Aspects of algebraic exponentiation

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Abstract

We analyse some aspects of the notion of *algebraic exponentiation* introduced by the second author [16] and satisfied by the category Gp of groups. We show how this notion provides a new approach to the categorical-algebraic question of the centralization. We explore, in the category Gp, the unusual universal properties and constructions determined by this notion, and we show how it is the origin of various properties of this category.

Introduction

In [16], the second author observed, by means of very straightforward Kan extension arguments, that, in the category Gp of groups, the change of base functor with respect to the fibration of points along any group homomorphism $h : X \to Y$:

$$h^* : Pt_Y Gp \to Pt_X Gp$$

has a right adjoint, revealing, for the category Gp, a property having a certain analogy with the property, for a category $\mathbb{E}$, of being *locally cartesian closed* [24], namely the property that the following change of base functor: $h^* : \mathbb{E}/Y \to \mathbb{E}/X$ has a right adjoint for any map h. On the other hand, he showed moreover that, in the algebraic context of unital categories $\mathbb{C}$, the condition that the change of base functor along the terminal map: $\tau_X^* : \mathbb{C} = Pt_1\mathbb{C} \to Pt_X\mathbb{C}$ has a right adjoint is related to the existence of some generalized notion of centralization.

Now, the property of local cartesian closedness is very powerful and well known to be shared, for instance, by any elementary topos. It is unnecessary

Received by the editors November 2011.
Communicated by F. Borceux.
2000 *Mathematics Subject Classification :* 18A40, 18C15, 18D15, 20A05, 13A99, 17B99.
Key words and phrases : Fibration of points, Beck-Chevalley condition, centralizer, unital, Mal'cev and protomodular categories, locally cartesian closed category.

to insist on its significance. We shall develop some aspects of this new concept of algebraic exponentiation.

In Section 1), we shall more deeply analyse the parallelism with cartesian closedness and we shall strictly elucidate the relationship with the classical notion of centralizer, in such a way that, *when a unital category $\mathbb{C}$ is regular, any change of base functor τ_X^*, as above, has a right adjoint if and only if any subobject has a centralizer*, revealing that, behind the notions of centre and centralizer, there was an unexpected wider-ranging phenomenon of functorial nature. In Section 2) we shall show that, in the Mal'cev context, algebraic exponentiation along split epimorphisms allows us to extend the existence of centralizers from subobjects to equivalence relations; accordingly, when the category $\mathbb{C}$ is moreover exact, we get a Schreier-Mac Lane extension theorem, according to [11]. In Section 3) we shall investigate the stability properties of algebraic exponentiation and in particular we shall show how, in the efficiently regular context, the existence of a right adjoint to:

$$h^* : Pt_Y\mathbb{C} \to Pt_X\mathbb{C}$$

can be extended from split epimorphims to regular epimorphisms. In Section 4), in the stricter context of protomodular categories, we give a detailed description of some constructions determined by the algebraic exponentiation of all morphisms, and in particular we shall investigate two main consequences, namely *strong protomodularity* (which guarantees, among other things, that the commutation of two equivalence relations (R, S) is characterized by the commutation of their associated normal subobjects (I_R, I_S) [9]) and *peri-abelianness* (which is strongly related to the cohomology of groups [10]). These last points show us how some well identified particular properties of the category Gp of groups originate from this algebraic exponentiation property. In this same category Gp, we shall explore in detail the very unusual universal properties and constructions involved in algebraic exponentiation.

On the other hand we shall enlarge the list of examples (Section 2.1) to some categories of topological models, such as topological groups and topological rings, and to non-pointed categories such as the fibres Grd_X above the set X of the fibration $(\ )_0 : Grd \to Set$ from groupoids to sets whose fibre Grd_1 above 1 is nothing but the category Gp itself.

1 Cartesian vs algebraic cartesian closedness

1.1 Slice categories and cartesian closedness

Let $\mathbb{E}$ be any finitely complete category and Y an object in $\mathbb{E}$. Any object $f : X \to Y$ in the slice category $\mathbb{E}/Y$ has a specific presentation as the domain of an equalizer of a split pair in $\mathbb{E}/Y$ which actually comes from a monad on the slice category $\mathbb{E}/Y$ (where the common splitting of the parallel pair below is the map $p_0 \times X$):

$$
\begin{tikzcd}
X \arrow[r, tail, "{(f,1)}"] \arrow[dr, "f"'] & Y \times X \arrow[r, shift left=2, "s_0 \times X"] \arrow[r, shift right=2, "Y \times (f,1)"'] \arrow[d, "p_Y"] & Y \times (Y \times X) \arrow[l] \arrow[dl, "p_Y"] \\
& Y &
\end{tikzcd}
$$

This presentation can be extended to the category $Pt_Y\mathbb{E}$ of points above Y, namely to the split epimorphisms, in the following way:

$$
\begin{tikzcd}
X \arrow[r, tail, "{(f,1)}"] \arrow[dr, shift right, "f"'] & Y \times X \arrow[r, shift left=2, "s_0 \times X"] \arrow[r, shift right=2, "Y \times (f,1)"'] \arrow[d, shift right, "p_Y"'] & Y \times (Y \times X) \arrow[l] \arrow[dl, shift right, "p_Y"'] \\
& Y \arrow[ul, shift right, "s"'] \arrow[u, shift right, "{(1,s)}"'] \arrow[ur, shift right, "{(s_0,s)}"'] &
\end{tikzcd}
$$

As a collateral consequence, when the category $\mathbb{E}$ is pointed, we get the kernel of f, from the previous diagram, by the following equalizer:

$$
\begin{tikzcd}
Ker f \arrow[r, tail, "k_f"] & X \arrow[r, shift left=2, "\iota_X"] \arrow[r, shift right=2, "{(f,1)}"'] & Y \times X \arrow[l]
\end{tikzcd}
$$

Now consider the change of base along the terminal map: $\tau_Y^* : \mathbb{E} \longrightarrow \mathbb{E}/Y$. According to our initial remark and because of the left exactness of right adjoints, the question of the existence of a right adjoint to τ_Y^* is reduced to the existence of cofree structures for the projections $p_Y : Y \times X \to Y$ in $\mathbb{E}/Y$; and this cofree structure is nothing but the exponential X^Y. In other words:

The functor $\tau_Y^* : \mathbb{E} \to \mathbb{E}/Y$ has a right adjoint if and only if the functor $Y \times - : \mathbb{E} \to \mathbb{E}$ has a right adjoint. The category $\mathbb{E}$ is cartesian closed if and only if any functor τ_Y^* has a right adjoint.

1.2 Fibration of points and algebraic cartesian closedness

In an algebraic context, no such exponential does exist in general, among other things because of the existence of a zero object in the main instances, and consequently no such right adjoint functor to τ_Y^*. However we have the possibility to consider the existence of a right adjoint to the "change of base" functor $\tau_Y^* : Pt_1\mathbb{E} \to Pt_Y\mathbb{E}$, i.e, here, "only" with respect to the points of $\mathbb{E}/Y$ and $\mathbb{E}$; and even more generally the existence of a right adjoint to the change of base functor $f^* : Pt_Y\mathbb{E} \to Pt_X\mathbb{E}$ for a any map $f : X \to Y$.

This idea was first introduced by the second author [16] who showed these right adjoints to f^* do exist in the categories *Gp* of groups and *R-Lie* of Lie R-algebras, for any commmutative ring R [17].

The previous observation above concerning the equalizer presentation of any split epimorphism applies now for the change of base functor: $\tau_Y^* : Pt_1\mathbb{E} \longrightarrow$

$Pt_Y\mathbb{E}$ along the terminal map. The question of the existence of a right adjoint to τ_Y^* is then reduced to the existence of cofree structures for the split epimorphisms $(p_Y, (1, u)) : Y \times X \to Y$ in $Pt_Y\mathbb{E}$ with a monomorphic $u : Y \rightarrowtail X$.

We shall work now more specifically in the algebraic context of a *unital* [4], or even *weakly unital* [22] category $\mathbb{C}$. Recall:

Definition 1.1. *A category $\mathbb{C}$ is unital (resp. weakly unital) when it is pointed, is finitety complete, and is such that any pair of maps of the following form:*

$$\begin{tikzcd} X \arrow[r, tail, "{(1,0)}"] & X \times Y & Y \arrow[l, tail, "{(0,1)}"'] \end{tikzcd}$$

is jointly strongly epic (resp. jointly epic).

