Rational involutive automorphisms related with standard representations of $\mathrm{SL}(2,\mathbb{R})$

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Abstract

Standard irreducible representations of the group $\mathrm{SL}(2,\mathbb{R})$ on coefficients of homogeneous polynomials in two variables are studied in a new context. It is proved that any standard representation of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^{n+1}$ induces an involutive rational mapping of an open dense subset of $\mathbb{R}^{n+1}$ onto itself. Examples in low dimensions are presented. We also construct formal involutive rational mappings with "arbitrary complexity".

1 Introduction

In [2], the first author studied an irreducible representation of $\mathrm{SL}(2,\mathbb{R})$ on the space of symmetric equiaffine connections with constant Christoffel symbols on $\mathbb{R}^2$. During the study of this representation and the attempts to find all invariants, a remarkable rational involutive map of an open dense subset of $\mathbb{R}^6$ onto itself appeared.

Involutive transformations play an important role in integrable dynamical systems, see e.g. [1], [3], [4], [7] and the references inside. Unfortunately, all the works known to the authors investigate in general, or apply to dynamics, involutive automorphisms of the type $\mathbb{R}^2 \to \mathbb{R}^2$ or involutive transformations of the real projective plane. In [6], involutive mappings appear as transformations of differential equations. Probably, no systematic studies in higher dimensions are known.

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It is well known that the group $\mathrm{SL}(2,\mathbb{R})$ admits an irreducible representation in any dimension, namely the representation on homogeneous polynomials of degree n in two variables ("binary forms of degree n"). In the present paper, these representations are studied. It is proved that each such representation induces an involutive mapping of an open dense subset of $\mathbb{R}^{n+1}$ onto itself. In dimensions 3, 4 and 5, corresponding involutive mappings are constructed explicitly.

2 Main result

We will consider spaces P_n of homogeneous polynomials of degree n in two variables (binary forms) denoted as

$$a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n,$$

and the corresponding spaces $\mathbb{R}^{n+1}[a_0, a_1, \ldots, a_n]$ of their coefficients. This notation is essential for the further considerations. Let the subgroup

$$g_1(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

of the group $\mathrm{SL}(2,\mathbb{R})$ act in the standard way on P_n for each $n \geq 2$. This determines the action of this group on the space $\mathbb{R}^{n+1}[a_0, a_1, \ldots, a_n]$ of coefficients. The induced Killing vector field is of the form

$$Z_n = n\,a_0\frac{\partial}{\partial a_1} + (n-1)\,a_1\frac{\partial}{\partial a_2} + \cdots + 2\,a_{n-2}\frac{\partial}{\partial a_{n-1}} + a_{n-1}\frac{\partial}{\partial a_n} \tag{1}$$

(cf. [5], Theorem 3.40, for instance). Let us define the *weight* of a monomial in the variables $a_0, \ldots, a_n$ as the sum of all indices contained in this monomial (counting also the multiplicities). On the ground of this we define a *polynomial of weight k*. Let us emphasize that the notion "weight" will be used in this particular meaning everywhere and it should not be confused with the standard meaning of "weight" as used in the representation theory. It is well known fact that the space of all invariants of the operator (1) admits a polynomial Hilbert basis.

Lemma 1. *Let J be a polynomial invariant of the operator (1) which is of the form $p_1J_1 + \cdots + p_kJ_k$, where $J_1, \ldots, J_k$ are homogeneous polynomials of mutually distinct constant weights or of mutually distinct degrees. Then $J_1, \ldots, J_k$ are invariants, too.*

Proof. The operator Z_n has the property that

$$Z_n(a_k) = (n-k+1)\,a_{k-1}, \qquad k = 1, \ldots, n.$$

Hence we see easily, that Z_n converts each polynomial of the constant weight s into a polynomial of the constant weight $s-1$ and each homogeneous polynomial of degree d again into a homogeneous polynomial of degree d. In particular, $p_1Z_n(J_1) + \cdots + p_kZ_n(J_k) = 0$ and hence $Z_n(J_1) = \cdots = Z_n(J_k) = 0$. ∎

