A Characterization of Dupin Hypersurfaces in $\mathbb{R}^4$

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Abstract

In this paper we study Dupin hypersurfaces in $\mathbb{R}^4$ parametrized by lines of curvature, with three distinct principal curvatures and $m_{jik} = 0$. We characterize locally a generic family of such hypersurfaces in terms of the principal curvatures and three vector valued functions of one variable, which are invariant under inversions and homotheties.

1 Introduction.

Dupin surfaces were first studied by Dupin in 1822 and more recently by many authors [1]-[3] and [6]-[15], which studied several aspects of Dupin hypersurfaces. The class of Dupin hypersurfaces is invariant under Lie transformations [8]. Therefore, the classification of Dupin hypersurfaces is considered up to these transformations. The local classification of Dupin surfaces in $\mathbb{R}^3$ is well known. Pinkall [9] gave a complete classification up to Lie equivalence for Dupin hypersurfaces $M^3 \subset \mathbb{R}^4$, with three distinct principal curvatures. Niebergall [7] and Cecil and Jensen [3] studied proper Dupin hypersurfaces with four distinct principal curvatures and constant Lie curvature.

Riveros [12] obtained a local characterization of the Dupin hypersurfaces in $\mathbb{R}^4$ parametrized by lines of curvature, with three distinct principal curvatures and $m_{jik} \neq 0$, in terms of the principal curvatures and three vector valued functions in $\mathbb{R}^4$ which are invariant under inversions and homotheties.

In this paper we consider Dupin hypersurfaces parametrized by lines of curvature and we ask if it is possible to obtain a similar result to that obtained in [12] with the condition $m_{jik} = 0$. The Theorem 3.1 gives an affirmative answer

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to this question, more precisely, we obtain a local characterization of a family of Dupin hypersurfaces parametrized by lines of curvature and $m_{jik} = 0$, in terms of the principal curvature functions and three vector valued functions of one variable. The characterization is based on the theory of higher-dimensional Laplace invariants introduced by Kamran-Tenenblat [4]-[5]. We consider generic hypersurfaces in the sense that suitable generic conditions on the Laplace invariants are required.

In section 2, we give some properties of hypersurfaces with distinct principal curvatures. Section 3, is devoted to showing our main results on Dupin hypersurfaces in $\mathbb{R}^4$ with three distinct principal curvatures. Moreover, we show that the vector valued functions, which appear in the characterization of Theorem 3.1 are invariant under inversions and homotheties and we conclude by applying the theory to a dupin hypersurface.

2 Preliminaries

Let Ω be an open subset of $\mathbb{R}^n$ and $x = (x_1, x_2, \cdots, x_n) \in \Omega$. Let $X : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$, $n \geq 3$, be a hypersurface parametrized by lines of curvature, with distinct principal curvatures λ_i, $1 \leq i \leq n$ and $N : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a unit normal vector field of X. Then

$$\begin{aligned} \langle X_{,i}, X_{,j} \rangle &= \delta_{ij} g_{ii} \,,\ 1 \leq i, j \leq n \,, \\ N_{,i} &= -\lambda_i X_{,i} \,, \end{aligned} \tag{1}$$

where the subscript $_{,i}$ denotes the derivative with respect to x_i.
Moreover,

$$X_{,ij} - \Gamma^i_{ij} X_{,i} - \Gamma^j_{ij} X_{,j} = 0 \,,\ \ 1 \leq i \neq j \leq n \,, \tag{2}$$

$$\Gamma^i_{ij} = \frac{\lambda_{i,j}}{\lambda_j - \lambda_i}, \quad 1 \leq i \neq j \leq n, \tag{3}$$

where Γ^k_{ij} are the Christoffel symbols.

