Extinction phenomenon for Spinor Ginzburg-Landau equations

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Abstract

Recent papers in physics literature have introduced spinor Ginzburg-Landau model for complex vector-valued order parameters in order to account for ferromagnetic or antiferromagnetic effects in high-temperature superconductors. In this paper, we study the spatial behavior of interacting components of Spinor Ginzburg-Landau model. We prove the interspecies interaction leads to extinction, that is, configurations where one or more densities are null.

1 Introduction

Recent papers in physics literature have introduced spin-coupled (or spinor) Ginzburg-Landau model for complex vector-valued order parameters in order to account for ferromagnetic or antiferromagnetic effects in high-temperature superconductors ([7]). This model can lead to new types of vortices, with fractional degree and non-trivial core structure ([1], [8], [9]).

A reduction of the full two dimensional evolutionary spinor Ginzburg-Landau model can be made which leads to a simplified model that retains the basic features ([1], [8], [9]), related to the superconductivity model introduced in [7]:

$$\begin{cases} \dfrac{\partial u_j}{\partial t} = \triangle u_j + \mu u_j + \displaystyle\sum_{i=1}^{2} U_{ij}|u_i|^2 u_j \quad \text{in } \mathbb{R}^2 \times (0,\infty)\,, \\ \qquad\qquad u_j(x,0) = u_{j0}(x) \ \text{ in } \ \mathbb{R}^2 \times \{0\},\ j = 1,2. \end{cases} \tag{1.1}$$

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u_j denotes the macroscopic wave function of the j^{th} (j=1, 2) component, $|u_j|^2$ is interpreted as the particle density of the j^{th} component. μ is the potential. The constant U_{jj} (j=1, 2) is the intraspecies scattering length of the j-th hyperfine state and $U_{ij}(i \neq j)$ is the interspecies scattering length. As $U_{jj} < 0(> 0)$, the self-interaction is repulsive (attractive). As $U_{ij} > 0(< 0)$, the interspecies interaction is attractive (repulsive).

In the understanding of the spatial behavior of interacting components, a central problem is to establish whether coexistence of all the components occurs, or the interspecies interaction leads to extinction, that is, configurations where one or more densities are null. As $|U_{12}| > \sqrt{|U_{11}||U_{22}|}$ and $U_{11} < 0, U_{22} < 0$, spontaneous symmetric breaking occurs, and the 1-th component and 2-th component are immiscible and separated in space called phase separation [14]. For this reason, we may set $U_{jj} = -\varepsilon^{-2} = -\mu, U_{ij} = -\varepsilon^{-2} - \beta(\varepsilon)$ in the system (1.1), and transform it into the following system

$$\begin{cases} \dfrac{\partial u_j^\varepsilon}{\partial t} - \triangle u_j^\varepsilon + \beta(\varepsilon) \sum\limits_{i \neq j} |u_i^\varepsilon|^2 u_j^\varepsilon = \dfrac{u_j^\varepsilon}{\varepsilon^2}(1 - \sum\limits_{i=1}^{2} |u_i^\varepsilon|^2) \quad \text{in } \mathbb{R}^2 \times (0, \infty)\ , \\ \qquad\qquad\qquad\qquad\qquad\qquad u_j^\varepsilon(x, 0) = u_{j0}^\varepsilon(x) \ \text{ on } \mathbb{R}^2 \times \{0\}. \end{cases} \tag{1.2}$$

The solution to (1.2) is the gradient flow of the following energy functional

$$E(u_1, u_2) = \frac{1}{2} \int_{\mathbb{R}^2} [\sum_{j=1}^{2} |\nabla u_j|^2 + \frac{1}{2\varepsilon^2}(1 - \sum_{i=1}^{2} |u_i|^2)^2] + \beta(\varepsilon)|u_1|^2|u_2|^2. \tag{1.3}$$

The following theorem is our main result concerning the extinction phenomenon of (1.2).

Theorem 1.1. *Let u_j^ε, $j = 1, 2$, be a solution to (1.2). Assume that $\sum_j |u_{j0}^\varepsilon|^2 \leq 1$ and*

$$E(u_{10}^\varepsilon, u_{20}^\varepsilon) \to \int_{\mathbb{R}^2} \frac{1}{2} \sum_j |\nabla w_{j0}|^2, \tag{1.4}$$

where $(w_{10}, w_{20}) \in V = \{(v_1, v_2) \in H^1(\mathbb{R}^2; \mathbb{C}^2) : |v_1|^2|v_2|^2 = 0,\ \sum_j |v_j|^2 = 1\}$. Assume $\beta(\varepsilon) = O(\varepsilon^{-2})$. There exist a (w_1, w_2) of complex-valued functions and $0 < \gamma < 1$ such that, up to a subsequence $\varepsilon \to 0$, we have

$$u_j^\varepsilon \to w_j \ \ in \ \ C_{loc}^{1+\gamma,(1+\gamma)/2}(\mathbb{R}^2 \times (0, \infty)),\ j = 1, 2, \tag{1.5}$$

and $w_1 = 0$ or $w_2 = 0$, that is, one component is an asymptotic null. Moreover, w_i satisfies the heat flow of the harmonic map from $\mathbb{R}^2$ to S^1 when $w_i \neq 0$.