Accordingly a finitely complete pointed category $\mathbb{C}$ is unital if and only if the supremum of the two previous subobjects is $1_{X\times Y}$. In these contexts, the functor τ_Y^* becomes fully faithful. Recall also that there is then an intrinsic notion of commutation for any pair of maps with same codomain. We say that a pair (f, g) commutes:

$$\begin{tikzcd} X \arrow[r, tail, "{(1,0)}"] \arrow[dr, "f"'] & X \times Y \arrow[d, dotted, "\phi"] & Y \arrow[l, tail, "{(0,1)}"'] \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

when there exists a factorization ϕ (called the *cooperator* of this pair), the uniqueness of ϕ making this existence a property of the pair (f, g), see also [18]. In these algebraic contexts, the meaning of the existence of a right adjoint to the functor τ_Y^* above can be made much more algebraically civilized:

Proposition 1.2. *Suppose $\mathbb{C}$ is a unital (resp. weakly unital) category. The change of base functor $\tau_Y^* : Pt_1\mathbb{E} \longrightarrow Pt_Y\mathbb{E}$ admits a right adjoint Φ_Y if and only if any subobject $u : Y \rightarrowtail X$ with domain Y admits a universal map $\zeta_u : Z[u] \to X$ commuting with it. This universal map ζ_u is necessarily a monomorphism.*

Proof. The universal property of ζ_u translates exactly the universal property of the cofree structure of the split epimorphism $(p_Y, (1, u)) : Y \times X \rightleftarrows Y$ with respect to τ_Y^*. Indeed, the natural transformation $\varepsilon : \tau_Y^*.\Phi_Y \Rightarrow Id$ is produced by a map in $Pt_Y\mathbb{C}$:

$$\begin{tikzcd} Y \times Z[u] \arrow[rr, "{(p_Y,\phi)}"] \arrow[dr, shift left, "p_Y"'] & & Y \times X \arrow[dl, shift left, "p_Y"'] \\ & Y \arrow[ul, shift left, "\iota_Y"'] \arrow[ur, shift left, "{(1,u)}"'] & \end{tikzcd}$$

which makes $\phi : Y \times Z[u] \to X$ the *cooperator* of the commuting pair (u, ζ_u), with $\zeta_u = \phi.\iota_{Z[u]}$. Consider now the kernel equivalence of the map ζ_u:

$$\begin{tikzcd} R[\zeta_u] \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & Z[u] \arrow[r, "\zeta_u"] & X \end{tikzcd}$$

The map $\zeta_u.p_i$ commutes with u by means of the cooperator $\phi.(Y \times p_i)$. Its factorization through ζ_u being unique, we get $p_0 = p_1$; and ζ_u is a monomorphism. ∎

Now, starting from any split epimorphism (f,s), the cofree structure $\Phi_Y[f,s]$ is the equalizer of the following upper parallel pair:

$$\begin{tikzcd}
\Phi_Y[f,s] \arrow[r, tail] \arrow[d] & Z[s] \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\zeta_s"'] & Z[(1,s)] \arrow[d, "\zeta_{(1,s)}"] \\
Kerf \arrow[r, tail, "k_f"'] & X \arrow[r, shift left, "\iota_X"] \arrow[r, shift right, "(f,1)"'] & Y \times X
\end{tikzcd}$$

Since the lower line is also an equalizer and the maps $(\zeta_s, \zeta_{(1,s)})$ are monomorphisms, the left-hand side square is a pullback and $\Phi_Y[f,s] = Kerf \cap Z[s]$. According to the previous proposition and the parallelism with cartesian closedness [1], we shall introduce the following:

Definition 1.3. *A category $\mathbb{E}$ with finite products is said to be algebraically cartesian closed (a.c.c.) when any functor $\tau_Y^* : Pt_1\mathbb{E} \longrightarrow Pt_Y\mathbb{E}$ has a right adjoint.*

On the other hand, we have the quite classical:

Definition 1.4. *Suppose $\mathbb{C}$ is a unital (resp. weakly unital) category. The centralizer of a subobject $u : Y \rightarrowtail X$ is the largest subobject commuting with it, i.e. the universal monomorphism commuting with u.*

So, *when $\mathbb{C}$ is a unital category which is algebraically cartesian closed, any subobject u has a centralizer ζ_u*. When $\mathbb{C}$ is regular in the sense of [1] (as it is the case for any variety of universal algebras) the converse is true:

Proposition 1.5. *Suppose $\mathbb{C}$ is a regular unital category. Then it is algebraically cartesian closed if and only if any subobject $u : Y \rightarrowtail X$ has a centralizer.*

Proof. Let $h : T \to X$ be any map commuting with u. Consider the canonical decomposition of h through a regular epimorphism $T \overset{\rho}{\twoheadrightarrow} V \overset{\bar{h}}{\rightarrowtail} X$. Then, since ρ is a regular epimorphism the pair $(u, \bar{h})$ of subobjects does commute in $\mathbb{C}$; so $\bar{h}$, and thus h, factorizes through the centralizer $\zeta_u : Z[u] \rightarrowtail X$ of u which therefore becomes also the universal map commuting with u. ■

1.3 Examples

The unital category *Mon* of monoids is unital and, as a variety of algebras, is exact and therefore regular. It has centralizers and thus is algebraically cartesian closed. More generally any unital variety of algebras with centralizers (as the categories *Gp* of groups or *CRg* of commutative rings) is algebraically cartesian closed. In the category *Gp* of groups any split epimorphism (f,s) above Y is of the kind $Y \ltimes_\psi K \rightleftarrows Y$, where ψ is the associated action of the group Y on the group K. Then $\Phi_Y[f,s]$ is nothing but the subgroup $\{k \in K / \forall y \in Y,\ {}^y k = k\}$ of K of the invariant elements under the action ψ.
The main consequence of the algebraic cartesian closedness is a commutation of limits: the functor τ_Y^*, having a right adjoint, preserves the colimits existing in $\mathbb{C}$.

1.4 Reflection of commuting pairs

There is a very simple result which will be of consequence later on:

Proposition 1.6. *Let $\mathbb{C}$ be a unital (resp. weakly unital) category and (Γ, ϵ, ν) a left exact comonad on it. Then the category $Coalg\Gamma$ of Γ-coalgebras is unital (resp. weakly unital) and the left exact forgetful functor $U : Coalg\Gamma \to \mathbb{C}$ reflects the commuting pairs.*

Proof. Since the comonad (Γ, ϵ, ν) is left exact and $\mathbb{C}$ is finitely complete, so is the category $Coalg\Gamma$, which is moreover pointed since so is $\mathbb{C}$. The forgetful functor $U : Coalg\Gamma \to \mathbb{C}$ is left exact. The category $Coalg\Gamma$ is unital (resp. weakly unital) since the functor U is conservative (resp. faithful). Now, suppose that we have a pair of morphisms in $Coalg\Gamma$:

$$(X, \xi) \xrightarrow{f} (Z, \zeta) \xleftarrow{g} (Y, v)$$

whose image by U is endowed with a cooperator ϕ. We have to show that this map ϕ is actually a map of coalgebras, namely that the following quadrangle commutes:

$$\begin{tikzcd}
X \arrow[r, tail, dotted, "{(1,0)}"] & X \times Y \arrow[d, "\phi"'] \arrow[dr, "\Gamma\xi \times \Gamma v"] & Y \arrow[l, tail, dotted, "{(0,1)}"'] \\
& Z \arrow[dr, "\zeta"'] & \Gamma X \times \Gamma Y \arrow[d, "\Gamma\phi"] \\
& & \Gamma Z
\end{tikzcd}$$

which can be done by composition with the two upper horizontal maps. ∎

1.5 Strongly unital categories

A unital category $\mathbb{C}$ is strongly unital [8], when in addition, for any object Y, the change of base functor $\tau_Y^* : \mathbb{C} \to Pt_Y\mathbb{C}$ is *saturated on subobjects*, namely such that any subobject $R \rightarrowtail \tau_Y^*(Z)$ is, up to isomorphism, the image by τ_Y^* of some subobject $S \rightarrowtail Z$. In this context, the idempotent comonad associated with the algebraic cartesian closedness has a specific property:

Proposition 1.7. *Suppose $\mathbb{C}$ is a strongly unital category. Suppose the functor $\tau_Y^* : Pt_1\mathbb{C} \longrightarrow Pt_Y\mathbb{C}$ admits a right adjoint Φ_Y. Then the natural transformation of the induced idempotent left exact comonad $\varepsilon_Y : \tau_Y^*.\Phi_Y \Rightarrow Id$ is monomorphic. Moreover any subobject $j : \bullet \rightarrowtail \bullet$ in $Pt_Y\mathbb{C}$ produces a pullback in $Pt_Y\mathbb{C}$:*

$$\begin{tikzcd}
\tau_Y^*.\Phi_Y(\bullet) \arrow[r, tail, "\tau_Y^*.\Phi_Y(j)"] \arrow[d, "\varepsilon_Y\bullet"'] & \tau_Y^*.\Phi_Y(\bullet) \arrow[d, "\varepsilon_Y\bullet"] \\
\bullet \arrow[r, tail, "j"'] & \bullet
\end{tikzcd}$$