Lemma 2. *For each fixed n, there is a substitution of the form $b_i = q_i a_i$, $q_i \in \mathbb{Q}$, $i = 0, 1, \ldots, n$, such that the operator Z_n takes on the form*

$$Z'_n = b_0 \frac{\partial}{\partial b_1} + b_1 \frac{\partial}{\partial b_2} + \cdots + b_{n-2} \frac{\partial}{\partial b_{n-1}} + b_{n-1} \frac{\partial}{\partial b_n}. \tag{2}$$

Proof. This is a routine computation. For example,

$$q_0 = 1, \quad q_1 = \frac{1}{n}, \quad q_2 = \frac{1}{n(n-1)}, \quad q_3 = \frac{1}{n(n-1)(n-2)},$$

and so on. ∎

The following Lemma is a basic one:

Lemma 3. *The space of all invariants with respect to the operator Z'_n on $\mathbb{R}^{n+1}[b_0, b_1, \ldots, b_n]$ admits a polynomial Hilbert basis $w_0, w_2, \ldots, w_n$, where $w_0 = b_0$ is of weight zero, and, for each $k = 2, \ldots, n$, w_k is a homogeneous polynomial of degree k with integral coefficients and of weight k in the variables $b_0, b_1, \ldots, b_k$. Its only summand involving the variable b_k is of the form $p{b_0}^{k-1} b_k$ for some integer p.*

Proof. We have $w_0 = b_0$ and for arbitrary $k \geq 2$ we define

$$w_k = {b_1}^k - \sum_{i=1}^{k-1} c_{ki} {b_0}^i {b_1}^{k-1-i} b_{i+1}, \qquad c_{ki} \in \mathbb{R}, \tag{3}$$

From the condition $Z'_n(w_k) = 0$ we obtain a simple system of equations for parameters c_{ki} with the solution $c_{k1} = k$ and $c_{ki} = -c_{k,i-1}(k-i)$ for $2 \leq i \leq k-1$. ∎

Example. Let us consider the operator Z'_6. According to the above formula we obtain invariants

$$\begin{aligned}
w_0 &= b_0, \\
w_2 &= {b_1}^2 - 2\, b_0 b_2, \\
w_3 &= {b_1}^3 - 3\, b_0 b_1 b_2 + 3\, {b_0}^2 b_3, \\
w_4 &= {b_1}^4 - 4\, b_0 {b_1}^2 b_2 + 8\, {b_0}^2 b_1 b_3 - 8\, {b_0}^3 b_4, \\
w_5 &= {b_1}^5 - 5\, b_0 {b_1}^3 b_2 + 15\, {b_0}^2 {b_1}^2 b_3 - 30\, {b_0}^3 b_1 b_4 + 30\, {b_0}^4 b_5, \\
w_6 &= {b_1}^6 - 6\, b_0 {b_1}^4 b_2 + 24\, {b_0}^2 {b_1}^3 b_3 - 72\, {b_0}^3 {b_1}^2 b_4 + 144\, {b_0}^4 b_1 b_5 - 144\, {b_0}^5 b_6.
\end{aligned}$$

Let us define now, for the sake of completeness, the quantity w_1 as $w_1 = b_1$. Herewith we obtain a system of polynomials $w_0, w_1, w_2, \ldots, w_n$, where, of course, w_1 is not an invariant. Then we have the following

Lemma 4. *The variables $b_0, b_1, \ldots, b_n$ can be expressed as (proper) rational functions of the quantities $w_0, w_1, w_2, \ldots, w_n$. It holds $w_0 = b_0$, $w_1 = b_1$ and for each integer $k \geq 2$, we have*

$$b_k = \frac{{w_1}^k + (-1)^{k-1}(k-1) w_k + Q_k(w_0, \ldots, w_{k-1})}{k!\, {w_0}^{k-1}}, \tag{4}$$

where the term $Q_k(w_0, \ldots, w_{k-1})$ is a polynomial in its variables which can involve only the powers ${w_1}^i$ for $i \leq k-2$. Each numerator in the formula (4) has constant weight k with respect to variables w_i.