We have a source of inspiration in our study, which is the corresponding theory for the elliptic case. When $u_1^\varepsilon(x)$ and $u_2^\varepsilon(x)$ are real functions, one investigates the phase separation phenomena ([2, 3, 11, 13]). In the recent paper [12], Terracini and Verzini extend this result of [2, 3, 11, 13] to the case of an arbitrary number of components $k \geq 3$. However, from a rigorous mathematical point of view, the phase separation is not well understood so far for the complex-valued solutions of (1.2). The method in [2, 3, 11, 12, 13] can not be applied to the this case.

In order to overcome the difficulty arising from complex-valued functions, we introduce the heat flow of the harmonic map to the singular space (see Section 3). We find that the solution of (1.2) converges to the solution of the heat flow of the harmonic map to the singular space in H^1-norm. The second important step in our proof is to prove $\sum_j |u_j^\varepsilon|^2 > 0$ provided ε is small. The third step is to prove the Bochner type inequality and small energy regularity theorem, which implies the uniformly Lipschitz estimate for $(u_1^\varepsilon, u_2^\varepsilon)$. The fourth step is to obtain $C^{1+\gamma,(1+\gamma)/2}$-estimates by Schauder theory.

The rest of this paper is organized as follows: In Section 2, we derive some basic lemmas. In Section 3, we prove the main Theorem 1.1.

2 Preliminaries

In this section, we will derive some basic lemmas. By maximum principles, we have the following lemmas.

Lemma 2.1. *Let $\Psi_\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ be the solution of (1.2) with $E(\Psi_\varepsilon^0) < +\infty$. Assume that $\beta(\varepsilon) = O(\varepsilon^{-2})$. Then there exists a constant $K > 0$ such that, for $t \geq \varepsilon^2$, we have*

$$|\Psi_\varepsilon(x,t)| \leq 3, \quad |\nabla \Psi_\varepsilon(x,t)| \leq \frac{K}{\varepsilon}, \quad |\frac{\partial \Psi_\varepsilon(x,t)}{\partial t}| \leq \frac{K}{\varepsilon^2} \tag{2.1}$$

and

$$|\Psi_\varepsilon(x,t)|^2 \leq 1 + K\exp(-\frac{t}{2\varepsilon^2}) \ \ for \ \ t \geq \varepsilon^2. \tag{2.2}$$

Moreover, if for some C_0 such that

$$|\Psi_\varepsilon^0(x)| \leq 1, \quad |\nabla \Psi_\varepsilon^0(x)| \leq \frac{C_0}{\varepsilon}, \quad |D^2\Psi_\varepsilon^0(x)| \leq \frac{C_0}{\varepsilon^2}, \quad \forall x \in \mathbb{R}^2. \tag{2.3}$$

Then, for any $x \in \mathbb{R}^2$ and $t > 0$,

$$|\Psi_\varepsilon(x,t)| \leq 1, \quad |\nabla \Psi_\varepsilon(x,t)| \leq \frac{K}{\varepsilon}, \quad |\frac{\partial \Psi_\varepsilon(x,t)}{\partial t}| \leq \frac{K}{\varepsilon^2}, \tag{2.4}$$

where K depends only on C_0.

Proof . *Set*

$$U_j(x,t) = u_j(\varepsilon x, \varepsilon^2 t), \quad \sigma_j(x,t) = |U_j(x,t)|^2.$$

Multiplying (1.2) by U_j we have

$$\partial_t \sigma_1 - \triangle \sigma_1 + 2|\nabla U_1|^2 + 2(\sum_j \sigma_j - 1)\sigma_1 + \varepsilon^2 \beta \sigma_2 \sigma_1 = 0,$$

$$\partial_t \sigma_2 - \triangle \sigma_2 + 2|\nabla U_2|^2 + 2(\sum_j \sigma_j - 1)\sigma_2 + \varepsilon^2 \beta \sigma_1 \sigma_2 = 0.$$