Proof. The comonad is idempotent even if $\mathbb{C}$ is only unital since the functor τ_Y^* is fully faithful. From the construction of $\Phi_Y[f,s]$, it is sufficient to prove the assertion for $Z[u] = \Phi_Y[p_Y,(1,u)]$. We observed that the natural map $\varepsilon_Y(p_Y,(1,u))$ is nothing but the map $Y \times Z[u] \overset{(p_Y,\phi)}{\rightarrow} Y \times X$, where the map ϕ is the cooperator of u and ζ_u. Now consider the following diagram in $\mathbb{C}$:

$$\begin{tikzcd}
Y \arrow[r, "\iota_Y"] \arrow[dr, shift right, "{(1,u)}"'] & Y \times Z[u] \arrow[d, "{(p_Y,\phi)}"] & Z[u] \arrow[l, "\iota_{Z[u]}"'] \arrow[dl, "{(0,\zeta_u)}"] \\
& Y \times X \arrow[ul, shift right, "p_Y"'] &
\end{tikzcd}$$

Then, according to Lemma 1.8.18 in [4] the vertical central map is a monomorphism if and only if so is the map $(0,\zeta_u)$, which is the case here. The end of the proof is a consequence of the following very general lemma: ∎

Lemma 1.8. *Let $U : \mathbb{E} \to \mathbb{F}$ be a left exact fully faithful functor between finitely complete categories. Suppose moreover that U has a right adjoint G such that the natural transformation of the induced idempotent left exact comonad $\varepsilon : U.G \Rightarrow Id$ is monomorphic. Then the functor U is saturated on subobjects if and only if, given any subobject $j : X' \rightarrowtail X$ in $\mathbb{F}$, the following square is a pullback in $\mathbb{F}$:*

$$\begin{tikzcd}
U.G(X') \arrow[r, tail, "U.G(j)"] \arrow[d, "\varepsilon_{X'}"'] & U.G(X) \arrow[d, "\varepsilon_X"] \\
X' \arrow[r, tail, "j"'] & X
\end{tikzcd}$$

Proof. Suppose the previous condition is satisfied. Given any subobject j, when ε_X is an isomorphism, so is $\varepsilon_{X'}$, and U is saturated on subobjects. Conversely suppose U saturated on subobjects, and consider the pullback of ε_X along j and the induced monomorphic factorization η:

$$\begin{tikzcd}
U.G(X') \arrow[dr, "\eta"] \arrow[drr, tail, bend left, "U.G(j)"] \arrow[ddr, bend right, "\varepsilon_{X'}"'] & & \\
& P \arrow[r, tail, "\bar{j}"'] \arrow[d, "\epsilon"'] & U.G(X) \arrow[d, "\varepsilon_X"] \\
& X' \arrow[r, tail, "j"'] & X
\end{tikzcd}$$

U being saturated on subobjects, we can choose an object $P = U(T)$. Accordingly there is a map $\bar{\eta} : U(T) = P \to U.G(X')$ with $\varepsilon_{X'}.\bar{\eta} = \epsilon$. Since $\varepsilon_{X'}$ and ϵ are monomorphisms, this $\bar{\eta}$ is η^{-1}; and the square in question is a pullback. ∎

2 Mal'cev context

We shall work now in the algebraic context of Mal'cev categories in the sense of [13] and [14]. One way of saying that a category $\mathbb{C}$ is a Mal'cev category is to say that any pointed fibre $Pt_Y\mathbb{C}$ is unital or, equivalently that any pointed fibre $Pt_Y\mathbb{C}$ is strongly unital, see [8]. Let us introduce the following:

Definition 2.1. *A finitely complete category $\mathbb{C}$ is said to be fiberwise algebraically cartesian closed (f.a.c.c.) when every fibre $Pt_Y\mathbb{C}$ is algebraically cartesian closed. A morphism $h : X \to Y$ in $\mathbb{C}$ is said to be algebraically exponentiable when the change of base functor $h^* : Pt_Y\mathbb{C} \to Pt_X\mathbb{C}$ along h admits a right adjoint. The category $\mathbb{C}$ is said to be locally algebraically cartesian closed (l.a.c.c) when any morphism h is algebraically exponentiable.*

So, a category $\mathbb{C}$ is fiberwise algebraically cartesian closed if and only if, given any split epimorphism $(f, s) : X \rightleftarrows Y$ the change of base functor $f^* : Pt_Y\mathbb{C} \to Pt_X\mathbb{C}$ has a right adjoint Φ_f. When the category $\mathbb{C}$ is a regular Mal'cev category, it is equivalent, according to Proposition 1.5, to saying that *in any fibre $Pt_Y\mathbb{C}$ there exist centralizers of subobjects*. We get immediately:

Proposition 2.2. *Let $\mathbb{C}$ be a fiberwise algebraically cartesian closed Mal'cev category. Let $(f, s) : X \rightleftarrows Y$ be any split epimorphism in $\mathbb{C}$. Then the change of base functor $f^* : Pt_Y\mathbb{C} \to Pt_X\mathbb{C}$ is such that the natural transformation $\varepsilon_f : f^*.\Phi_f \Rightarrow Id$ of the induced idempotent left exact comonad is monomorphic. Moreover any subobject $j : \bullet \rightarrowtail \bullet$ in $Pt_X\mathbb{C}$ produces a pullback in $Pt_X\mathbb{C}$:*

$$\begin{tikzcd}
f^*.\Phi_f(\bullet) \arrow[r, tail, "f^*.\Phi_f(j)"] \arrow[d, "\varepsilon_f\bullet"'] & f^*.\Phi_f(\bullet) \arrow[d, "\varepsilon_f\bullet"] \\
\bullet \arrow[r, tail, "j"'] & \bullet
\end{tikzcd}$$

Proof. As we recalled above, the category $\mathbb{C}$ being Mal'cev, any fibre $Pt_Y\mathbb{C}$ is not only unital but also strongly unital. Accordingly, just apply Proposition 1.7. ∎

2.1 Examples

In [16] and [17], it was shown that: the category CRg of commutative rings is fiberwise algebraically cartesian closed but not locally algebraically cartesian closed; the categories Gp of groups and R-Lie of Lie R-algebras, for any commmutative ring R, are locally algebraically cartesian closed; when a category $\mathbb{E}$ is a cartesian closed category with pullbacks, the category $Gp\mathbb{E}$ of internal groups in $\mathbb{E}$ is locally algebraically cartesian closed. A category $\mathbb{A}$ was defined as essentially affine in [7] when any change of base functor $h^* : Pt_Y\mathbb{A} \to Pt_X\mathbb{A}$ is an equivalence of categories; accordingly any essentially affine category is locally algebraically cartesian closed. In particular any additive category is locally algebraically cartesian closed.

Non-pointed examples 1: slice and coslice categories

Lemma 2.3. *Let $U : \mathbb{C} \to \mathbb{D}$ be a pullback preserving functor which is moreover a discrete fibration (resp. discrete cofibration). When $U(f) : U(X) \to U(Y)$ is algebraically exponentiable in $\mathbb{D}$, so is $f : X \to Y$ in $\mathbb{C}$.*

Proof. Straightforward. ■

Let $\mathbb{C}$ be any category. Then, for any object Y in $\mathbb{C}$, the domain functor $\mathbb{C}/Y \to \mathbb{C}$ is a discrete fibration which preserves pullbacks. Still, for any object Y in $\mathbb{C}$, the codomain functor $Y/\mathbb{C} \to \mathbb{C}$ is a left exact discrete cofibration. Accordingly fiberwise algebraic cartesian closedness (resp. locally algebraic cartesian closedness) is stable under slicing and coslicing, giving rise to non-pointed examples. As a consequence, when $\mathbb{C}$ is fiberwise algebraically cartesian closed (resp. locally algebraically cartesian closed), so is any fibre $Pt_Y\mathbb{C}$.

Non-pointed examples 2: the fibres of the fibration $Grd \to Set$

Let us denote by Grd the category of groupoids and by $(\ )_0 : Grd \to Set$ the forgetful functor associating with any groupoid $\underline{Y}_1$ the set Y_0 of its objects; it is a fibration whose cartesian maps in Grd are the fully faithful functors. The fibre above 1 is clearly the pointed category Gp of groups. We shall denote by Grd_X the fibre above the set X: its objects are the groupoids whose set of objects is X and its arrows are those functors between such groupoids which are bijective on objects. We know that these fibres Grd_X are protomodular [7] and thus Mal'cev categories, and they are no longer pointed. The aim of this section is to show that any fibre Grd_X is locally algebraically cartesian closed; the proof will be a slight generalization of the proof for Gp.