Proof. For $k = 2$, we have a special formula $w_2 = {b_1}^2 - 2b_0b_2$, from which

$$b_2 = \frac{{b_1}^2 - w_2}{2\,b_0} = \frac{{w_1}^2 - w_2}{2\,w_0}.$$

Thus, $Q_2(w_0, w_1) = 0$. Now let us fix a number $n > 2$ and suppose that (4) holds for $k = 2, \dots, n-1$. Then we get

$$\begin{aligned} w_n &= {b_1}^n - \sum_{i=1}^{n-1} c_{ni}{b_0}^i{b_1}^{n-1-i}b_{i+1} = \\ &= {b_1}^n - \sum_{i=1}^{n-2} c_{ni}{b_0}^i{b_1}^{n-1-i}b_{i+1} - c_{n,n-1}{b_0}^{n-1}b_n. \end{aligned}$$

Hence

$$b_n = \frac{-w_n + {b_1}^n - \sum_{i=1}^{n-2} c_{ni}{b_0}^i{b_1}^{n-1-i}b_{i+1}}{c_{n,n-1}{b_0}^{n-1}}. \tag{5}$$

Now, we have $b_0 = w_0$, $b_1 = w_1$ and for $b_2, \dots, b_{n-1}$, we substitute from the formulas (4). This means that, we can put

$$b_{i+1} = \frac{{w_1}^{i+1} + (-1)^i i w_{i+1} + Q_{i+1}(w_0, \dots, w_i)}{(i+1)!\,{w_0}^i}$$

for $i = 1, \dots, n-2$, where $Q_{i+1}(w_0, \dots, w_i)$ is a polynomial which can involve only the powers of w_1 until the degree $i - 1$. The formula (5) is now in the form

$$b_n = \frac{-w_n + {w_1}^n - \sum_{i=1}^{n-2} \frac{c_{ni}}{(i+1)!}{w_1}^{n-1-i}\big({w_1}^{i+1} + (-1)^i i w_{i+1} + Q_{i+1}(w_0, \dots, w_i)\big)}{c_{n,n-1}{w_0}^{n-1}}.$$

Now we multiply the numerator and the denominator by $n - 1$. In the denominator, we obtain $(n-1)c_{n,n-1}{w_0}^{n-1} = (-1)^n n!\,{w_0}^{n-1}$. A small technical calculation shows that $(n-1)\frac{c_{ni}}{(i+1)!} = (-1)^{i+1}\binom{n}{i+1}$. In particular, these coefficients are integers. Another combinatorial calculation shows that the coefficient by ${w_1}^n$ is

$$(n-1) - \sum_{i=1}^{n-2}(n-1)\frac{c_{ni}}{(i+1)!} = (n-1) - \sum_{i=1}^{n-2}(-1)^{i+1}\binom{n}{i+1} = (-1)^n.$$

Hence, after simplifying the signs, we can write

$$b_n = \frac{{w_1}^n + (-1)^{n-1}(n-1)w_n + Q_n(w_0, \dots, w_{n-1})}{n!\,{w_0}^{n-1}},$$

where $Q_n(w_0, \dots, w_{n-1})$ is a polynomial which can involve only the powers of w_1 until the degree $n - 2$. The last statement follows easily from Lemma 3. ∎

It is easy to see that Lemma 3 is still valid for the original operator Z_n from (1) and for the original variables a_i, because it is obviously invariant with respect to the transformation of variables from Lemma 2. In particular, this transformation puts every polynomial $P(b_0, b_1, \dots, b_n)$ with integral coefficients into a polynomial $P'(a_0, a_1, \dots, a_n)$ with rational coefficients. The polynomials $w_k(b_0, b_1, \dots, b_n)$, $k = 0, 2, \dots, n$ change into new polynomials $v_k(a_0, a_1, \dots, a_n)$, which form a Hilbert basis of invariants for the original operator Z_n. If we multiply all of them by proper integers, all invariants become those with integral coefficients. We should stress here that the sequence of invariants w_i does not depend on the dimension n. However, the substitution from Lemma 2 is different for different n. Hence, coefficients by monomials in polynomials v_i depend on the dimension n. See the examples in the next Section.