Now we consider

$$y'(t) + (y(t) + 1)y(t) = 0,$$

which has an explicit solution

$$y_0 = \frac{\exp(-t)}{1 - \exp(-t)} \ \textit{for} \ \ t > 0,$$

which blows-up as t tends to zero. Set $\tilde{\sigma}(x, t) = y_0(t)$, $\sigma = \sigma_1 + \sigma_2 - 1$. *Then*

$$\partial_t \sigma - \triangle \sigma + (1 + \sigma)\sigma + \varepsilon^2 \beta \sigma_1 \sigma_2 + 2(|\nabla U_1|^2 + |\nabla U_2|^2) = 0,$$

$$\partial_t (\tilde{\sigma} - \sigma) - \triangle (\tilde{\sigma} - \sigma) + 2(\tilde{\sigma} + \sigma + 1)(\tilde{\sigma} - \sigma) \geq 0. \tag{2.5}$$

The maximum principle implies that

$$\tilde{\sigma}(x, t) - \sigma(x, t) \geq 0 \ \textit{for all} \ t > 0 \ \textit{and} \ x \in \mathbb{R}^2,$$

which implies

$$\sum_j |U_j(x, t)|^2 = \sum_j \sigma_j(x, t) \leq 9 \ \textit{for} \ t \geq \frac{1}{4} \ \textit{and} \ x \in \mathbb{R}^2.$$

Note that

$$\partial_t U_j - \triangle U_j = 2U_j(1 - |U_j|^2) - \varepsilon^2 \beta \sum_{i \neq j} |U_i|^2 U_j \ \ \textit{on} \ \ \mathbb{R}^2 \times [0, \infty), \ j = 1, 2.$$

Since $|U_j(x, t)| \leq 3$ *for* $t \geq \frac{1}{4}$, *we have*

$$\Big| 2U_j(1 - |U_j|^2) - \varepsilon^2 \beta \sum_{i \neq j} |U_i|^2 U_j \Big| \leq C \ \ \textit{for} \ \ t \geq \frac{1}{4}, \ \ j = 1, 2.$$

By the standard parabolic equation theory, we have

$$||U_j||_{C^{1,\alpha/2}(\mathbb{R}^2 \times [1, \infty))} \leq K, \ \ j = 1, 2$$

where $0 < \alpha < 1$. *The conclusions (2.1), (2.2) of Lemma 2.1 follow. (2.3) and (2.4) follow as in the proof of (2.1) and (2.2).* ∎

Now we give the energy estimate.

Lemma 2.2. *Let* u_j^ε, $j = 1, 2$, *be a solution of (1.2). Then, we have*

$$\int_0^t \int_{\mathbb{R}^2} \sum_j |\partial_t u_j^\varepsilon|^2 + E(u_1^\varepsilon(t), u_2^\varepsilon(t)) = E(u_{10}^\varepsilon, u_{20}^\varepsilon). \tag{2.6}$$

Proof . *Multiply (1.2) by* u_{jt}^ε *and integrate by parts.* ∎

3 Proof of Theorem 1.1

In this section, we will prove Theorem 1.1.

Lemma 3.1. *Let* $\Psi_\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ *be the solutions to (1.2). Assume that*

$$E(u_{10}^\varepsilon, u_{20}^\varepsilon) \to \int_{\mathbb{R}^2} \frac{1}{2} \sum_j |\nabla w_{j0}|^2 \tag{3.1}$$

and $\sum_j |w_{j0}^\varepsilon|^2 = 1$*. Assume* $\beta(\varepsilon) \to +\infty$ *as* $\varepsilon \to 0$*. Then there exist a subsequence of* $\{(u_1^\varepsilon, u_2^\varepsilon)\}$ *and a function* $W = (w_1, w_2) \in H^1(\mathbb{R}^2 \times (0,T))$*, such that*

$$u_1^\varepsilon \to w_1, \quad u_2^\varepsilon \to w_2 \ \textit{ strongly in } \ H^1(\mathbb{R}^2 \times (0,T)) \ \textit{ as } \ \varepsilon \to 0; \tag{3.2}$$

$$\int_0^T \int_{\mathbb{R}^2} \beta(\varepsilon)|u_1^\varepsilon|^2|u_2^\varepsilon|^2 + \frac{1}{\varepsilon^2}(1-|\Psi_\varepsilon|^2)^2 \to 0 \ \textit{ as } \ \varepsilon \to 0. \tag{3.3}$$

Proof . *By Lemma 2.2, we obtain*

$$\int_0^T \int_{\mathbb{R}^2} \sum_j |\frac{\partial}{\partial t} u_j^\varepsilon|^2 + E(\Psi(\cdot,t)) = E(\Psi_0^\varepsilon). \tag{3.4}$$