Lemma 2.4. *Let be given a groupoid $\underline{Y}_1$. The fibre $Pt_{\underline{Y}_1}(Grd_{Y_0})$ is in bijection with the functor category $\mathcal{F}(\underline{Y}_1, Gp)$. Suppose $\underline{F}_1 : \underline{Y}_1 \to \underline{Z}_1$ be any functor. Then the change of base functor $\underline{F}_1^* : Pt_{\underline{Z}_1}(Grd_{Z_0}) \to Pt_{\underline{Y}_1}(Grd_{Y_0})$ is naturally isomorphic to the functor $\mathcal{F}(\underline{F}_1, Gp) : \mathcal{F}(\underline{Z}_1, Gp) \to \mathcal{F}(\underline{Y}_1, Gp)$.*

Proof. The category Gp can be considered as the full subcategory of the category Cat (of categories) whose objects are the groupoids with only one object. The lemma is a specification of the Grothendieck construction. From any functor $H : \underline{Y}_1 \to Gp$ we get a bijective on objects split cofibration $\underline{H}_1 : \underline{X}_1 \to \underline{Y}_1$ where a map $y \to y'$ in $\underline{X}_1$ is a pair (f, γ) with $f : y \to y'$ is a map in $\underline{Y}_1$ and $\gamma \in H(y')$. The composition is defined by: $(f', \gamma').(f, \gamma) = (f'.f, \gamma'.H(f')(\gamma))$. The functor $\underline{H}_1$, defined by $\underline{H}_1(f, \gamma) = f$, has a splitting $\underline{T}_1$ defined by $\underline{T}_1(f) = (f, 1_{H(y')})$. Conversely any split bijective on objects functor $\underline{H}_1 : \underline{X}_1 \to \underline{Y}_1$ is necessarily a split cofibration and determines a functor $H : \underline{Y}_1 \to Gp$. The end of the proof is straightforward. ■

Theorem 2.5. *Consider the fibration $(\ )_0 : Grd \to Set$; any of its fibres Grd_X is locally algebraically cartesian closed.*

Proof. Given any functor $\underline{F}_1 : \underline{Y}_1 \to \underline{Z}_1$ between two groupoids, the functor $\mathcal{F}(\underline{F}_1, Gp) : \mathcal{F}(\underline{Z}_1, Gp) \to \mathcal{F}(\underline{Y}_1, Gp)$ admits a right adjoint, given by the right Kan extension along the functor $\underline{F}_1$. Then, according to the previous lemma, any change of base functor $\underline{F}_1^* : Pt_{\underline{Z}_1}(Grd_{Z_0}) \to Pt_{\underline{Y}_1}(Grd_{Y_0})$ has a right adjoint. The theorem holds by taking $\underline{F}_1$ a bijective on objects functor with $Z_0 = X = Y_0$. ∎

Topological models

In this section we shall make explicit some topological examples. Let $\mathbb{T}$ be a Mal'cev theory, $\mathbb{V}(\mathbb{T})$ the corresponding variety of $\mathbb{T}$-algebras and $Top(\mathbb{T})$ the category of topological $\mathbb{T}$-algebras. Recall that $Top(\mathbb{T})$ is then a regular Mal'cev category, see [19], whose regular epimorphisms are the open surjective maps. Moreover, for every surjective homomorphism $g : X \twoheadrightarrow Y$ in $\mathbb{V}(\mathbb{T})$ with X in $Top(\mathbb{T})$, endowing Y with the quotient topology makes Y an object in $Top(\mathbb{T})$ and makes g an open map. Recall also the following:

Lemma 2.6. *Let $\mathbb{T}$ be a Mal'cev theory. Then the forgetful left exact functor $U : Top(\mathbb{T}) \to \mathbb{V}(\mathbb{T})$ reflects the pullback of split epimorphisms along regular epimorphisms.*

Proposition 2.7. *Let $\mathbb{T}$ be a Mal'cev theory such that $\mathbb{V}(\mathbb{T})$ is fiberwise algebraically cartesian closed. Then the category $Top(\mathbb{T})$ of topological $\mathbb{T}$-algebras is fiberwise algebraically cartesian closed.*

Proof. Let $(f, s) : X \rightleftarrows Y$ be a split epimorphism in $Top(\mathbb{T})$ and $(g, t) : V \rightleftarrows X$ an object in $Pt_X Top(\mathbb{T})$. First consider the following diagram given by the algebraic exponentiation in $\mathbb{V}(\mathbb{T})$:

$$\begin{tikzcd}
 & U(f)^*\Phi_{U(f)}[U(g),U(t)] \arrow[dl, "\varepsilon_{d_1}(U(g),U(t))"'] \arrow[ddl, shift right] \arrow[r, shift right, ->>, "\phi"'] & \Phi_{U(f)}[U(g),U(t)] \arrow[l, shift right, tail, "\sigma"'] \arrow[ddl, shift right, "\gamma"'] \\
U(V) \arrow[d, shift left, "U(g)"] & & \\
U(X) \arrow[u, shift left, "U(t)"] \arrow[uur, shift right, "\beta"'] \arrow[r, shift left, ->>, "U(f)"] & U(Y) \arrow[l, shift left, tail, "U(s)"] \arrow[uur, shift right, "\tau"'] &
\end{tikzcd}$$

and then the following one in $Top(\mathbb{T})$ where V' is the algebra $U(f)^*\Phi_{U(f)}[U(g), U(t)]$ equipped with the topology induced by the one on V:

$$\begin{tikzcd}
 & V' \arrow[dl, "\bar{\varepsilon}"'] \arrow[ddl, shift right] \arrow[r, dotted, shift right, ->>, "\bar{\phi}"'] & W \arrow[l, dotted, shift right, tail, "\bar{\sigma}"'] \arrow[ddl, dotted, shift right, "\bar{\gamma}"'] \\
V \arrow[d, shift left, "g"] & & \\
X \arrow[u, shift left, "t"] \arrow[uur, dotted, shift right, "\bar{\beta}"'] \arrow[r, shift left, ->>, "f"] & Y \arrow[l, shift left, tail, "s"] \arrow[uur, dotted, shift right, "\bar{\tau}"'] &
\end{tikzcd}$$

The map t in $Top(\mathbb{T})$ and the factorization β in $\mathbb{V}(\mathbb{T})$ produce the factorization $\bar{\beta}$. Then put the quotient topology on $\Phi_{U(f)}[U(g), U(t)]$ to produce the object W in $Top(\mathbb{T})$. Then we get the dotted maps above the quadrangle pullback of our initial diagram. The previous lemma asserts that it is a pullback. From this situation, it is straightforward to check that the split epimorphism $(\bar{\gamma}, \bar{\tau})$ has the desired universal property with respect to the change of base functor f^*. ∎

Accordingly the categories *TopGp* and *TopCRg* of topological groups and topological commutative rings are fiberwise algebraically cartesian closed.

2.2 Abelian split extension

A split epimorphism $(f,s) : X \rightleftarrows Y$ is said to be abelian in a Mal'cev category $\mathbb{C}$ when it is an abelian object in the fibre $Pt_Y\mathbb{C}$. Since any right adjoint functor is left exact, any algebraically exponentiable map $h : Y \to Y'$ is such that the restriction of $\Phi_h : Pt_Y\mathbb{C} \to Pt_{Y'}\mathbb{C}$ to the abelian objects determines a functor:

$$\Phi_h : AbPt_Y\mathbb{C} \to AbPt_{Y'}\mathbb{C}$$

In particular, when $\mathbb{C}$ is pointed and fiberwise algebraically cartesian closed, when (f,s) is abelian, so is the object $\Phi_Y[f,s]$. Recall that, when $\mathbb{C}$ is the category Gp of groups, a split epimorphism is abelian if and only if it has an abelian kernel A:

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, tail] & Y \ltimes_\psi A \arrow[r, two heads, shift left, "\pi"] & Y \arrow[l, shift left, tail, "\sigma"] \arrow[r] & 1 \end{tikzcd}$$

The subgroup $\Phi_Y[\pi,\sigma]$ of the invariant elements of A under the action ψ was denoted A^Y in [20] and shown to be the 0-dimensional cohomology group $H^0_\psi(Y,A)$. This was used in [15] to introduce in the Mal'cev context a notion of internal cohomology.

2.3 Centralizer of equivalence relations

In the Mal'cev context, there exists also an intrinsic notion of commutation at the level of equivalence relations, see [12] and also [25] and [23]. First, the subobjects of the object $(p_0,s_0) : X \times X \rightleftarrows X$ in the fibre $Pt_X\mathbb{C}$ coincide exactly with the reflexive relations on X, hence, in the Mal'cev context, with the equivalence relations on X. Recall that two equivalence relations R and S on an object X commute in $\mathbb{C}$ if and only if the two following subobjects in the fibre $Pt_X\mathbb{C}$ do commute in $Pt_X\mathbb{C}$, see Proposition 2.6.12 in [4]:

$$\begin{tikzcd} & & S \arrow[dl, tail, "(d_0,d_1)"'] \arrow[dl, shift right=3, bend left=0, "d_0" description] \\ R \arrow[r, tail, "(d_1,d_0)"] \arrow[dr, shift right, "d_1"'] & X \times X \arrow[d, shift right, "p_0"'] & \\ & X \arrow[ul, shift right, "s_0"'] \arrow[u, shift right, "s_0"'] \arrow[uur, shift right, "s_0"'] & \end{tikzcd}$$

the choice of this presentation (R^{op} rather than R) being made for technical reasons related to the classical presentation of the axioms of a Mal'cev operation. So, in the fiberwise algebraically cartesian closed context, the existence of centralizers can be immediately transfered to the level of equivalence relations.