For the invariants v_i, we cannot provide an explicit analogue of the formula (4), because of the complicated coefficients. But, we easily get the following

Lemma 5. *The expressions of a_k through v_i are rational mappings with certain powers of v_0 in the denominators. Each numerator in the formula for a_k has constant weight k with respect to variables v_i.*

Let now the subgroup

$$g_2(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

of $\mathrm{SL}(2, \mathbb{R})$ act in a standard way on the space $\mathbb{R}^{n+1}[a_0, a_1, \dots, a_n]$ of coefficients. The induced Killing vector field is of the form

$$Y_n = a_1 \frac{\partial}{\partial a_0} + 2\, a_2 \frac{\partial}{\partial a_1} + \cdots + (n-1)\, a_{n-1} \frac{\partial}{\partial a_{n-2}} + n\, a_n \frac{\partial}{\partial a_{n-1}}. \tag{6}$$

We can see that the operator (1) is transformed into the operator (6) via the involutive permutation $p\colon (a_0, a_1, \dots, a_n) \mapsto (a_n, a_{n-1}, \dots, a_1, a_0)$. The corresponding invariants $v_0 \circ p, v_2 \circ p, \dots, v_n \circ p$ of (6) will be denoted as $u_0, u_2, \dots, u_n$. These invariants are new functions of $a_0, a_1, \dots, a_n$ and they form a Hilbert basis for the polynomial invariants with respect to the action of $g_2(t)$. For the completeness, we put $u_1 = a_{n-1}$.

Lemma 6. *The invariants $u_0, u_2 \dots, u_n$ are also homogeneous polynomials of constant weights.*

Proof. Let v_k be an invariant in P_n which is homogeneous of degree d and weight s. Because the permutation p changes each variable a_i into the variable a_{n-i}, the term $u_k = v_k \circ p$ is again homogeneous of degree d and it has constant weight $nd - s$. ■

Now we can formulate our basic result:

Theorem 7. *To each standard representation of the group $\mathrm{SL}(2, \mathbb{R})$ on the space $\mathbb{R}^{n+1}$ of parameters we can attach at least one involutive rational mapping of the set $\mathbb{R}^{n+1} \setminus D$ onto itself, where D is a subset of measure zero.*

Proof. Express all variables $a_0, a_1, \dots, a_n$ through the quantities $v_0, v_1, \dots, v_n$ as in Lemma 5 and substitute into the functions $u_0, u_1, \dots, u_n$. We see easily that the corresponding expressions $u_k = R_k(v_0, v_1, \dots, v_n)$, $k = 0, 1, \dots, n$, are rational functions whose denominators are all just powers of the variable $v_0 = a_0$. Using the involutive permutation $p\colon (a_0, a_1, \dots, a_n) \mapsto (a_n, a_{n-1}, \dots, a_1, a_0)$, we see that, conversely, the variables $v_0, v_1, \dots, v_n$ can be expressed through the variables $u_0, u_1, \dots, u_n$ exactly in the same form, i.e. $v_k = R_k(u_0, u_1, \dots, u_n)$. In the denominators of $R_k(u_0, u_1, \dots, u_n)$, there are just powers of the variable $u_0 = a_n$. It is obvious that both mappings are involutive and mutually inverse. To ensure the correctness, we have to consider our maps just on the set $\mathbb{R}^{n+1} \setminus D$, where D is the union of the hyperplanes defined by $v_0 = 0$ and $v_n = 0$. ■

Lemma 8. *The polynomials in the numerators of components $u_k = R_k(v_0, \dots, v_n)$ have constant weights with respect to variables v_i.*

Proof. Each a_k has weight k with respect to variables v_i and each u_l has weight $nd - s$ with respect to variables a_k, according to the proof of Lemma 6. After the substitution, the weight remains $nd - s$ with respect to variables v_i. ■