Hence, from (3.1), there exist $w_1, w_2 \in H^1(\mathbb{R}^2 \times (0,T), \mathbb{C})$ *such that, up to a subsequence,*

$$u_1^\varepsilon \rightharpoonup w_1, \quad u_2^\varepsilon \rightharpoonup w_2 \ \textit{ weakly-* in } \ L^2(0,\infty; H^1(\mathbb{R}^2)) \ \textit{ as } \ \varepsilon \to 0, \tag{3.5}$$

$$u_{1t}^\varepsilon \rightharpoonup w_{1t}, \quad u_{2t}^\varepsilon \rightharpoonup w_{2t} \ \textit{ weakly in } \ L^2(0,\infty; L^2(\mathbb{R}^2)) \ \textit{ as } \ \varepsilon \to 0, \tag{3.6}$$

$$u_1^\varepsilon \to w_1, \quad u_2^\varepsilon \to w_2 \ \textit{ strongly in } \ L^2(0,\infty; L^2(\mathbb{R}^2)) \ \textit{ as } \ \varepsilon \to 0. \tag{3.7}$$

Note that

$$\int_0^T \int_{\mathbb{R}^2} \frac{(1-\sum_j |u_j^\varepsilon|^2)^2}{\varepsilon^2} \le C; \quad \beta(\varepsilon) \int_0^T \int_{\mathbb{R}^2} |u_1^\varepsilon|^2 |u_2^\varepsilon|^2 \le C, \tag{3.8}$$

we obtain

$$\sum_j |w_j|^2 = 1 \quad \textit{a.e.} \ (x,t) \in \mathbb{R}^2 \times [0,T], \quad \int_0^T \int_{\mathbb{R}^2} |w_1|^2 |w_2|^2 = 0. \tag{3.9}$$

Taking the exterior product of (1.2) with u_j^ε *, we get*

$$\Psi_{\varepsilon t} \wedge \Psi_\varepsilon - \nabla \cdot (\nabla \Psi_\varepsilon \wedge \Psi_\varepsilon) = 0. \tag{3.10}$$

In view of (3.5)-(3.7), we get by passing to the limit in (3.10), denoting $W = (w_1, w_2)$*, that*

$$W_t \wedge W - \nabla \cdot (\nabla W \wedge W) = 0. \tag{3.11}$$

From [5] we know that W *is a weak solution of the following problem*

$$\begin{cases} \dfrac{\partial W}{\partial t} - \triangle W = W|\nabla W|^2 \quad \textit{in } \mathbb{R}^2 \times (0,\infty)\ , \\ |W| = 1 \quad \textit{on } \mathbb{R}^2 \times (0,\infty)\ , \\ W = W_0 \quad \textit{on } \mathbb{R}^2 \times \{0\}, \end{cases} \tag{3.12}$$

and

$$\int_0^T \int_0^t \int_{\mathbb{R}^2} |\frac{\partial}{\partial t}W|^2 + \frac{1}{2}\int_0^T \int_{\mathbb{R}^2} |\nabla W|^2 = \frac{1}{2}T \int_{\mathbb{R}^2} |\nabla W_0|^2. \tag{3.13}$$

Using (3.1), (3.4), (3.13), and a lower semi-continuity argument, one may deduce the strong convergence

$$\Psi_\varepsilon \to W \text{ strongly in } H^1(\mathbb{R}^2 \times (0,T)), \tag{3.14}$$

$$\int_0^T \int_{\mathbb{R}^2} \frac{(1-|\Psi_\varepsilon|^2)^2}{\varepsilon^2} \to 0. \tag{3.15}$$

The conclusion of Lemma 3.1 follows. ∎

Proposition 3.2. *Under the assumption of Lemma 3.1, $\sum_j |u^\varepsilon_{j0}|^2 \leq 1$ and $\beta(\varepsilon) = O(\varepsilon^{-1})$. Then, $|w_1|$ and $|w_2|$ are continuous functions. Denote $\Omega_j = \{(x,t) \in \mathbb{R}^2 \times (0,T) : |w_j(x,t)| > 0\}$, $j = 1,2$. We have*

$$\| |\Psi_\varepsilon(x,t)| \|_{C^{1+\gamma,(1+\gamma)/2}_{loc}(\mathbb{R}^2\times(0,\infty))} \leq C \tag{3.16}$$

and

$$\|u^\varepsilon_j\|_{C^{1+\gamma,(1+\gamma)/2}_{loc}(\Omega_j)} \leq C. \tag{3.17}$$

Proof . **Step 1**: $|\Psi_\varepsilon(x)|^2 \to 1$ *uniformly on* $\mathbb{R}^2 \times (0,T)$ *as* $\varepsilon \to 0$.