We now consider the higher-dimensional Laplace invariants of the system of equations (2) (see [4]-[5] for the definition of these invariants),

$$\begin{aligned} m_{ij} &= -\Gamma^i_{ij,i} + \Gamma^i_{ij} \Gamma^j_{ij} \,, \\ m_{ijk} &= \Gamma^i_{ij} - \Gamma^k_{kj} \,,\ \ k \neq i, j \,,\ \ 1 \leq k \leq n. \end{aligned} \tag{4}$$

As a consequence of (3) and the Lemma obtained in [5], we obtain the following identities, valid for distinct i, j, k, l, $1 \leq i, j, k, l \leq n$:

$$\begin{aligned} m_{ijk} + m_{kji} &= 0, \\ m_{ijk,k} - m_{ijk} m_{jki} - m_{kj} &= 0, \\ m_{ij,k} + m_{ijk} m_{ik} + m_{ikj} m_{ij} &= 0, \\ m_{ijk} - m_{ijl} - m_{ljk} &= 0, \\ m_{lik,j} + m_{ijl} m_{kil} + m_{ljk} m_{kij} &= 0. \end{aligned} \tag{5}$$

Definition 2.1 An immersion $X : \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ is a parametrized *Dupin hypersurface* if each principal curvature is constant along its corresponding lines of curvature. If the multiplicities of the principal curvatures are constant then the Dupin hypersurface is said to be *proper*.

For Dupin hypersurfaces, the Laplace invariants m_{ij} are equal to zero and from Remark 2.2 in [14], it follows that for $n \geq 3$, the higher-dimensional Laplace invariants do not change under inversions in spheres centered at the origin and homotheties.

For Dupin hypersurfaces with distinct principal curvatures, the *Möbius curvature* is defined, for distinct i, j, k, by

$$C^{ijk} = \frac{\lambda_i - \lambda_j}{\lambda_k - \lambda_j}. \tag{6}$$

Since all λ_i are distinct we conclude that $C^{ijk} \neq 0$ and $C^{ijk} \neq 1$. Möbius curvatures are invariant under Möbius transformations.

The following result extends Lemma 2.3 in [14], which provides some properties which are satisfied by the principal curvatures of a hypersurface in $\mathbb{R}^{n+1}$ parametrized by lines of curvature.

Lemma 2.2 *Let* $\lambda_r : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, $n \geq 3$, *be smooth functions distinct at each point. Consider functions* m_{ijk} *defined by (3) and (4). Then for* i, j *fixed,* $1 \leq i \neq j \leq n$, *the following properties hold*

$$\left[C^{kji} m_{jki}\right]_{,i} = m_{jki,i} + \left[\frac{\lambda_{i,i}}{\lambda_j - \lambda_i}\right]_{,k}, \tag{7}$$

$$\left[C^{kji} m_{jki}\right]_{,j} = -\left[\frac{\lambda_{j,j}}{\lambda_j - \lambda_i}\right]_{,k}, \tag{8}$$

$$\left[C^{kji} m_{jki}\right]_{,l} = \left[C^{lji} m_{jli}\right]_{,k}, \tag{9}$$

where C^{kji} *is the Möbius curvature and* $1 \leq k \neq l \leq n$ *are distinct from* i *and* j.

Proof: The result it follows from (5) and the equations

$$\begin{aligned}
C^{kji}_{,i} &= \frac{C^{kji}}{\lambda_j - \lambda_i}\left[\lambda_{i,i} + (\lambda_k - \lambda_i) C^{ijk} m_{kij}\right], \\
C^{kji}_{,j} &= \frac{C^{kij}}{\lambda_j - \lambda_i}\left[\lambda_{j,j} + (\lambda_k - \lambda_j) C^{jik} m_{kji}\right], \\
C^{kji}_{,k} &= \frac{1}{\lambda_i - \lambda_j}\left[\lambda_{k,k} + (\lambda_k - \lambda_i) C^{kji} m_{jki}\right], \\
C^{kji}_{,l} &= \frac{\lambda_k - \lambda_l}{\lambda_i - \lambda_j}\left[m_{jlk} + C^{ilk} C^{kji} m_{ilj}\right].
\end{aligned}$$