Proposition 2.8. *Suppose $\mathbb{C}$ is a Mal'cev category which is fiberwise algebraically cartesian closed. Let R be any equivalence relation on the object X. Then the centralizer $Z(R)$ of the equivalence relation R does exist in $\mathbb{C}$.*

Proof. Since the category $\mathbb{C}$ is fiberwise algebraically cartesian closed, the unital fibre $Pt_X\mathbb{C}$ has centralizers of subobjects, and according to the previous recall about equivalence relations and their commutations, the centralizer of R in $\mathbb{C}$ is nothing but the centralizer of the subobject $R \rightarrowtail X \times X$ in the fibre $Pt_X\mathbb{C}$ (see the diagram above). According to Proposition 1.2, it is given by $\Phi_{d_1}[p_R, (1, d_0)]$. ∎

So, when the category $\mathbb{C}$ is exact, Mal'cev and fiberwise algebraically cartesian closed, the existence of centralizers makes the Schreier- Mac Lane extensions classification theorem hold, see [11].

3 Algebraic exponentiable morphisms

3.1 Stability under pullback along split epimorphisms

We show that the algebraically exponentiable morphisms are stable under pullback along split epimorphisms. It is a consequence of the following very general lemma:

Lemma 3.1. *Let $\mathbb{E}$ be a category with pullbacks and $U : \mathbb{E} \to \mathbb{F}$ a functor which admits a right adjoint G. Then, for any object $X \in \mathbb{E}$, the induced functor:*

$$U_X : Pt_X\mathbb{E} \longrightarrow Pt_{UX}\mathbb{F}$$

has a right adjoint G_X. When moreover the category $\mathbb{F}$ has pullbacks, any map $f : X \to X'$ makes the following leftward diagram commute up to a natural isomorphism:

$$\begin{tikzcd} Pt_{X'}\mathbb{E} \arrow[d, "f^*"'] \arrow[r, shift right, dotted, "U_{X'}"'] & Pt_{UX'}\mathbb{F} \arrow[l, shift right, "G_{X'}"'] \arrow[d, "Uf^*"] \\ Pt_X\mathbb{E} \arrow[r, shift left, dotted, "U_X"] & Pt_{UX}\mathbb{F} \arrow[l, shift left, "G_X"] \end{tikzcd}$$

When, in addition, U is left exact the previous diagram also commutes at the level of dotted arrows.

Proof. Let $(\tau, \sigma) : T \rightleftarrows UX$ be an object of $Pt_{UX}\mathbb{F}$. It is straightforward to check that $G_X(\tau, \sigma)$ is given by the following pullback in $\mathbb{E}$, where η_X is the unit of the adjunction:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, shift right] & GT \arrow[d, shift right, "G\tau"'] \\ X \arrow[r, "\eta_X"'] \arrow[u, shift right] & GUX \arrow[u, shift right, "G\sigma"'] \end{tikzcd}$$

The second point is a consequence of the naturality of the unit η and of the fact that the right adjoint functor G preserves pullbacks. From that, the last point is straightforward. ∎

Proposition 3.2. *Let $\mathbb{C}$ be a finitely complete category. Then the algebraically exponentiable morphisms in $\mathbb{C}$ are stable under pullback along split epimorphisms. Moreover any pullback in the category $\mathbb{C}$ with y algebraically exponentiable:*

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, shift right, "f"'] & X' \arrow[d, shift right, "f'"'] \\
Y \arrow[r, "y"'] \arrow[u, shift right, "s"'] & Y' \arrow[u, shift right, "s'"']
\end{tikzcd}$$

satisfies the following Beck-Chevalley conditions, i.e. makes the following diagrams commute up to natural isomorphisms:

$$\begin{tikzcd}
Pt_X\mathbb{C} \arrow[r, shift left, "\Phi_x"] & Pt_{X'}\mathbb{C} \arrow[l, shift left, dotted, "x^*"] \\
Pt_Y\mathbb{C} \arrow[u, "f^*"] \arrow[r, shift right, "\Phi_y"'] & Pt_{Y'}\mathbb{C} \arrow[l, shift right, dotted, "y^*"'] \arrow[u, "f'^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Pt_X\mathbb{C} \arrow[r, shift left, "\Phi_x"] \arrow[d, "s^*"'] & Pt_{X'}\mathbb{C} \arrow[l, shift left, dotted, "x^*"] \arrow[d, "s'^*"] \\
Pt_Y\mathbb{C} \arrow[r, shift right, "\Phi_y"'] & Pt_{Y'}\mathbb{C} \arrow[l, shift right, dotted, "y^*"']
\end{tikzcd}$$

Proof. Apply the previous lemma to the functor $y^* : Pt_{Y'}\mathbb{C} \to Pt_Y\mathbb{C}$ and notice that we have $Pt_{(f',s')}(Pt_{Y'}\mathbb{C}) = Pt_{X'}\mathbb{C}$ and $Pt_{(f,s)}(Pt_Y\mathbb{C}) = Pt_X\mathbb{C}$. Then consider the following morphisms in $Pt_{Y'}\mathbb{C}$:

$$\begin{tikzcd}
X' \arrow[rr, "f'"] \arrow[dr, shift right, "f'"'] & & Y' \arrow[dl, shift right, "1"'] \\
& Y' \arrow[ul, shift right, "s'"'] \arrow[ur, shift right, "1"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
Y' \arrow[rr, "s'"] \arrow[dr, shift right, "1"'] & & X' \arrow[dl, shift right, "f'"'] \\
& Y' \arrow[ul, shift right, "1"'] \arrow[ur, shift right, "s'"'] &
\end{tikzcd}$$

∎

Then we get immediately the following:

Corollary 3.3. *When a split epimorphism $(f,s) : X \rightleftarrows Y$ is algebraically exponentiable, the induced endofunctor $f^*.\Phi_f$ on $Pt_X\mathbb{C}$ is (up to a natural isomorphism) equal to the endofunctor $\Phi_{p_0}.p_1^*$, where p_0 and p_1 are given by the kernel equivalence relation:*

$$\begin{tikzcd}
R[f] \arrow[r, shift left=2, "p_0"] \arrow[r, shift right=2, "p_1"'] & X \arrow[l, "s_0"'] \arrow[r, two heads, "f"'] & Y
\end{tikzcd}$$

This endofunctor $\Phi_{p_0}.p_1^*$ on $Pt_X\mathbb{C}$ inherits the left exact comonad structure induced by the adjoint pair (f^*, Φ_f).

3.2 The efficiently regular context

A regular category $\mathbb{C}$ is said to be efficiently regular, when, in addition, any equivalence relation S, on an object X, which is included in an effective equivalence relation $S \overset{m}{\rightarrowtail} R[f]$ by an effective monomorphism m, is itself effective. The main examples are the categories $TopGp$ and $TopAb$ of topological groups and abelian groups. Any exact category is efficiently regular. When the category $\mathbb{C}$ is moreover efficiently regular, we can extend algebraic exponentiability from split epimorphisms to regular epimorphisms and have a kind of converse to Proposition 3.2. For that, let us begin by the following:

Proposition 3.4. *Let $\mathbb{C}$ be regular. Consider an internal discrete cofibration: $\underline{f}_1 : \underline{X}_1 \to \underline{Y}_1$ between two groupoids:*

$$\begin{tikzcd}
R[d_0] \arrow[r, shift left=2, "d_0^1"] \arrow[r] \arrow[r, shift right=2, "d_2^1"'] \arrow[d, "R(f_1)"'] & X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] \arrow[d, "f_1"'] & X_0 \arrow[l, "s_0"'] \arrow[d, "f_0"] \\
R[d_0] \arrow[r, shift left=2, "d_0^1"] \arrow[r] \arrow[r, shift right=2, "d_2^1"'] & Y_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & Y_0 \arrow[l, "s_0"']
\end{tikzcd}$$

Suppose the morphism f_0 is algebraically exponentiable. Then the functor $\underline{f}_1^* : SCof_{\underline{Y}_1} \to SCof_{\underline{X}_1}$ from the split discrete cofibrations above $\underline{Y}_1$ to the split discrete cofibrations above $\underline{X}_1$ defined by pulling back along the functor $\underline{f}_1$ admits a right adjoint which is constructed levelwise.

