Radius of convexity for certain multivalent functions with missing coefficients

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Abstract

The object of the present paper is to derive the radius of convexity for certain multivalent functions with missing coefficients.

1 Introduction

Let $A_p(n)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k} \quad (p, n \in N = \{1, 2, 3, \cdots\}), \tag{1.1}$$

which are p-valent analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$. A function $f(z) \in A_p(n)$ is said to be p-valently convex of order ρ in U if it satisfies

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \rho \quad (0 \leq \rho < p;\ z \in U). \tag{1.2}$$

For functions $f(z)$ and $g(z)$ analytic in U, we say that $f(z)$ is subordinate to $g(z)$ in U, and we write $f(z) \prec g(z)$ $(z \in U)$, if there exists an analytic function $w(z)$ in U such that

$$|w(z)| \leq |z| \quad \text{and} \quad f(z) = g(w(z)) \quad (z \in U).$$

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Furthermore, if the function $g(z)$ is univalent in U, then

$$f(z) \prec g(z) \quad (z \in U) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

A number of results for p-valently convex functions have been obtained by several authors (see, e.g., [1,2,3,4,5,6]). In this note we shall derive the radius of convexity for certain p-valent functions with missing coefficients.

2 Main results

Theorem 2.1. Let $f(z)$ belong to the class $A_p(n)$ and satisfy

$$\frac{f'(z)}{pz^{p-1}} \prec \left(\frac{1+z}{1-z}\right)^{\gamma} \quad (0 < \gamma \leq 1;\ z \in U). \tag{2.1}$$

Then

$$\operatorname{Re}\left\{(1-\delta)\left(\frac{f'(z)}{pz^{p-1}}\right)^{\frac{1}{\gamma}} + \delta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \rho \quad (|z| < r_n(p,\gamma,\delta,\rho)), \tag{2.2}$$

where $0 < \delta \leq 1$, $0 \leq \rho < 1$ and $r_n(p,\gamma,\delta,\rho)$ is the root in $(0,1)$ of the equation

$$[1+\rho-(p+1)\delta]r^{2n} - 2(1-\delta+n\delta\gamma)r^n + 1 - \rho + (p-1)\delta = 0.$$

The result is sharp.

Proof. From (2.1) we can write

$$\left(\frac{f'(z)}{pz^{p-1}}\right)^{\frac{1}{\gamma}} = \frac{1+z^n\varphi(z)}{1-z^n\varphi(z)}, \tag{2.3}$$

where $\varphi(z)$ is analytic and $|\varphi(z)| \leq 1$ in U. Differentiating both sides of (2.3) logarithmically, we arrive at

$$1 + \frac{zf''(z)}{f'(z)} = p + \frac{2n\gamma z^n\varphi(z)}{1-(z^n\varphi(z))^2} + \frac{2\gamma z^{n+1}\varphi'(z)}{1-(z^n\varphi(z))^2} \quad (z \in U). \tag{2.4}$$

Put $|z| = r < 1$ and $\left(\frac{f'(z)}{pz^{p-1}}\right)^{\frac{1}{\gamma}} = u + iv$ $(u, v \in R)$. Then (2.3) implies that

$$z^n\varphi(z) = \frac{u-1+iv}{u+1+iv} \tag{2.5}$$

and

$$\frac{1-r^n}{1+r^n} \leq u \leq \frac{1+r^n}{1-r^n}. \tag{2.6}$$

With the help of the Carathéodory inequality:

$$|\varphi'(z)| \leq \frac{1-|\varphi(z)|^2}{1-r^2},$$

it follows from (2.5) and (2.6) that

$$\begin{aligned}
&\mathrm{Re}\left\{(1-\delta)\left(\frac{f'(z)}{pz^{p-1}}\right)^{\frac{1}{\gamma}}+\delta\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}\\
&\geq (1-\delta)u+p\delta+2n\delta\gamma\mathrm{Re}\left\{\frac{z^n\varphi(z)}{1-(z^n\varphi(z))^2}\right\}-2\delta\gamma\left|\frac{z^{n+1}\varphi'(z)}{1-(z^n\varphi(z))^2}\right|\\
&\geq (1-\delta)u+p\delta+\frac{n\delta\gamma}{2}\left(u-\frac{u}{u^2+v^2}\right)+\frac{\delta\gamma}{2}\frac{(u-1)^2+v^2-r^{2n}((u+1)^2+v^2)}{r^{n-1}(1-r^2)(u^2+v^2)^{1/2}}\\
&= F_n(u,v) \quad (say) \qquad (2.7)
\end{aligned}$$

and

$$\frac{\partial}{\partial v}F_n(u,v)=\delta\gamma v G_n(u,v), \qquad (2.8)$$

where $0<r<1, 0<\delta\leq 1$ and

$$G_n(u,v)=\frac{nu}{(u^2+v^2)^2}+\frac{1-r^{2n}}{r^{n-1}(1-r^2)(u^2+v^2)^{\frac{1}{2}}}+\frac{r^{2n}((u+1)^2+v^2)-((u-1)^2+v^2)}{2r^{n-1}(1-r^2)(u^2+v^2)^{\frac{3}{2}}}>0 \qquad (2.9)$$

because of (2.5). Since $F_n(u,v)$ is a dual function of v, from (2.7), (2.8) and (2.9), we see that

$$F_n(u,v)\geq F_n(u,0)=(1-\delta)u+p\delta+\frac{n\delta\gamma}{2}\left(u-\frac{1}{u}\right)+\frac{\delta\gamma}{2r^{n-1}(1-r^2)}\left[(1-r^{2n})\left(u+\frac{1}{u}\right)-2(1+r^{2n})\right]. \qquad (2.10)$$