■

Remark 2.3 Let $X : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^4$, be a Dupin hypersurface parametrized by lines of curvature, with three distinct principal curvatures λ_r, $1 \le r \le 3$. Using Lemma 2.4 in [14], for i, j, k fixed, $1 \le i \ne j \ne k \le 3$, the transformation

$$X = V\bar{X}\,, \quad \text{where} \quad V = \frac{e^{-\int C^{kji} m_{jki} dx_k}}{\lambda_j - \lambda_i}\,, \tag{10}$$

transforms system (2) into

$$\begin{array}{rcl} \bar{X}_{,ij} + A\bar{X}_{,j} & = & 0, \\ \bar{X}_{,ik} + (A + m_{jik})\bar{X}_{,k} & = & 0, \\ \bar{X}_{,jk} + m_{ikj}\bar{X}_{,j} + m_{ijk}\bar{X}_{,k} & = & 0, \end{array} \tag{11}$$

where

$$A_{,j} = 0\,,\; A_{,k} = -m_{jki,i}. \tag{12}$$

It follows from Lemma 2.2 that, the derivatives of the function V defined by (10) are given by,

$$\begin{array}{rcl} V_{,i} & = & \left(A + \Gamma^j_{ji}\right) V, \\ V_{,j} & = & \Gamma^i_{ij} V, \\ V_{,k} & = & \Gamma^i_{ik} V, \end{array} \tag{13}$$

where A is given by (12).

3 Main results

In this section, we prove our main result which provides a local characterization of generic Dupin hypersurfaces parametrized by lines of curvature in $\mathbb{R}^4$, with three distinct principal curvatures and Laplace invariant $m_{jik} = 0$.

Theorem 3.1 *Let $X : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^4$, be a Dupin hypersurface parametrized by lines of curvature, with three distinct principal curvatures λ_r. For i, j, k distinct fixed indices, suppose $m_{jik} = 0$ then*

$$X = V\bar{X}, \tag{14}$$

where

$$\bar{X} = e^{-\int A dx_i} \left\{ \int e^{\int A dx_i} G_i(x_i) dx_i + C^{ikj} \left[\int C^{jik} G_k(x_k) dx_k + G_j(x_j) \right] \right\}, \tag{15}$$

V is defined by (10), $G_r(x_r)$, $r = i, j, k$, are vector valued functions into $\mathbb{R}^4$ and $A_{,j} = 0$, $A_{,k} = 0$.
Moreover, considering

$$\beta^i = \left(A + \frac{\lambda_{j,i}}{\lambda_i - \lambda_j}\right)\bar{X} + \bar{X}_{,i}\,, \qquad \beta^s = \frac{\lambda_{i,s}}{\lambda_s - \lambda_i}\bar{X} + \bar{X}_{,s}\;,\; s \ne i, \tag{16}$$

the functions $G_r(x_r)$ satisfy the following properties in Ω, for $1 \le r \ne t \le 3$:

a) $\beta^r \neq 0$,

b) $\langle \beta^r, \beta^t \rangle = 0, \;\; r \neq t$,

c) $\lambda_r = \dfrac{\langle \beta^r_{,r}\,, \beta^i \times \beta^j \times \beta^k \rangle}{V|\beta^r|^2\, |\beta^i|\, |\beta^j|\, |\beta^k|}$.

Conversely, let $\lambda_r : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$, $r = 1, 2, 3$ be real functions, distinct at each point. Assume that the functions m_{rts} and m_{rt} defined by

$$\begin{aligned} m_{rts} &= \frac{\lambda_{r,t}}{\lambda_t - \lambda_r} - \frac{\lambda_{s,t}}{\lambda_t - \lambda_s}\;,\; 1 \leq r \neq t \neq s \leq 3, \\ m_{rt} &= -\left[\frac{\lambda_{r,t}}{\lambda_t - \lambda_r}\right]_{,r} - \frac{\lambda_{r,t}\lambda_{t,r}}{(\lambda_t - \lambda_r)^2}\;,\; 1 \leq r \neq t \leq 3, \end{aligned} \tag{17}$$

satisfy (5), and for i, j, k distinct fixed indices, $m_{jik} = 0$. Then for any vector valued functions $G_r(x_r)$ satisfying properties a) b) c), where β^r is defined by (16), the function $X : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^4$ given by (14) describes a Dupin hypersurface parametrized by lines of curvature whose principal curvatures are the functions λ_r.