Let us also remark that we can reduce the degree of polynomials w_i, or v_i, respectively, in the Hilbert basis. For example, we obtain

$$\begin{aligned}
\widetilde{w}_4 &= ({w_2}^2 - w_4)/4\,{w_0}^2 = \\
&= 2\,b_0b_4 - 2\,b_1b_3 + {b_2}^2, \\
\widetilde{w}_5 &= (w_2w_3 - w_5)/6\,{w_0}^2 = \\
&= -5\,{b_0}^2b_5 + 5\,b_0b_1b_4 - b_0b_2b_3 - 2\,{b_1}^2b_3 + b_1{b_2}^2, \\
\widetilde{w}_{61} &= (({w_3}^2 - w_6)/{w_0}^2 - 9w_2\widetilde{w}_4)/9\,w_0 = \\
&= 16\,{b_0}^2b_6 - 16\,b_0b_1b_5 + 4\,b_0b_2b_4 + b_0{b_3}^2 + 6\,{b_1}^2b_4 - 6\,b_1b_2b_3 + 2\,{b_2}^3, \\
\widetilde{w}_{62} &= (({w_2}^3 - w_6)/4\,{w_0}^2 - 3w_2\widetilde{w}_4)/4\,w_0 = \\
&= 9\,{b_0}^2b_6 - 9\,b_0b_1b_5 + 3\,b_0b_2b_4 + 3\,{b_1}^2b_4 - 3\,b_1b_2b_3 + {b_2}^3.
\end{aligned}$$

The original invariants w_i and the reduced invariants $\widetilde{w}_i$ have the same weight and, still, the only summand involving the variable b_i is of the form $p{b_0}^q b_i$ for some integers p and q. We also see from the invariants $\widetilde{w}_{61}$ and $\widetilde{w}_{62}$ that this reduced basis is not uniquely determined. The reduced invariants are more suitable for the simplicity of the calculations which will follow.

3 Examples

In this Section, we construct involutive mappings for the examples up to dimension 5. We start with the operator

$$Z'_4 \;=\; b_0\frac{\partial}{\partial b_1} + b_1\frac{\partial}{\partial b_2} + b_2\frac{\partial}{\partial b_3} + b_3\frac{\partial}{\partial b_4}. \tag{7}$$

The invariants with respect to this operator are

$$\begin{aligned}
w_0 &= b_0, \\
w_2 &= 2\,b_0b_2 - {b_1}^2,
\end{aligned}$$

$$\begin{aligned} w_3 &= 3\,{b_0}^2 b_3 - 3\,b_0 b_1 b_2 + {b_1}^3, \\ \widetilde{w}_4 &= 2\,b_0 b_4 - 2\,b_1 b_3 + {b_2}^2. \end{aligned} \quad (8)$$

We have changed conveniently the sign of w_2. In the following, we will denote the reduced invariant $\widetilde{w}_4$ simply by w_4 and the corresponding invariant $\widetilde{v}_4$ also simply by v_4. For the completeness, we have $w_1 = b_1$. Now we will start from dimension 3.

3.1 Dimension 3

We will use polynomials $w_0, \dots, w_2$ after the transformation

$$a_0 = b_0, \quad a_1 = 2\,b_1, \quad a_2 = 2\,b_2.$$

The new polynomials are

$$\begin{aligned} v_0 &= a_0, \\ v_1 &= a_1, \\ v_2 &= 4\,a_0 a_2 - {a_1}^2. \end{aligned}$$

Using the permutation $p\colon (a_0, a_1, a_2) \mapsto (a_2, a_1, a_0)$, we obtain polynomials $u_0, \dots, u_2$ in the form

$$\begin{aligned} u_0 &= a_2, \\ u_1 &= a_1, \\ u_2 &= 4\,a_0 a_2 - {a_1}^2. \end{aligned}$$

From these formulas, we obtain the expressions of a_i using v_i, or u_i, respectively, in the form

$$\begin{aligned} a_0 &= v_0, & a_0 &= \frac{{u_1}^2 + u_2}{4\,u_0}, \\ a_1 &= v_1, & a_1 &= u_1, \\ a_2 &= \frac{{v_1}^2 + v_2}{4\,v_0}, & a_2 &= u_0. \end{aligned}$$

By the substitution of these formulas into the formulas above, we obtain easily the involutive mappings of $\mathcal{D} = \{(a_0, a_1, a_2) \in \mathbb{R}^3, a_0 \neq 0 \neq a_2\}$ onto itself in the form

$$\begin{aligned} u_0 &= \frac{{v_1}^2 + v_2}{4\,v_0}, & v_0 &= \frac{{u_1}^2 + u_2}{4\,u_0}, \\ u_1 &= v_1, & v_1 &= u_1, \\ u_2 &= v_2, & v_2 &= u_2. \end{aligned}$$