Let $(x_0,t_0) \in \mathbb{R}^2 \times (0,T)$ and set $\alpha = |\Psi_\varepsilon(x_0,t_0)|$. By Lemma 2.1 then $\alpha \leq 1$, and we have

$$|\Psi_\varepsilon(x,t)| \leq \alpha + \frac{C}{\varepsilon}\rho + \frac{C}{\varepsilon^2}\rho^2 \text{ if } |x-x_0| < \rho, |t-t_0| < \rho^2. \tag{3.18}$$

Thus

$$(1-|\Psi_\varepsilon(x,t)|^2)^2 \geq (1-\alpha-\frac{C}{\varepsilon}\rho - \frac{C}{\varepsilon^2}\rho^2)^2 \text{ provided } \frac{C\rho}{\varepsilon} + \frac{C}{\varepsilon^2}\rho^2 \leq 1-\alpha. \tag{3.19}$$

By (3.3), we obtain

$$\varepsilon^2 o(1) = \int_{t_0-\rho^2}^{t_0+\rho^2} \int_{B(x_0,\rho)} (1-|\Psi_\varepsilon|^2)^2 \geq \pi\rho^4(1-\alpha-\frac{C\rho}{\varepsilon}-\frac{C}{\varepsilon^2}\rho^2)^2. \tag{3.20}$$

Let ε be small such that

$$\rho = \frac{\varepsilon(1-\alpha)}{4C}. \tag{3.21}$$

Hence

$$\varepsilon^2 o(1) \geq \pi\frac{\varepsilon^2(1-\alpha)^2}{4C^2}\frac{(1-\alpha)^2}{16} \tag{3.22}$$

and therefore

$$(1-\alpha)^4 \leq o(1) \tag{3.23}$$

i.e., $|\Psi_\varepsilon| \to 1$ *uniformly on* $\mathbb{R}^2 \times (0, T)$*. The proof of Step 1 is completed.*

Step 2*: (Bochner type inequality) Let*

$$e(\Psi_\varepsilon) = \frac{1}{2}|\nabla\Psi_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(1 - |\Psi_\varepsilon|^2)^2 + \frac{1}{2}\beta(\varepsilon)|u_1|^2|u_2|^2.$$

We have the following Bochner type inequality

$$(\partial_t - \triangle)e(\Psi_\varepsilon) \le C(1 + e(\Psi_\varepsilon))e(\Psi_\varepsilon). \tag{3.24}$$

Now we prove (3.24). Note that

$$(\partial_t - \triangle)(\frac{1}{2}|\nabla\Psi_\varepsilon|^2) = -|\nabla^2\Psi_\varepsilon|^2 + \nabla(\partial_t\Psi_\varepsilon - \triangle\Psi_\varepsilon) \cdot \nabla\Psi_\varepsilon. \tag{3.25}$$

Using equation (1.2) we find

$$u_{jtx_i} - \triangle u_{jx_i} = -\beta(\varepsilon)\sum_{i\neq j}(|u_i|^2u_j)_{x_i} - \frac{2}{\varepsilon^2}(\Psi_\varepsilon\Psi_{\varepsilon x_i})u_j + \frac{1}{\varepsilon^2}(1 - |\Psi_\varepsilon|^2)u_{jx_i}.$$

Inserting this into (3.25) and using (1.2) we see that

$$\begin{aligned}
&(\partial_t - \triangle)(\frac{1}{2}|\nabla\Psi_\varepsilon|^2) = -|\nabla^2\Psi_\varepsilon|^2 - \frac{1}{\varepsilon^2}\sum_k(\Psi_\varepsilon \cdot \Psi_{\varepsilon x_k})^2 \\
&-\beta(\varepsilon)\sum_k\sum_{i\neq j}(|u_i|^2u_j)_{x_k}u_{jx_k} + |\nabla\Psi_\varepsilon|^2\frac{1}{\varepsilon^2}(1 - |\Psi_\varepsilon|^2) \\
&\le -|\nabla^2\Psi_\varepsilon|^2 - \frac{1}{\varepsilon^2}\sum_k(\Psi_\varepsilon \cdot \Psi_{\varepsilon x_k})^2 - \beta(\varepsilon)\sum_k\sum_{i\neq j}(|u_i|^2u_j)_{x_k}u_{jx_k} \qquad (3.26)\\
&+\frac{|\nabla\Psi_\varepsilon|^2}{|\Psi_\varepsilon|}(|\partial_t\Psi_\varepsilon| + |\triangle\Psi_\varepsilon| + \beta(\varepsilon)\sum_{i\neq j}|u_i||u_j||\Psi_\varepsilon|).
\end{aligned}$$