Proof: From equation (2) we have,

$$X_{,sr} - \Gamma^s_{sr} X_{,s} - \Gamma^r_{sr} X_{,r} = 0\;,\;\; 1 \leq s \neq r \leq 3. \tag{18}$$

For fixed distinct indices $i,\; j,\; k$, we consider the transformation

$$X = V\bar{X}, \tag{19}$$

as in Remark 2.3, where V is given by (10). Then system (18) reduces to

$$\begin{gathered} \bar{X}_{,ij} + A\bar{X}_{,j} = 0, \\ \bar{X}_{,ik} + A\bar{X}_{,k} = 0, \\ \bar{X}_{,jk} + m_{ikj}\bar{X}_{,j} + m_{ijk}\bar{X}_{,k} = 0, \end{gathered} \tag{20}$$

where

$$A_{,j} = 0, \quad A_{,k} = 0. \tag{21}$$

We observe that the condition $m_{jik} = 0$ was used to obtain the second equation of (20). It follows from the first two equations of (20) that

$$\bar{X}_{,i} + A\bar{X} = G_i(x_i), \tag{22}$$

whose integration with respect x_i, provides

$$\bar{X} = e^{-\int A dx_i} \left[\int e^{\int A dx_i} G_i(x_i) dx_i + F(x_j, x_k)\right]. \tag{23}$$

The substitution of $\bar{X}_{ij}, \bar{X}_j$ and $\bar{X}_{ik}, \bar{X}_k$ in the first and second equations of (20), respectively, gives identities.

Using $\bar{X}_{jk}, \bar{X}_j$ and $\bar{X}_k$ in the third equation of (20), we obtain

$$F_{,kj} + m_{ikj}F_{,j} + m_{ijk}F_{,k} = 0. \tag{24}$$

Next we compute the Laplace invariant $\tilde{m}_{jk}$ of equation (24),

$$\tilde{m}_{jk} = m_{ikj,j} + m_{ikj} m_{ijk}.$$

Using third equation of (5), we obtain

$$\tilde{m}_{jk} = 0.$$

Therefore, the solution of equation (24) is given by

$$F(x_j, x_k) = e^{-\tilde{S}} \left[\int e^{\tilde{S}-S} G_k(x_k) dx_k + G_j(x_j) \right], \tag{25}$$

where

$$\tilde{S} = \int m_{ikj} dx_k \ , \ S = \int m_{ijk} dx_j.$$

By integration, we obtain

$$\tilde{S} = \log(C^{jki}) \ , \ S = \log(-C^{kji}). \tag{26}$$

Using (26) in (25) we obtain

$$F(x_j, x_k) = C^{ikj} \left[\int C^{jik} G_x(x_k) dx_k + G_j(x_j) \right]. \tag{27}$$

The substitution of (27) in (23) gives (15).

Considering β^i and $\beta^s, \ s = j, k$ defined by (16), it follows from (7), (8), (13) and (4) that

$$X_{,r} \ = \ V\beta^r \ , \ r = i, j, k. \tag{28}$$

Differentiating (28), we have

$$X_{,rr} = V_{,r}\beta^r + V\beta^r_{,r} \ , \ r = i, j, k. \tag{29}$$

It follows from (28) that the metric of $X_{,r}$ is given by

$$g_{rr} = (V)^2 |\beta^r|^2 \ \ , \ g_{rt} = 0 \ , \ r \neq t. \tag{30}$$

A unit vector field normal to X is given by

$$N = \frac{\beta^i \times \beta^j \times \beta^k}{|\beta^i| \ |\beta^j| \ |\beta^k|}. \tag{31}$$

Since X is a hypersurface parametrized by orthogonal curvature lines, with λ_s, as principal curvature we have, for $1 \leq r \neq s \leq 3$

$$\langle N, X_{,rs} \rangle = 0, \qquad \lambda_s = \frac{\langle X_{,rr}, N \rangle}{g_{rr}} \ .$$

Hence from (29) and (31) we obtain for $r = i, j, k$,

$$\lambda_r = \frac{\langle \beta^r_{,r} \ , \beta^i \times \beta^j \times \beta^k \rangle}{V |\beta^r|^2 |\beta^i| \ |\beta^j| \ |\beta^k|}.$$

Therefore, we conclude that conditions a), b) and c) are satisfied.

