\varepsilon_2(1-p)+\varepsilon_2 q}{(1-p)(1-q)-pq}. \tag{3.12}$$

Furthermore, $0 < l_1, l_2 < 1$. Therefore, we can choose k sufficiently small such that

$$k < \min\Big\{\Big(\frac{l_1}{|\Omega|^{p/\alpha}}(1+C)^{-1-\varepsilon_2}\Big)^{1/\varepsilon_2}, \Big(\frac{l_2}{|\Omega|^{q/\beta}}(1+C)^{-1-\varepsilon_2}\Big)^{1/\varepsilon_2}\Big\}. \tag{3.13}$$

And provided $u_0(x), v_0(x)$ are also sufficiently small to satisfy (3.9). Then, from (3.2)-(3.4) and (3.9), (3.10)-(3.13), we know that $(\bar{u}, \bar{v})$ is a positive super-solution of (1.1).

(3) Finally, if $p,q < 1$ and $pq = (1-p)(1-q)$, then we may choose positive constants $l_1, l_2 < 1$ such that

$$\frac{p}{1-p} = \frac{l_1}{l_2} = \frac{1-q}{q} \tag{3.14}$$

That is $l_1 = p(l_1 + l_2), l_2 = q(l_1 + l_2)$. Without loss of generality, we may assume that $\Omega \subset\subset B$, where B is a sufficiently large ball. And denote $\varphi_B(x)$ is the unique positive solution of the following linear elliptic problem

$$-\Delta\varphi(x) = 1,\ x \in B; \quad \varphi(x) = 0,\ x \in \partial B.$$

Let $C_0 = \max_{x\in B} \varphi_B(x)$, then $C \leq C_0$. Therefore, as long as Ω is sufficiently small and such that

$$|\Omega| < \min\{\big(\frac{l_1}{C_0+1}\big)^{\alpha/p},\quad \big(\frac{l_2}{C_0+1}\big)^{\beta/q}\}. \tag{3.15}$$

Furthermore, choose k large enough to satisfy (3.9). Then, it follows from (3.2)-(3.4) and (3.14)-(3.15) that $(\bar{u}, \bar{v})$ is a positive super-solution of (1.1).

Thus, according to the Lemma 2, the proof of Theorem 1 is completed.

4 Blow-up results

In this section, we prove Theorem 2, to this end, we only need to construct blowing-up positive sub-solutions.

Denote by $\lambda_1 > 0$ and $\phi_1(x)$ the first eigenvalue and the corresponding eigenfunction of the following eigenvalue problem

$$-\Delta\phi(x) = \lambda\phi(x),\ x \in \Omega; \quad \phi(x) = 0,\ x \in \partial\Omega. \tag{4.1}$$

It is well known that $\phi_1(x)$ may be normalized as $\phi_1(x) > 0$ in Ω and $\max_\Omega \phi_1(x) = 1$.

Now, we define the functions $\tilde{u}(x,t), \tilde{v}(x,t)$ as follows

$$\tilde{u}(x,t) = \big(s(t)\phi_1(x)\big)^{l_1}, \quad \tilde{v}(x,t) = \big(s(t)\phi_1(x)\big)^{l_2}, \tag{4.2}$$

where $l_1, l_2 > 1$, and $s(t)$ is the unique positive solution of the following Cauchy problem

$$\begin{cases} s'(t) = k_1 s^{r_1}(t)\big(s^{r_2}(t) - k_2\big), & t > 0, \\ s(0) = \delta > 0. \end{cases} \tag{4.3}$$

where constants $k_1, k_2, r_2 > 0$ and $r_1 \geq 1$ to be fixed later. Clearly, $s(t) \geq \delta$ and become unbounded in finite time $T(\delta)$. Next, we will show that $(\tilde{u}, \tilde{v})$ is a sub-solution of problem (1.1). A series of computations yields