Proof. According to Proposition 3.2, since the vertical square with the d_0 is a pullback (the functor $\underline{f}_1$ being a discrete cofibration), the maps f_1 and $R(f_1)$ are also algebraically exponentiable. Let $(\underline{\alpha}_1, \underline{\beta}_1) : \underline{T}_1 \rightleftarrows \underline{X}_1$ be a split discrete fibration above $\underline{X}_1$. We are going to show that the split epimorphisms $\Phi_{f_0}(\alpha_0, \beta_0) = (\bar{\alpha}_0, \bar{\beta}_0) : W_0 \rightleftarrows Y_0$ and $\Phi_{f_1}(\alpha_1, \beta_1) = (\bar{\alpha}_1, \bar{\beta}_1) : W_1 \rightleftarrows Y_1$ are actually underlying a discrete cofibration above $\underline{Y}_1$, which will determine the construction of the levelwise right adjoint in question. For that, let us consider the following diagram:

$$\begin{tikzcd}
 & & T_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2] \arrow[lld, "\alpha_1"'] & T_0 \arrow[l] \arrow[lld, "\alpha_0"] \\
X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] \arrow[dd, "f_1"'] & X_0 \arrow[l] \arrow[dd, "f_0"] & & \\
 & & W_1 \arrow[r, shift left=2, "d_0"] \arrow[r, dotted, shift right=2] \arrow[lld, "\bar{\alpha}_1"'] & W_0 \arrow[l] \arrow[lld, "\bar{\alpha}_0"] \\
Y_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & Y_0 \arrow[l] & &
\end{tikzcd}$$

Since the square with the d_0 in the statement is a pullback underlying a pullback of split epimorphisms, then, according to Proposition 3.2, the Beck-Chevalley condition holds for this square. Consequently the lower quadrangle with d_0 above is underlying a pullback of split epimorphisms. But a discrete cofibration between groupoids is also a discrete fibration and the square with the d_1 in the statement is also a pullback. Moreover, the Beck-Chevalley condition not only says that the co-free object are preserved by pullbacks, but also the universal natural transformation $f_i^*.\Phi_{f_i} \Rightarrow 1_{Pt_{Y_i}\mathbb{C}}$, $i \in \{0,1\}$. This determines an arrow $d_1 : W_1 \to Y_1$ which makes also the lower quadrangle with d_1 a pullback, and produces a reflexive graph $W_1 \rightrightarrows W_0$. The same Beck-Chevalley condition makes this reflexive graph underlying a groupoid structure and $\underline{\alpha}_1$ a discrete fibration which is, by construction, a levelwise cofree structure with respect to the pulling back along the functor $\underline{f}_1$. ■

Whence the following "converse" to Proposition 3.2:

Proposition 3.5. *Let $\mathbb{C}$ be an efficiently regular. Consider the following pullback with f' a regular epimorphism:*

$$\begin{tikzcd} X \arrow[r, two heads, "f"] \arrow[d, "x"'] & Y \arrow[d, "y"] \\ X' \arrow[r, two heads, "f'"'] & Y' \end{tikzcd}$$

Then, when the morphism x is algebraically exponentiable, so is the morphism y, and we have the Beck-Chevalley commutation:

$$\begin{tikzcd} Pt_X \arrow[d, "\Phi_x"'] & Pt_Y \arrow[l, "f^*"'] \arrow[d, "\Phi_y"] \\ Pt_{X'} & Pt_{Y'} \arrow[l, "f'^*"] \end{tikzcd}$$

Proof. Complete the previous pullback by the following diagram:

$$\begin{tikzcd} R^2[f] \arrow[r, shift left=2, "p_0^1"] \arrow[r, "p_1^1"'] \arrow[r, shift right=2, "p_2^1"'] \arrow[d, "R^2(x)"'] & R[f] \arrow[r, shift left=2, "p_0"] \arrow[r, shift right=2, "p_1"'] \arrow[d, "R(x)"'] & X \arrow[l, "s_0"'] \arrow[r, two heads, "f"] \arrow[d, "x"'] & Y \arrow[d, "y"] \\ R^2[f'] \arrow[r, shift left=2, "p_0^1"] \arrow[r, "p_1^1"'] \arrow[r, shift right=2, "p_2^1"'] & R[f'] \arrow[r, shift left=2, "p_0"] \arrow[r, shift right=2, "p_1"'] & X' \arrow[l, "s_0"'] \arrow[r, two heads, "f'"'] & Y' \end{tikzcd}$$

which determines a discrete cofibration $\underline{R}_1(x) : \underline{R}_1[f] \to \underline{R}_1[f']$ between the left hand side induced horizontal groupoids. According to the previous proposition the change of base functor $\underline{R}_1(x)^* : SCof_{\underline{R}_1[f']} \to SCof_{\underline{R}_1[f]}$ admits a right adjoint. Now consider the following commutative square:

$$\begin{tikzcd}
Pt_{Y'}\mathbb{C} \arrow[r, "y^*"] \arrow[d, "F_{Y'}"'] & Pt_Y\mathbb{C} \arrow[d, "F_Y"] \\
SCof_{\underline{R}_1[f']} \arrow[r, "\underline{R}_1(x)^*"'] & SCof_{\underline{R}_1[f]}
\end{tikzcd}$$

where the functors F_Y and $F_{Y'}$ are the canonical straightforward functors which are fully faithful, since f and f' are regular epimorphisms. They are also essentially surjective, since, in an efficiently regular category, any equivalence fibration which is discretely cofibered above an effective equivalence relation is itself effective. Accordingly the functors F_Y and $F_{Y'}$ are equivalences of categories, and the change of base functor y^* admits a right adjoint. This construction of the right adjoint to y^* imposes the Beck-Chevalley condition. ■

Corollary 3.6. *Let $\mathbb{C}$ be efficiently regular and $f : X \twoheadrightarrow Y$ a regular epimorphism such that the map $p_0 : R[f] \to X$ is algebraically exponentiable, then the map f is itself algebraically exponentiable and we have: $f^*.\Phi_f \simeq \Phi_{p_0}.p_1^*$.*

When $\mathbb{C}$ is efficiently regular and fiberwise algebraically cartesian closed, then any regular epimorphism $f : X \twoheadrightarrow Y$ is algebraically exponentiable.

Proof. Consider the following pullback:

$$\begin{tikzcd}
R[f] \arrow[r, two heads, "p_1"] \arrow[d, "p_0"'] & x \arrow[d, "f"] \\
X \arrow[r, two heads, "f"'] & Y
\end{tikzcd}$$

and apply the previous proposition. ■

4 Protomodular context

We shall now work in the stronger context of a protomodular category $\mathbb{C}$ [7], which means that any (left exact) change of base functor: $h^* : Pt_Y\mathbb{C} \to Pt_X\mathbb{C}$ is conservative. We get immediately the following:

Proposition 4.1. *Suppose $\mathbb{C}$ is protomodular, then any change of base functor h^* along an algebraic exponentiable map $h : X \to Y$ reflects commuting pairs.*

Proof. In the protomodular context, any change of base functor being left exact and conservative, any algebraic exponentiable map $h : X \to Y$ makes this functor h^* immediately comonadic [16]. Accordingly, the assertion in question is a direct consequence of Proposition 1.6. ■

4.1 Lacc pointed protomodular categories

On the one hand, in [5] and [6], the notion of *action representative* category was introduced, i.e. a pointed protomodular category $\mathbb{C}$ in which each object X admits a universal split extension with kernel X (=split extension classifier):

$$\begin{tikzcd} X \arrow[r, tail, "\gamma"] & D_1(X) \arrow[r, shift left, two heads, "d_0"] & D(X) \arrow[l, shift left, tail, "s_0"] \end{tikzcd}$$

in the sense that any other split extension with kernel X determines a unique morphism $\chi : G \to D(X)$ such that the following diagram commutes and the right hand side squares are pullbacks:

$$\begin{tikzcd} X \arrow[r, tail, "k"] \arrow[d, "1_X"'] & H \arrow[r, shift left, two heads, "f"] \arrow[d, "\chi_1"'] & G \arrow[l, shift left, tail, "s"] \arrow[d, "\chi"] \\ X \arrow[r, tail, "\gamma"'] & D_1(X) \arrow[r, shift left, two heads, "d_0"] & D(X) \arrow[l, shift left, tail, "s_0"] \end{tikzcd}$$

On the other hand, in [16], the second author showed that *when the category* $\mathbb{C}$ *is pointed protomodular, it is locally algebraically cartesian closed if and only if the change of base functors along the initial maps have a right adjoint*.