Since $|\triangle\Psi_\varepsilon| \le |\nabla^2\Psi_\varepsilon|$ *and* $|\Psi_\varepsilon| \ge \frac{1}{2}$ *when* ε *is small, using the Hölder inequality, we have*

$$\begin{aligned}
&(\partial_t - \triangle)(\frac{1}{2}|\nabla\Psi_\varepsilon|^2) \le -\frac{7}{8}|\nabla^2\Psi_\varepsilon|^2 + C(1 + e(\Psi_\varepsilon))e(\Psi_\varepsilon) \qquad (3.27)\\
&+\frac{1}{8}\beta^2(\varepsilon)\sum_{i\neq j}|u_i|^4|u_j|^2 + \frac{1}{64}|\partial_t\Psi_\varepsilon|^2.
\end{aligned}$$

Similarly, using (1.2), we have

$$\begin{aligned}
(\partial_t - \triangle)(\frac{(1 - |\Psi_\varepsilon|^2)^2}{\varepsilon^2}) &= -\frac{1}{\varepsilon^2}\sum_k(\Psi_\varepsilon\Psi_{\varepsilon x_k})^2 + |\nabla\Psi_\varepsilon|^2\frac{1}{\varepsilon^2}(1 - |\Psi_\varepsilon|^2) \\
-\frac{1}{\varepsilon^2}(1 - |\Psi_\varepsilon|^2)\Psi_\varepsilon(\partial_t\Psi_\varepsilon - \triangle\Psi_\varepsilon) &\le \frac{3}{16}|\triangle\Psi_\varepsilon|^2 + C(1 + e(\Psi_\varepsilon))e(\Psi_\varepsilon) \\
&+ \frac{5}{8}\beta^2(\varepsilon)\sum_{i\neq j}|u_i|^4|u_j|^2 - \frac{1}{16}|\partial_t\Psi_\varepsilon|^2. \qquad (3.28)
\end{aligned}$$

Using equation (1.2), the same computing of (3.27) gives

$$
\begin{aligned}
&((\partial_t - \triangle)\beta(\varepsilon)\sum_{i\neq j}|u_i|^2))|u_j|^2\\
&\quad= -2\beta(\varepsilon)\sum_{i\neq j}|u_i|^2|\nabla u_j|^2 - 2\beta(\varepsilon)\sum_{i\neq j}u_i\nabla u_i u_j\nabla u_j - \beta^2(\varepsilon)\sum_{i\neq j}|u_i|^4|u_j|^2\\
&\qquad\qquad + \frac{2}{\varepsilon^2}(1-|\Psi_\varepsilon|^2)\beta(\varepsilon)\sum_{i\neq j}|u_i|^2|u_j|^2\\
&\quad\le -2\beta(\varepsilon)\sum_{i\neq j}|u_i|^2|\nabla u_j|^2 + \frac{1}{4}|\nabla^2\Psi_\varepsilon|^2 - \frac{7}{8}\beta^2(\varepsilon)\sum_{i\neq j}|u_i|^4|u_j|^2 +\\
&\qquad\qquad C(1+e(\Psi_\varepsilon))e(\Psi_\varepsilon) + \frac{1}{64}|\partial_t\Psi_\varepsilon|^2. \quad (3.29)
\end{aligned}
$$

Combining (3.27), (3.28) with (3.29) we obtain (3.24).

Step 3*: (Small energy regularity theorem) Let* $z = (x,t), z_0 = (x_0,t_0), R, \lambda > 0$ *and*

$$P_R(z) = \{z = (x,t) : |x-x_0| < R, |t-t_0| < R^2\}. \quad (3.30)$$

There are two positive constants $\theta_0 \in (0,1)$ *and* K_0 *such that*

$$\frac{1}{R^2}\int_{P_R(z)} e(\Psi_\varepsilon) \le \theta_0 \quad (3.31)$$

then

$$(\frac{1}{2}R)^2 \sup_{P_{R/2}(z)} e(\Psi_\varepsilon)(x,t) \le K_0\frac{1}{R^2}\int_{P_R(z)} e(\Psi_\varepsilon). \quad (3.32)$$

The proof of [[4]; Lemma 2.4] carries over almost literally.