3.2 Dimension 4

We will use polynomials $w_0, \ldots, w_3$ after the transformation

$$a_0 = b_0, \quad a_1 = 3\,b_1, \quad a_2 = 6\,b_2, \quad a_3 = 6\,b_3.$$

The new polynomials are

$$\begin{aligned} v_0 &= a_0, & u_0 &= a_3, \\ v_1 &= a_1, & u_1 &= a_2, \\ v_2 &= 3\,a_0 a_2 - a_1{}^2, & u_2 &= 3\,a_3 a_1 - a_2{}^2, \\ v_3 &= 27\,a_0{}^2 a_3 - 9\,a_0 a_1 a_2 + 2\,a_1{}^3, & u_3 &= 27\,a_3{}^2 a_0 - 9\,a_3 a_2 a_1 + 2\,a_2{}^3. \end{aligned}$$

For the inverse expressions of a_i using v_i, we obtain

$$\begin{aligned} a_0 &= v_0, \\ a_1 &= v_1, \\ a_2 &= \frac{v_1{}^2 + v_2}{3\,v_0}, \\ a_3 &= \frac{v_1{}^3 + 3\,v_1 v_2 + v_3}{27\,v_0{}^2}. \end{aligned}$$

By the substitution of these formulas into the formulas for u_i above, we obtain the transformation from v_i to u_i. The transformation from u_i to v_i is obtained analogously. We write down only the first transformation, the formulas for the second one differ just by interchanging u_i and v_i.

$$\begin{aligned} u_0 &= \frac{v_1{}^3 + 3\,v_1 v_2 + v_3}{27\,v_0{}^2}, \\ u_1 &= \frac{v_1{}^2 + v_2}{3\,v_0}, \\ u_2 &= \frac{v_1{}^2 v_2 + v_1 v_3 - v_2{}^2}{9\,v_0{}^2}, \\ u_3 &= \frac{-v_1{}^3 v_3 + 6\,v_1{}^2 v_2{}^2 + 3\,v_1 v_2 v_3 + 2\,v_2{}^3 + v_3{}^2}{27\,v_0{}^3}. \end{aligned}$$

This involutive transformation maps the set
$\mathcal{D} = \{(a_0, \ldots, a_3) \in \mathbb{R}^4, a_0 \neq 0 \neq a_3\}$ onto itself.

3.3 Dimension 5

Here we will use polynomials $w_0, \ldots, w_4$ after the transformation

$$a_0 = b_0, \quad a_1 = 4\,b_1, \quad a_2 = 12\,b_2, \quad a_3 = 24\,b_3, \quad a_4 = 24 b_4.$$

For the new polynomials, we choose

$$\begin{aligned} v_0 &= 4\,a_0, & u_0 &= 4\,a_4, \\ v_1 &= a_1, & u_1 &= a_3, \end{aligned}$$

$$\begin{aligned} v_2 &= 8\,a_0a_2 - 3\,a_1{}^2, & u_2 &= 8\,a_4a_2 - 3\,a_3{}^2,\\ v_3 &= 8\,a_0{}^2a_3 - 4\,a_0a_1a_2 + a_1{}^3, & u_3 &= 8\,a_4{}^2a_1 - 4\,a_4a_3a_2 + a_3{}^3,\\ v_4 &= 12\,a_0a_4 - 3\,a_1a_3 + a_2{}^2, & u_4 &= 12\,a_0a_4 - 3\,a_1a_3 + a_2{}^2, \end{aligned}$$

because coefficients in components of involutive transformation will be simpler with this choice for v_0. We obtain the inverse transformation

$$\begin{aligned} a_0 &= \frac{1}{4}\,v_0,\\ a_1 &= v_1,\\ a_2 &= \frac{3\,v_1{}^2 + v_2}{2v_0},\\ a_3 &= \frac{v_1{}^3 + v_1v_2 + 2v_3}{v_0{}^2},\\ a_4 &= \frac{4\,v_0{}^2v_4 + 24\,v_1v_3 + 6\,v_1{}^2v_2 + 3\,v_1{}^4 - v_2{}^2}{12\,v_0{}^3}. \end{aligned}$$