It is worth translating in detail what this means, and, rather surprisingly, we shall observe that this means a kind of extended dual of the action representativity. So let $\mathbb{C}$ be a locally algebraically cartesian closed pointed protomodular category. Let Y be an object of $\mathbb{C}$ and $\alpha_Y : 1 \rightarrowtail Y$ its associated initial map. We shall denote by $ß_Y$ the right adjoint of α_Y^*. Starting with any object T in $\mathbb{C}$, the object $ß_Y(T)$ is a split epimorphism above Y which is equipped with a (universal) map from its kernel towards T. In other words it produces a universal split exact sequence we shall denote this way:

$$\begin{tikzcd} Ł(Y,T) \arrow[r, tail, "\kappa_T^Y"] \arrow[d, "l_T^Y"'] & Y \ltimes Ł(Y,T) \arrow[r, shift left, two heads, "\psi_T^Y"] & Y \arrow[l, shift left, tail, "\zeta_T^Y"] \\ T & & \end{tikzcd}$$

which from any given similar situation, i.e. a split exact sequence with codomain

Y and a comparison map h:

$$\begin{tikzcd}
K \arrow[r, tail, "k"] \arrow[d, dotted, "\bar{h}"'] \arrow[dd, bend right=60, "h"'] & X \arrow[r, shift left, two heads, "f"] \arrow[d, dotted, "Y\ltimes\bar{h}"'] & Y \arrow[l, shift left, "s"] \arrow[d, equal] \\
\text{Ł}(Y,T) \arrow[r, tail, "\kappa_T^Y"] \arrow[d, "l_T^Y"'] & Y \ltimes \text{Ł}(Y,T) \arrow[r, shift left, two heads, "\psi_T^Y"] & Y \arrow[l, shift left, "\zeta_T^Y"] \\
T & &
\end{tikzcd}$$

produces a unique dotted factorization $\bar{h}$. In particular the following upper canonical split exact sequence determines a factorization which will be denoted by $\S_T^Y$:

$$\begin{tikzcd}
T \arrow[r, tail, "\iota_T"] \arrow[d, dotted, "\S_T^Y"'] \arrow[dd, bend right=60, "1_T"'] & Y \times T \arrow[r, shift left, two heads, "p_Y"] \arrow[d, dotted, "Y\ltimes\S_T^Y"'] & Y \arrow[l, shift left, "\iota_Y"] \arrow[d, equal] \\
\text{Ł}(Y,T) \arrow[r, tail, "\kappa_T^Y"] \arrow[d, "l_T^Y"'] & Y \ltimes \text{Ł}(Y,T) \arrow[r, shift left, two heads, "\psi_T^Y"] & Y \arrow[l, shift left, "\zeta_T^Y"] \\
T & &
\end{tikzcd}$$

Starting from the more specific one with the diagonal s_0 as section, we have also the following factorization:

$$\begin{tikzcd}
Y \arrow[r, tail, "\iota_1"] \arrow[d, dotted, "\varpi_Y"'] \arrow[dd, bend right=60, "1_Y"'] & Y \times Y \arrow[r, shift left, two heads, "p_0"] \arrow[d, dotted, "Y\ltimes\varpi_Y"'] & Y \arrow[l, shift left, "s_0"] \arrow[d, equal] \\
\text{Ł}(Y,Y) \arrow[r, tail, "\kappa_Y^Y"] \arrow[d, "l_Y^Y"'] & Y \ltimes \text{Ł}(Y,Y) \arrow[r, shift left, two heads, "\psi_Y^Y"] & Y \arrow[l, shift left, "\zeta_Y^Y"] \\
Y & &
\end{tikzcd}$$

which we shall analyse below more precisely in the category Gp.

4.2 The category Gp of groups

We shall explore in detail here the very unusual constructions involved in the local exponentiation property of the category Gp. In this case, we have $\text{Ł}(Y,T) = \mathcal{F}(\underline{Y},T)$, namely $\text{Ł}(Y,T)$ is the set of functions from the underlying set of the group Y to the underlying set of the group T equipped with the group structure determined by the group structure on T. The action of the group Y on this group $\mathcal{F}(\underline{Y},T)$ associates with the pair (y,ϕ) the function:

$$\phi \circ \tau_y : Y \to T \; ; \quad z \mapsto \phi(z.y)$$

where τ_y is the translation on the right in the group Y (in other words we get: $({}^y\phi)(z) = \phi(z.y)$). So, in the category Gp, the parallelism between cartesian closedness and algebraic cartesian closedness is not only simply formal, but a kind of strong memory of the underlying exponentiation in *Set*.

The homomorphism $l^Y_T : \mathcal{F}(\underline{Y}, T) \to T$ is the evaluation at the unit element of Y. Given any split extension with a map h:

$$\begin{tikzcd} K \arrow[r, tail] \arrow[d, "h"'] & Y \ltimes_\alpha K \arrow[r, shift left, two heads] & Y \arrow[l, shift left, tail] \\ T & & \end{tikzcd}$$

the group homomorphism $\bar{h} : K \to \mathcal{F}(\underline{Y}, T)$ is then defined by $\bar{h}(k)(y) = h({}^y k)$. In particular we get the group homomorphism $\S^Y_T : T \to \text{Ł}(Y, T)$ defined by $\S^Y_T(t)(y) = t$, in other words $\S^Y_T(t)$ is the function constant on t. Also, we get $\varpi_Y : Y \to \text{Ł}(Y, Y)$ defined by $\varpi_Y(y)(z) = z.y.z^{-1}$ which is a very awkward way to integrate the "inner action" of Y inside the category Gp.

If we start from:

$$\begin{tikzcd} K \arrow[r, tail] \arrow[d, "h"'] & X \arrow[r, shift left, two heads, "f"] & Y \arrow[l, shift left, tail, "s"] \\ T & & \end{tikzcd}$$

we get: $(Y \ltimes \bar{h})(x) = (f(x), \bar{h}(x.s \circ f(x^{-1})))$.

4.3 Consequences of local algebraic cartesian closedness

In this section, we shall investigate two important consequences of local algebraic cartesian closedness, namely strong protomodularity and peri-abelianness. These well identified properties in the category Gp now clearly appear to have originated from locally algebraic cartesian closedness.

Normal functor and strong protomodularity

Recall that a left exact functor $U : \mathbb{C} \to \mathbb{D}$ is called *normal* when it is conservative and it reflects the normal monomorphisms in the sense of [4]. A protomodular category $\mathbb{C}$ is said to be *strongly protomodular* [4] when any change of base functor $h^* : Pt_Y\mathbb{C} \to Pt_X\mathbb{C}$ with respect to the fibration of points is not only conservative but also normal. The categories Gp of groups and R-*Lie* of Lie R-algebras, for any commmutative ring R, are strongly protomodular. In this section we shall show that, when a protomodular category $\mathbb{C}$ is locally algebraically cartesian closed, it is necessarily strongly protomodular. Let us begin by the following observation:

Lemma 4.2. *Let $U : \mathbb{C} \to \mathbb{D}$ be a left exact conservative functor. Suppose moreover that $\mathbb{D}$ is protomodular. If it has a right adjoint G, then U is normal.*

Proof. The right adjoint G is left exact and consequently preserves the monomorphims and the equivalence relations. Now let $m : X' \rightarrowtail X$ be a monomorphism in $\mathbb{C}$ such that the monomorphism $U(m)$ is normal to the equivalence relation R

in $\mathbb{D}$, namely such that we have a discrete fibration in $\mathbb{D}$:

$$\begin{tikzcd}
U(X') \times U(X') \arrow[r, tail, "\mu"] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "p_1"] & R \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \\
U(X') \arrow[r, tail, "U(m)"'] \arrow[u, dotted] & U(X) \arrow[u, dotted]
\end{tikzcd}$$

Since $U(X') \times U(X') = U(X' \times X')$, by adjunction we get a map $\bar{\mu}$ in $\mathbb{C}$ such that $\epsilon_R.U(\bar{\mu}) = \mu$:

$$\begin{tikzcd}
X' \times X' \arrow[r, tail, dotted, "\beta"'] \arrow[rr, tail, bend left, "\bar{\mu}"] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "p_1"] & T \arrow[r, "\bar{\eta}"'] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & G(R) \arrow[d, shift right=2, "G(d_0)"'] \arrow[d, shift left=2, "G(d_1)"] \\
X' \arrow[r, tail, "m"'] \arrow[u] & X \arrow[r, "\eta_X"'] \arrow[u] & G.U(X) \arrow[u]
\end{tikzcd}$$

which determines a morphism between the equivalence relations $\nabla_{X'}$ and $G(R)$. We shall set $T = \eta_X^{-1}(G(R))$ and denote by β the induced factorization. We are going to show that m is normal to the equivalence relation T and that $U(T) \simeq R$. For that, consider the following diagram in $\mathbb{D}$:

$$\begin{tikzcd}
U(X') \times U(X') \arrow[r, tail, dotted, "U(\beta)"'] \arrow[rr, tail, bend left, "U(\bar{\mu})"] \arrow[rrr, bend left=40, "\mu"] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "p_1"] & U(T) \arrow[r, "U(\bar{\eta})"'] \arrow[d, shift right=2, "U(d_0)"'] \arrow[d, shift left=2, "U(d_1)"] & U.G(R) \arrow[r, "\epsilon_R"'] \arrow[d, shift right=2] \arrow[d, shift left=2] & R \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \\
U(X') \arrow[r, tail, "U(m)"'] \arrow[u] & U(X) \arrow[r, "U(\eta_X)"'] \arrow[u] \arrow[rr, bend right, "1_{U(X)}"'] & U.G.U(X) \arrow[r, "\epsilon_{U(X)}"'] \arrow[u] & U(X) \arrow[u]
\end{tikzcd}$$