$$\begin{aligned} \Delta\tilde{u} + \|\tilde{u}\tilde{v}\|_\alpha^p &= l_1(l_1-1)s^{l_1}\phi_1^{l_1-2}|\nabla\phi_1|^2 - \lambda_1 l_1 s^{l_1}\phi_1^{l_1} + c_1 s^{p(l_1+l_2)} \\ &\geq -\lambda_1 l_1 s^{l_1}\phi_1^{l_1} + c_1 s^{p(l_1+l_2)} \\ &= l_1 s^{l_1-1}\phi_1^{l_1}\frac{c_1}{l_1}s\big(s^{p(l_1+l_2)-l_1}\phi_1^{-l_1} - \frac{\lambda_1 l_1}{c_1}\big) \\ &\geq l_1 s^{l_1-1}\phi_1^{l_1}\frac{c_1}{l_1}s\big(s^{p(l_1+l_2)-l_1} - \frac{\lambda_1 l_1}{c_1}\big), \end{aligned} \tag{4-4}$$

$$\tilde{u}_t = l_1 s^{l_1-1}\phi_1^{l_1}s'(t), \tag{4-5}$$

$$\Delta\tilde{v} + \|\tilde{u}\tilde{v}\|_\beta^q \geq l_2 s^{l_2-1}\phi_1^{l_2}\frac{c_2}{l_2}s\big(s^{q(l_1+l_2)-l_2} - \frac{\lambda_1 l_2}{c_2}\big), \quad \tilde{v}_t = l_2 s^{l_2-1}\phi_1^{l_2}s'(t), \tag{4-6}$$

where

$$c_1 = \|\phi_1^{l_1+l_2}\|_\alpha^p > 0, \quad c_2 = \|\phi_1^{l_1+l_2}\|_\beta^q > 0.$$

(1) If $p, q < 1$ and $pq > (1-p)(1-q)$, then there exist constant $0 < \varepsilon_3 < 1$ such that

$$(1+\varepsilon_3)(1-q) > (1-\varepsilon_3)p, \quad (1+\varepsilon_3)(1-p) > (1-\varepsilon_3)q. \tag{4-7}$$

Hence,$AL = (-\varepsilon_3, -\varepsilon_3)^T$ yields

$$(1-p)l_1 - pl_2 = -\varepsilon_3, \quad -ql_1 + (1-q)l_2 = -\varepsilon_3. \tag{4-8}$$

Namely,

$$l_1 = -\frac{\varepsilon_3(1-q) + \varepsilon_3 p}{(1-p)(1-q) - pq}, \quad l_2 = -\frac{\varepsilon_3(1-p) + \varepsilon_3 q}{(1-p)(1-q) - pq}. \tag{4-9}$$

Moreover, $l_1, l_2 > 1$. Therefore, if we choose

$$k_1 = \min\{\frac{c_1}{l_1}, \frac{c_2}{l_2}\}, \quad k_2 = \max\{\frac{\lambda_1 l_1}{c_1}, \frac{\lambda_1 l_2}{c_2}\}, \quad r_1 = 1, \quad r_2 = \varepsilon_3. \tag{4-10}$$

Then, $k_1, k_2, r_2 > 0, r_1 \geq 1$. Thus, assume that $u_0(x), v_0(x)$ large enough to satisfy

$$\tilde{u}(x,0) = (\delta\phi_1)^{l_1} \leq u_0(x), \quad \tilde{v}(x,0) = (\delta\phi_1)^{l_2} \leq v_0(x). \tag{4-11}$$

Now, it follows from (4.1)-(4.11) that $(\tilde{u}, \tilde{v})$ is a positive sub-solution of (1.1), which blows up in finite time since $s(t)$ does.

(2) Next, we consider the case $p, q < 1$ and $pq = (1-p)(1-q)$. Clearly, there exist positive constants $l_1, l_2 > 1$ such that

$$p(l_1 + l_2) - l_1 = 0, \quad q(l_1 + l_2) - l_2 = 0. \tag{4-12}$$

Without loss of generality, we may assume that $0 \in \Omega$, and let $B_R(0)$ be a ball such that $B_R(0) \subset\subset \Omega$. In the following, we will prove that $(\tilde{u}, \tilde{v})$ blows up in finite time in the ball B_R. Because of so, $(\tilde{u}, \tilde{v})$ does blow up in the larger domain Ω.