Step 4*: We choose* $r_0 > 0$ *such that*

$$\frac{1}{r_0^2}\int_{P_{r_0}(z)} |\nabla W|^2 \le \theta_0/2. \quad (3.33)$$

By Lemma 3.1, for all ε *small,*

$$\frac{1}{r_0^2}\int_{P_{r_0}(z)} e(\Psi_\varepsilon) \le \theta_0. \quad (3.34)$$

Now using the small energy regularity theorem we have that

$$e(\Psi_\varepsilon)(z) \le C\theta_0, \;\; x \in P_{r_0/2}(z). \quad (3.35)$$

Then, using the finite covering theorem, for any compact subset $K \subset \mathbb{R}^2 \times (0,\infty)$*, we have*

$$e(\Psi_\varepsilon) \le C_K \;\; in \;\; K. \quad (3.36)$$

Step 5*: Let* $Q_{r,s} = B_r(x_0) \times [t_0 - s; t_0 + s]$*. Then for any* $q > 2$*, there are a constant* $C_q > 0$ *independent of* ε *and a constant* $\varepsilon_0 > 0$ *such that*

$$\||u_j^\varepsilon|^2\|_{W_q^{2,1}(Q_{r/2,s/2})} \le C_q, \; j = 1,2, \; \varepsilon < \varepsilon_0. \quad (3.37)$$

First of all, we have from Step 4 that $\|\Psi_\varepsilon\|_L^q(Q_{r,s}) \leq C_q$. *Moreover, we have for* $\Phi = \frac{(1-|\Psi_\varepsilon|^2)}{\varepsilon^2}$

$$\varepsilon^2\Phi_t - \varepsilon^2\triangle\Phi + \frac{1}{2}\Phi \leq 2\beta|u_1|^2|u_2|^2 + 2|\nabla\Psi_\varepsilon|^2 \text{ in } Q_{r,s}. \tag{3.38}$$

Here we have used the fact that $|\Psi_\varepsilon| \geq 1/2$.

Take cut-off function $\xi(x) \in C_0^\infty(B_r(x_0)), \xi = 1$ *in* $B_{r/2}(x_0), \eta(t) \in C_0^\infty([t_0-s, t_0+s]), \eta = 1$ *in* $[t_0-s/2, t_0+s/2], |\nabla\xi| \leq C/r, |\nabla\eta| \leq C/s$, $0 \leq \xi \leq 1, 0 \leq \eta \leq 1$. *Multiply (3.38) by* $\xi^2(x)\eta^2(t)\Phi^{q-1}$ *and integrate it over* $Q_{r,s}$ *to give*

$$\begin{aligned}\frac{\varepsilon^2}{q}\int_{B_r}\xi^2(x)\eta^2(t)\Phi^q|_{t_0-s}^{t_0+s} - \varepsilon^2\int_{Q_{r,s}}\xi^2(x)\eta^2(t)\Phi^{q-1}\triangle\Phi + \frac{1}{2}\int_{Q_{r,s}}\xi^2(x)\eta^2(t)\Phi^q \\ \leq \int_{Q_{r,s}}\xi^2(x)\eta^2(t)(\beta|u_1|^2|u_2|^2 + |\nabla\Psi_\varepsilon|^2)\Phi^{q-1}.\end{aligned} \tag{3.39}$$

i.e.

$$\begin{aligned}&\frac{1}{2}\varepsilon^2(q-1)\int_{Q_{r,s}}\xi^2(x)\eta^2(t)\Phi^{q-2}|\nabla\Phi|^2 + \frac{1}{2}\int_{Q_{r,s}}\xi^2(x)\eta^2(t)\Phi^q \\ &\quad\leq \sigma\int_{Q_{r,s}}\xi^2(x)\eta^2(t)\Phi^q + C_\sigma\int_{Q_{r,s}}\xi^2(x)\eta^2(t)\Phi^{q-1}(\beta|u_1|^2|u_2|^2 + |\nabla\Psi_\varepsilon|^2)^{q/2} \\ &\qquad + \frac{2\varepsilon^2}{q}\int_{Q_{r,s}}\xi(x)\eta(t)|\eta_t|\Phi^q + \frac{2\varepsilon^2}{q-1}\int_{Q_{r,s}}|\nabla\xi(x)|^2\eta^2(t)\Phi^q\end{aligned} \tag{3.40}$$

Set $\sigma = \frac{1}{4}$, *we have*

$$\begin{aligned}\frac{1}{4}\int_{Q_{r,s}}\xi^2(x)\eta^2(t)\Phi^q &\leq C\int_{Q_{r,s}}\xi^2(x)\eta^2(t)\Phi^{q-1}(\beta|u_1|^2|u_2|^2 + |\nabla\Psi_\varepsilon|^2)^{q/2} \\ &\quad + \frac{2\varepsilon^2}{q}\int_{Q_{r,s}}\xi(x)\eta(t)|\eta_t|\Phi^q + \frac{2\varepsilon^2}{q-1}\int_{Q_{r,s}}|\nabla\xi(x)|^2\eta^2(t)\Phi^q\end{aligned} \tag{3.41}$$