The map $\gamma = \epsilon_R.U(\bar{\eta})$ produces an inclusion $U(T) \subset R$ of equivalence relations. Since the whole diagram is a discrete fibration and γ is a monomorphism, then the left hand side part of the diagram is a discrete fibration. Accordingly $U(m)$ is normal to the equivalence relation $U(T)$. Now, when $\mathbb{D}$ is protomodular, a monomorphism is normal to at most one equivalence relation, and γ is necessarily an isomorphism. On the other hand, since U is left exact and conservative it reflects the pullbacks, so that m is normal to T in $\mathbb{C}$. ∎

Theorem 4.3. *Let $\mathbb{C}$ be a protomodular category which is locally algebraically cartesian closed. Then $\mathbb{C}$ is strongly protomodular.*

Proof. Since $\mathbb{C}$ is protomodular and locally algebraically cartesian closed, so is any fibre $Pt_Y\mathbb{C}$. Moreover any change of base functor:

$$h^* : Pt_Y\mathbb{C} \to Pt_X\mathbb{C}$$

is left exact and conservative since $\mathbb{C}$ is protomodular; it has a right adjoint since $\mathbb{C}$ is locally algebraically cartesian closed. By the previous lemma it is normal, and $\mathbb{C}$ is strongly protomodular. ∎

Now suppose, in addition, that $\mathbb{C}$ is pointed. Being pointed and strongly protomodular, it is such that two equivalence relations (R, S) centralize if and only if their associated normal subobjects (I_R, I_S) commute, see [9]; in other words the category $\mathbb{C}$ is such that we have the so-called equation "Smith=Huq".

The assertion of the theorem above was mentioned by G. Janelidze during the CT 2010 conference in Genova, but in the much stricter context of semi-abelian categories, as an immediate consequence of Proposition 9 in [3], which deals with preservation of colimits and cannot be used in our context.

Peri-abelian categories

When $\mathbb{C}$ is a regular strongly unital category with finite colimits, the inclusion functor $Ab\mathbb{C} \rightarrowtail \mathbb{C}$ from the full subcategory of abelian objects in $\mathbb{C}$ admits a left adjoint which is given by the cokernel of the diagonal $s_0 : X \rightarrowtail X \times X$, or by the coequalizer of the pair $(\iota_0, \iota_1); X \rightrightarrows X \times X$. Recall now the following [10]:

Definition 4.4. *We shall say that a finitely cocomplete, regular Mal'cev category $\mathbb{D}$ is peri-abelian when the change of base functor along any map $h : Y \to Y'$ with respect to the fibration of points preserves the associated abelian object.*

If $(AbPt)\mathbb{D}$ denotes the subcategory of the abelian objects in the fibres of the fibration of points, it is equivalent to saying that the reg-epi reflection $A_{()}$ is cartesian, i.e. it preserves the cartesian maps:

$$\begin{tikzcd}
(AbPt)\mathbb{D} \arrow[rr, tail] \arrow[dr] & & Pt\mathbb{D} \arrow[ll, bend right, "A_{()}"'] \arrow[dl, "¶_{\mathbb{D}}"] \\
& \mathbb{D} &
\end{tikzcd}$$

The categories Gp of groups, Rg of non unitary commutative rings and $\mathbb{K}$-*Lie* of Lie $\mathbb{K}$-algebras are peri-abelian. The previous definition was introduced in [10] as a tool to produce some cohomology isomorphisms which hold in the Eilenberg-Mac Lane cohomology of groups and in the cohomology of Lie $\mathbb{K}$-algebras when $\mathbb{K}$ is a field.

Theorem 4.5. *Let $\mathbb{C}$ be a finitely cocomplete regular Mal'cev category which is locally algebraically cartesian closed. Then $\mathbb{C}$ is peri-abelian.*

Proof. This is a straightforward consequence of the fact that the change of base-functors h^*, having a right adjoint, preserve cokernels or coequalizers. ∎

4.4 The non-pointed protomodular case

We shall suppose here that the category is still protomodular, but no longer pointed. We shall show that the algebraic exponentiability of any split monomorphism implies the algebraic exponentiability of any of its retractions, and from that, in the efficiently regular context, of a large class of morphisms. This will be the consequence of a very general result:

Proposition 4.6. *Let $\mathbb{C}$ be a protomodular category. For morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathbb{C}$, if $g.f$ and f are algebraically exponentiable, then g is algebraically exponentiable.*

Proof. Recall that for any morphism $p : E \to B$ in $\mathbb{C}$, p^* preserves all limits and reflects isomorphisms. Therefore if p^* has a right adjoint, then by the dual of the Weak Tripleability Theorem [21], p^* is comonadic. The result follows from the well-known *adjoint functor lifting theorem* (see e.g. [2]) applied to the diagram of functors:

$$\begin{tikzcd} Pt_Z\mathbb{C} \arrow[rr, "g^*"] \arrow[dr, shift left, "(g.f)^*"] & & Pt_Y\mathbb{C} \arrow[dl, shift left, "f^*"] \\ & Pt_X\mathbb{C} \arrow[ul, shift left, "\Phi_{gf}"] \arrow[ur, shift left, "\Phi_f"] & \end{tikzcd}$$

in which Φ_{gf} and Φ_f are the right adjoints to the functors $(g.f)^*$ and f^* respectively. ∎

Whence the following theorem:

Theorem 4.7. *Let $\mathbb{C}$ be a protomodular category such that any split monomorphism s is algebraically exponentiable. Then any of its retractions f is algebraically exponentiable. If, in addition, $\mathbb{C}$ is efficiently regular, any map $h : X \to Y$ is algebraically exponentiable provided that its domain X has a global support.*

Proof. The first point is a straightforward consequence of the previous proposition. Moreover, given any map h in $\mathbb{C}$, we get $h = p_Y.(1_X, h)$, where the map $(1_X, h)$ is a monomorphism split by p_X, and consequently the change of base functor along it admits a right adjoint by assumption. When $\mathbb{C}$ is regular and X has global support, the map p_Y is a regular epimorphism. Now when $\mathbb{C}$ is efficiently regular, the change of base functor along p_Y admits a right adjoint by Proposition 3.6. ∎

4.5 Back to the pointed case

The previous construction of the right adjoint obviously applies in the pointed case and gives an alternative description of the centralizers. First, let $(f, s) : X \rightleftarrows$

Y be any split epimorphism; we get a classifying map $\gamma_{(f,s)}$:

$$\begin{tikzcd}
K \arrow[r, tail, "k"] \arrow[d, dotted, "\gamma_{(f,s)}"'] \arrow[dd, bend right=60, "1_K"'] & X \arrow[r, shift left, two heads, "f"] \arrow[d, dotted, "Y \ltimes \gamma_{(f,s)}"'] & Y \arrow[l, shift left, tail, "s"] \arrow[d, equal] \\
Ł(Y,K) \arrow[r, tail, "\kappa_K^Y"'] \arrow[d, "l_K^Y"'] & Y \ltimes Ł(Y,K) \arrow[r, shift left, two heads, "\psi_K^Y"] & Y \arrow[l, shift left, tail, "\zeta_K^Y"] \\
K & &
\end{tikzcd}$$

Then $\Phi_Y[f,s]$ is given by the following equalizer:

$$\begin{tikzcd}
\Phi_Y[f,s] \arrow[r, tail] & K \arrow[r, shift left, "\S_K^Y"] \arrow[r, shift right, "\gamma_{(f,s)}"'] & Ł(Y,K)
\end{tikzcd}$$

In particular the centralizer of a subobject $u : Y \rightarrowtail X$ is given by the following equalizer:

$$\begin{tikzcd}
Z[u] \arrow[r, tail, "\zeta_u"] & X \arrow[r, shift left, "\S_X^Y"] \arrow[r, shift right, "\gamma_{(p_Y,(1,u))}"'] & Ł(Y,X)
\end{tikzcd}$$

Let us make explicit these constructions in the category Gp. First we have: $\gamma_{(f,s)}(k)(y) = s(y).k.s(y^{-1})$, and consequently: $\Phi_Y[f,s] = \{k \in K / \forall y \in Y,\ k = s(y).k.s(y^{-1})\}$, and thus, of course, $Z[u] = \{x \in X / \forall y \in Y,\ x = y.x.y^{-1}\}$.

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