Denote by $\lambda_{B_R} > 0$ and $\phi_R(r)$ the first eigenvalue and the corresponding eigenfunction of the following eigenvalue problem

$$-\phi''(r) - \frac{N-1}{r}\phi'(r) = \lambda\phi(r), \quad r \in (0, R); \quad \phi'(0) = 0, \quad \phi(R) = 0.$$

It is well known that $\phi_R(r)$ can be normalized as $\phi_R(r) > 0$ in B_R and $\phi_R(0) = \max_{B_R} \phi_R(r) = 1$. By the scaling property (let $\tau = r/R$) of eigenvalues and eigenfunctions we see that $\lambda_{B_R} = R^{-2}\lambda_{B_1}$ and $\phi_R(r) = \phi_1(r/R) = \phi_1(\tau)$, where λ_{B_1} and $\phi_1(\tau)$ are the first eigenvalue and the corresponding normalized eigenfunction of the eigenvalue problem in the unit ball $B_1(0)$. Moreover,

$$\max_{B_1} \phi_1(\tau) = \phi_1(0) = \phi_R(0) = \max_{B_R} \phi_R(r) = 1.$$

Now, we define the functions $\tilde{u}(x,t), \tilde{v}(x,t)$ as follows

$$\tilde{u}(x,t) = \big(s(t)\phi_R(|x|)\big)^{l_1}, \quad \tilde{v}(x,t) = \big(s(t)\phi_R(|x|)\big)^{l_2},$$

where $s(t)$ is confined as in (4.3). Then, a similar calculation as that of (4.4)-(4.6) yields

$$\Delta\tilde{u} + \|\tilde{u}\tilde{v}\|_\alpha^p \geq l_1 s^{l_1-1}\phi_R^{l_1}\frac{c_1}{l_1}s\big(1 - \frac{\lambda_{B_R} l_1}{c_1}\big),\ \tilde{u}_t = l_1 s^{l_1-1}\phi_R^{l_1}s'(t); \tag{4-13}$$

$$\Delta\tilde{v} + \|\tilde{u}\tilde{v}\|_\beta^q \geq l_2 s^{l_2-1}\phi_R^{l_2}\frac{c_2}{l_2}s\big(1 - \frac{\lambda_{B_R} l_2}{c_2}\big),\ \tilde{v}_t = l_2 s^{l_2-1}\phi_R^{l_2}s'(t); \tag{4-14}$$

where

$$c_1 = \|\phi_R^{l_1+l_2}\|_\alpha^p = \big(\int_\Omega \phi_R^{\alpha(l_1+l_2)}(|x|)\mathrm{d}x\big)^{\frac{p}{\alpha}} = R^{\frac{Np}{\alpha}}\big(\int_\Omega \phi_1^{\alpha(l_1+l_2)}(|y|)\mathrm{d}y\big)^{\frac{p}{\alpha}} \leq K_1 R^{\frac{Np}{\alpha}},$$
$$c_2 = \|\phi_R^{l_1+l_2}\|_\beta^q \leq K_2 R^{\frac{Nq}{\beta}}$$

and K_1, K_2 are constants independent of R. Then, in view of $\lambda_{B_R} = R^{-2}\lambda_{B_1}$, we may assume that R, that is , the ball $B_R(0)$, is sufficiently large that

$$\lambda_{B_R} < \min\{\frac{c_1}{l_1}, \frac{c_2}{l_2}\}.$$

Hence,

$$1 - \frac{\lambda_{B_R} l_1}{c_1} > 0, \quad 1 - \frac{\lambda_{B_R} l_2}{c_2} > 0. \tag{4-15}$$

We choose

$$k_1 = \min\{\frac{c_1}{l_1}, \frac{c_2}{l_2}\}, \quad k_2 = \max\{\frac{\lambda_{B_R} l_1}{c_1}, \frac{\lambda_{B_R} l_2}{c_2}\}, \quad r_1 = 1, \quad r_2 = 0. \tag{4-16}$$

Then, $k_1, k_2 > 0$. On the other hand, assume that $u_0(x), v_0(x)$ large enough to satisfy

$$\tilde{u}(x,0) = (\delta\phi_R)^{l_1} \leq u_0(x), \quad \tilde{v}(x,0) = (\delta\phi_R)^{l_2} \leq v_0(x). \tag{4-17}$$

It follows from (4.12)-(4.17) that $(\tilde{u}, \tilde{v})$ is a positive sub-solution of (1.1) in the ball $B_R(0)$, which blows up in finite time since $s(t)$ does.

Thus, according to the Lemma 2, the proof of Theorem 2 is completed. ∎

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