Hence

$$\frac{1}{4}\int_{Q_{r,s}}\xi^2(x)\eta^2(t)\Phi^q \leq C_q + C\varepsilon^2\int_{Q_{r,s}\setminus Q_{r/2,s/2}}(\frac{1}{r^2}\Phi^q + \frac{1}{s}\Phi^q). \tag{3.42}$$

Fixing r, s *and taking* ε *small enough such that*

$$\frac{C\varepsilon^2}{r^2} \leq \frac{1}{16}, \frac{C\varepsilon^2}{s} \leq \frac{1}{16}. \tag{3.43}$$

We have

$$\frac{1}{4}\int_{Q_{r,s}}\xi^2(x)\eta^2(t)\Phi^q \leq C_q + \frac{1}{16}\int_{Q_{r,s}\setminus Q_{r/2,s/2}}\Phi^q. \tag{3.44}$$

It follows that

$$\int_{Q_{r/2,s/2}}\Phi^q \leq C_q \quad \forall q > 2. \tag{3.45}$$

Note that

$$(\partial_t - \triangle)|u_i|^2 = -2(|\nabla u_i|^2 + \beta \sum_{i\neq j} |u_i|^2|u_j|^2) + 2\Phi|u_i|^2. \tag{3.46}$$

From Step 4, (3.45) and L^p-theory of parabolic equations, we obtain (3.37).

From the Sobolev imbedding,we have, for some $\gamma \in (0,1)$, that

$$\| |u_j|^2 \|_{C^{1+\gamma,(1+\gamma)/2}_{loc}(\mathbb{R}^2\times(0,\infty))} \leq C,\ j = 1,2. \tag{3.47}$$

Hence, $|w_1|$ and $|w_2|$ are continuous functions. From (3.9), we have $|w_1||w_2| = 0$.

Step 6: *Let $K \subseteq \Omega_j$ be any compact subdomain. By step 5, we have $w_i = 0$ in Ω_j, $i \neq j$, and*

$$|u^\varepsilon_i| \to 0 \quad \text{uniformly in } K \subset \Omega_j \text{ as } \varepsilon \to 0. \tag{3.48}$$

We may assume that ε is sufficiently small so that

$$|u^\varepsilon_j| \geq 1/4 \ \text{ in } \ K \subset \Omega_j. \tag{3.49}$$

Thus we may write

$$u^\varepsilon_j(x,t) = \rho_\varepsilon(x,t)\exp(i\varphi_\varepsilon(x,t)) \ \text{ in } K,$$

and we may assume

$$\frac{1}{|K|}\int_K \varphi_\varepsilon \in [0,2\pi). \tag{3.50}$$

Using (1.2), we have

$$\rho^2_\varepsilon \frac{\partial\varphi_\varepsilon}{\partial t} - \operatorname{div}(\rho^2_\varepsilon \nabla\varphi_\varepsilon) = 0 \quad \text{in } K, \tag{3.51}$$

$$\frac{\partial\rho_\varepsilon}{\partial t} - \triangle\rho_\varepsilon + \rho_\varepsilon|\nabla\varphi|^2 + 2\beta\sum_{i\neq j}|u_i|^2\rho_\varepsilon = \frac{1}{\varepsilon^2}(1-|\Psi|^2)\rho_\varepsilon \quad \text{in } K. \tag{3.52}$$

By Step 5, we have, for $0 < \gamma < 1$,

$$\|\rho_\varepsilon\|_{C^{1+\gamma,(1+\gamma)/2}(K)} \leq C. \tag{3.53}$$

Using Schauder theory [6], it follows that, for $\varepsilon < \varepsilon_0$ and $K_1 \subset K$,

$$\|\varphi_\varepsilon\|_{C^{2+\gamma,1+\gamma/2}(K_1)} \leq C\|\varphi_\varepsilon\|_{C^{\gamma,\gamma/2}(K)} \leq C. \tag{3.54}$$

Combing (3.53) with (3.54), we obtain (3.17). ∎

Lemma 3.3. *Under the assumption of Proposition 3.2, we have $w_1 = 0$ or $w_2 = 0$.*

Proof . *By Lemma 3.1 and Proposition 3.2, we have $|w_1|^2 + |w_2|^2 = 1$, $|w_1||w_2| = 0$, $|w_1|$ and $|w_2|$ are continuous. Hence, $w_1 = 0$ or $w_2 = 0$.* ∎

From Lemma 3.1, Proposition 3.2 and Lemma 3.3, we prove Theorem 1.1.

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