Let us now calculate the minimum value of $F_n(u,0)$ on the closed interval $\left[\frac{1-r^n}{1+r^n},\frac{1+r^n}{1-r^n}\right]$. Noting that

$$\frac{1-r^{2n}}{r^{n-1}(1-r^2)}\geq n \quad (see\ [7])$$

and (2.6), we deduce from (2.10) that

$$\begin{aligned}
\frac{d}{du}F_n(u,0)&=1-\delta+\frac{\delta\gamma}{2}\left[\left(\frac{1-r^{2n}}{r^{n-1}(1-r^2)}+n\right)-\frac{1}{u^2}\left(\frac{1-r^{2n}}{r^{n-1}(1-r^2)}-n\right)\right]\\
&\geq 1-\delta+\frac{\delta\gamma}{2}\left[\left(\frac{1-r^{2n}}{r^{n-1}(1-r^2)}+n\right)-\left(\frac{1+r^n}{1-r^n}\right)^2\left(\frac{1-r^{2n}}{r^{n-1}(1-r^2)}-n\right)\right]\\
&=1-\delta+\frac{2\delta\gamma I_n(r)}{(1-r^n)^2}, \qquad (2.11)
\end{aligned}$$

where

$$I_n(r)=\frac{n}{2}(1+r^{2n})-r(1+r^2+\cdots+r^{2n-2}).$$

Also

$$I_n'(r) = n^2 r^{2n-1} - (1 + 3r^2 + \cdots + (2n-1)r^{2n-2}).$$

$I_1'(r) = r - 1 < 0$. Suppose that $I_n'(r) < 0$. Then

$$\begin{aligned} I_{n+1}'(r) &= (n+1)^2 r^{2n+1} - (2n+1)r^{2n} - (1 + 3r^2 + \cdots + (2n-1)r^{2n-2}) \\ &< n^2 r^{2n} - (1 + 3r^2 + \cdots + (2n-1)r^{2n-2}) \\ &< I_n'(r) < 0. \end{aligned}$$

Hence, by virtue of the mathematical induction, we have $I_n'(r) < 0$ for all $n \in N$ and $0 \le r < 1$. This implies that

$$I_n(r) > I_n(1) = 0 \quad (n \in N;\ 0 \le r < 1). \tag{2.12}$$

In view of (2.11) and (2.12), we see that

$$\frac{d}{du} F_n(u, 0) > 0 \quad \left(\frac{1 - r^n}{1 + r^n} \le u \le \frac{1 + r^n}{1 - r^n} \right). \tag{2.13}$$

Further it follows from (2.7), (2.10) and (2.13) that

$$\begin{aligned} &\operatorname{Re}\left\{ (1-\delta) \left(\frac{f'(z)}{pz^{p-1}} \right)^{\frac{1}{\gamma}} + \delta \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} - \rho \ge F_n\left(\frac{1 - r^n}{1 + r^n}, 0 \right) - \rho \\ &= (1-\delta) \frac{1 - r^n}{1 + r^n} + \delta \frac{p - 2n\gamma r^n - pr^{2n}}{1 - r^{2n}} - \rho \\ &= \frac{J_n(r)}{1 - r^{2n}}, \end{aligned} \tag{2.14}$$

where $0 \le \rho < 1$ and

$$J_n(r) = [1 + \rho - (p+1)\delta] r^{2n} - 2(1 - \delta + n\delta\gamma) r^n + 1 - \rho + (p-1)\delta.$$

Note that $J_n(0) = 1 - \rho + (p-1)\delta > 0$ and $J_n(1) = -2n\delta\gamma < 0$. If we let $r_n(p, \gamma, \delta, \rho)$ denote the root in $(0, 1)$ of the equation $J_n(r) = 0$, then (2.14) yields the desired result (2.2).

To see that the bound $r_n(p, \gamma, \delta, \rho)$ is the best possible, we consider the function

$$f(z) = p \int_0^z t^{p-1} \left(\frac{1 + t^n}{1 - t^n} \right)^{\gamma} dt. \tag{2.15}$$

It is clear that for $z = r \in (r_n(p, \gamma, \delta, \rho), 1)$,

$$(1-\delta) \left(\frac{f'(r)}{pr^{p-1}} \right)^{\frac{1}{\gamma}} + \delta \left(1 + \frac{rf''(r)}{f'(r)} \right) - \rho = \frac{J_n(r)}{1 - r^{2n}} < 0,$$

which shows that the bound $r_n(p, \gamma, \delta, \rho)$ can not be increased.

Setting $\delta = 1$, Theorem 2.1 reduces to the following result.

Corollary 2.2. Let $f(z)$ satisfy the condition (2.1) and $0 \le \rho < 1$. Then $f(z)$ is convex of order ρ in

$$|z| < \left[\frac{((n\gamma)^2 + (p-\rho)^2)^{\frac{1}{2}} - n\gamma}{p - \rho} \right]^{\frac{1}{n}}. \tag{2.16}$$

The result is sharp.

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