Conversely, let λ_r be real functions distinct at each point. Assume that the functions m_{rts} and m_{rt}, defined by (17), satisfy (5) and suppose $G_r(x_r)$, $1 \leq r \leq 3$, are vector valued functions satisfying properties a), b) and c). Defining X by (14), it follows from Lemma 2.2 and properties a) and b), that X is an immersion, whose coordinates curves are orthogonal. Moreover, the induced metric is given by (30) and a unit normal vector field by (31).

Differentiating (28) with respect to x_t, using Lemma 2.2, the expressions (5), (13) and (16) we obtain

$$X_{,rt} = V\left(\frac{\lambda_{r,t}}{\lambda_t - \lambda_r}\beta^r + \frac{\lambda_{t,r}}{\lambda_r - \lambda_t}\beta^t\right) \; , \; r \neq t.$$

From (31), it follows that $\langle X_{,rt} \, , N\rangle = 0$. Hence the second fundamental form is diagonal and therefore the coordinates curves are lines of curvature. Moreover, it follows from (29) - (31) and from property c) that for $r = i, j, k$,

$$\frac{\langle X_{,rr}, N\rangle}{g_{rr}} = \frac{\langle \beta^r_{,r} \, , \beta^i \times \beta^j \times \beta^k\rangle}{V|\beta^r|^2|\beta^i| \; |\beta^j| \; |\beta^k|} \;\; = \;\; \lambda_r,$$

which concludes the proof. ∎

Now we show that the vector valued functions which appear in Theorem 3.1 are invariant under inversions and homotheties.

Theorem 3.2 *Let* $X : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^4$ *be a Dupin hypersurface with three distinct principal curvatures* λ_r*, parametrized by lines of curvature as in the Theorem 3.1. Then the vector valued functions* $G_r(x_r), 1 \leq r \leq 3$ *are invariants under inversions and homotheties.*

Proof: a) Assuming without loss of generality that $0 \notin X(\Omega)$, we consider $\tilde{X} = I^4(X)$ a Dupin hypersurface parametrized by lines of curvature, obtained by composing X with the inversion defined by $I^4(X) = \dfrac{X}{\langle X, X\rangle}$ and whose distinct principal curvatures are given by

$$\tilde{\lambda}_r = \langle X, X\rangle\lambda_r + 2\langle X, N\rangle \; , \; r = i, j, k. \tag{32}$$

Applying the Theorem 3.1 to $\tilde{X}$, we have for i, j, k fixed distinct indices

$$\tilde{X} = \tilde{V}\bar{\tilde{X}},$$

where

$$\begin{aligned}
\bar{\tilde{X}} &= e^{-\int \tilde{A}dx_i}\left\{\int e^{\int \tilde{A}dx_i}\tilde{G}_i(x_i)dx_i + \tilde{C}^{ikj}\left[\int \tilde{C}^{jik}\tilde{G}_k(x_k)dx_k + \tilde{G}_j(x_j)\right]\right\}, \\
\tilde{V} &= \frac{e^{-\int \tilde{C}^{kji}\tilde{m}_{jki}dx_k}}{\tilde{\lambda}_j - \tilde{\lambda}_i} \; , \; \tilde{C}^{kji} = \frac{\tilde{\lambda}_k - \tilde{\lambda}_j}{\tilde{\lambda}_i - \tilde{\lambda}_j}, \\
\tilde{A} &= -\int \tilde{m}_{jki,i}dx_k \; ,
\end{aligned} \tag{33}$$

and $\tilde{G}_r(x_r),\ r = i, j, k$ are vector valued functions in $\mathbb{R}^4$.

Since $\tilde{X}$ and X have the same higher-dimensional Laplace invariants, we obtain

$$\tilde{A} = A. \tag{34}$$

Substituting (32) in (33), we have

$$\tilde{V} = \frac{V}{\langle X, X \rangle}. \tag{35}$$

On the other hand,

$$\tilde{X} = \frac{X}{\langle X, X \rangle}. \tag{36}$$

We will show that $\tilde{G}_r(x_r) = G_r(x_r),\ r = i, j, k$. It follows from (35) and (36) that

$$\bar{\tilde{X}} = \bar{X}. \tag{37}$$

Differentiating (37) with respect to x_i, we get

$$\tilde{G}_i(x_i) = G_i(x_i). \tag{38}$$

From (37) and (38), we have

$$\left[\int \tilde{C}^{jik}\tilde{G}_k(x_k)dx_k + \tilde{G}_j(x_j)\right] = \left[\int C^{jik}G_k(x_k)dx_k + G_j(x_j)\right] \tag{39}$$

Differentiating with respect to x_k, we get $\tilde{G}_k(x_k) = G_k(x_k)$, hence it follows that $\tilde{G}_j(x_j) = G_j(x_j)$, which concludes the proof of a).

b) Let $\hat{X} = aX$ be a homothety of X. Then $\hat{X}$ is a Dupin hypersurface parametrized by lines of curvature, with distinct principal curvatures given by

$$\hat{\lambda}_r = \frac{\lambda_r}{a},\ \ r = i, j, k. \tag{40}$$

Applying Theorem 3.1 to $\hat{X}$, we have for i, j, k distinct fixed indices

$$\hat{X} = \hat{V}\bar{\hat{X}}.$$

where

$$\begin{aligned}
\bar{\hat{X}} &= e^{-\int \hat{A}dx_i}\left\{\int e^{\int \hat{A}dx_i}\hat{G}_i(x_i)dx_i + \hat{C}^{ikj}\left[\int \hat{C}^{jik}\hat{G}_k(x_k)dx_k + \hat{G}_j(x_j)\right]\right\},\\
\hat{V} &= \frac{e^{-\int \hat{C}^{kji}\hat{m}_{jki}dx_k}}{\hat{\lambda}_j - \hat{\lambda}_i},\ \ \hat{C}^{kji} = \frac{\hat{\lambda}_k - \hat{\lambda}_j}{\hat{\lambda}_i - \hat{\lambda}_j},\\
\hat{A} &= -\int \hat{m}_{jki,i}dx_k\ ,
\end{aligned} \tag{41}$$

and $\hat{G}_r(x_r),\ r = i, j, k$ are vector valued functions in $\mathbb{R}^4$.

Since $\hat{X}$ and X have the same Laplace invariants. Therefore, it follows that

$$\hat{A} = A. \tag{42}$$

We will show that $\hat{G}_r(x_r) = G_r(x_r)$. Substituting (40) in (41), we have

$$\hat{V} = aV. \tag{43}$$

Since

$$X = V\bar{X} \ , \ \ \hat{X} = \hat{V}\bar{\hat{X}}. \tag{44}$$

Substituting (43) and (44) in $\bar{X} = aX$ we have,

$$\bar{\hat{X}} = \bar{X}.$$

The same argument of item a) proves that $\hat{G}r(x_r) = G_r(x_r)$, $\forall\ r$. ∎

Example Now we will give an example: We consider a Dupin hypersurface $X : \Omega \to \mathbb{R}^4$ given by

$$X(x_1, x_2, x_3) = ((a + r\cos x_3)\cos x_1, (a + r\cos x_3)\sin x_1, r\sin x_3, x_2)$$

where $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 \in (0, 2\pi),\ x_2 \in \mathbb{R},\ x_3 \in (0, \pi/2)\}$
The principal curvatures of X is given by

$$\lambda_1 = -\frac{\cos x_3}{a + r\cos x_3},\ \lambda_2 = 0,\ \lambda_3 = -\frac{1}{r}$$

and for $i = 1,\ j = 2,\ k = 3$; the Laplace invariant $m_{213} = 0$, therefore, using Theorem 3.1, we obtain

$$X = Ve^{-\int Adx_1}\left\{\int e^{\int Adx_1} G_1(x_1)dx_1 + C^{132}\left[\int C^{213} G_3(x_3)dx_3 + G_2(x_2)\right]\right\},$$

where

$$\begin{aligned} V &= a + r\cos x_3, \\ A &= 0. \end{aligned} \tag{45}$$

The vector valued functions are given by

$$\begin{aligned} G_1(x_1) &= (-\sin x_1, \cos x_1, 0, 0), \\ G_2(x_2) &= (0, 0, 0, \frac{x_2}{a}), \\ G_3(x_3) &= (0, 0, -1, 0). \end{aligned}$$

We observe that in this example, using (6) we can show that the Möbius curvature C^{ijk} is not constant.

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