By the substitution of these formulas into the formulas for u_i above, we obtain again an involutive mapping which maps the set $\mathcal{D} = \{(a_0, \ldots, a_4) \in \mathbb{R}^5, a_0 \neq 0 \neq a_4\}$ onto itself and which is given by the formulas

$$\begin{aligned} u_0 &= \frac{4\,v_0{}^2v_4 + 3\,v_1{}^4 + 6\,v_1{}^2v_2 + 24\,v_1v_3 - v_2{}^2}{3\,v_0{}^3},\\ u_1 &= \frac{v_1{}^3 + v_1v_2 + 2\,v_3}{v_0{}^2},\\ u_2 &= \Big[12\,v_0{}^2v_1{}^2v_4 + 4\,v_0{}^2v_2v_4 + 3\,v_1{}^4v_2 + 36\,v_1{}^3v_3\\ &\quad -6\,v_1{}^2v_2{}^2 - 12\,v_1v_2v_3 - v_2{}^3 - 36\,v_3{}^2\Big]/3\,v_0{}^4,\\ u_3 &= \Big[8\,v_0{}^4v_1v_4{}^2 - 6\,v_0{}^2v_1{}^5v_4 + 60\,v_0{}^2v_1{}^2v_3v_4 - 10\,v_0{}^2v_1v_2{}^2v_4\\ &\quad -12\,v_0{}^2v_2v_3v_4 - 9\,v_1{}^6v_3 + 6\,v_1{}^5v_2{}^2 + 45\,v_1{}^4v_2\,v_3 + 180\,v_1{}^3v_3{}^2\\ &\quad -15\,v_1{}^2v_2{}^2v_3 + 2\,v_1v_2{}^4 + 36\,v_1v_2v_3{}^2 + 3\,v_2{}^3v_3 + 72\,v_3{}^3\Big]/9\,v_0{}^6,\\ u_4 &= v_4. \end{aligned}$$

4 Complexity of involutive rational mappings

Let us first stress the geometrical aspect of involutive mappings constructed in the previous Section. It follows from the fact that polynomials v_i and u_i were invariants of the operators generating the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. The involutive property was induced by the involutive permutation which interchanged the variables of these polynomials.

Definition 9. *Let ϕ be a rational mapping of $\mathbb{R}^{n+1} \setminus D$, where D is a subset of measure zero, to itself. Let ϕ_i, $i = 0, 1, \ldots, n$ be its components expressed in a reduced form (i.e., there are no nontrivial common factors in numerators and corresponding denominators). By the complexity of the mapping ϕ we mean the highest degree occurring among monomials involved in the numerators of the components ϕ_i.*

It can be proved that, with increasing dimension of the representation, the complexity of corresponding geometrical involutive mappings also increases. But, the exact estimate from below is not important, because, formally, we are able to construct rational involutive mapping with any complexity, as the following example shows. Let us consider polynomials

$$\begin{aligned} y_0 &= c_0, & z_0 &= c_3, \\ y_1 &= c_1, & z_1 &= c_2, \\ y_2 &= {c_1}^k - 2\,c_0c_2, & z_2 &= {c_2}^k - 2\,c_3c_1, \\ y_3 &= {c_1}^m - 3\,{c_0}^2c_3, & z_3 &= {c_2}^m - 3\,{c_3}^2c_0, \end{aligned}$$

where k and m are integers, $k \geq 2$, $m \geq 3$. We obtain the expressions of c_i via y_i in the form

$$\begin{aligned} c_0 &= y_0, \\ c_1 &= y_1, \\ c_2 &= \frac{{y_1}^k - y_2}{2\,y_0}, \\ c_3 &= \frac{{y_1}^m - y_3}{3\,{y_0}^2} \end{aligned}$$

and the involutive mapping

$$\begin{aligned} z_0 &= \frac{{y_1}^m - y_3}{3\,{y_0}^2}, \\ z_1 &= \frac{{y_1}^k - y_2}{2\,y_0}, \\ z_2 &= \frac{3\,({y_1}^k - y_2)^k + 2^{k+1}{y_0}^{k-2}y_1y_3 - 2^{k+1}{y_0}^{k-2}{y_1}^{m+1}}{3\cdot 2^k{y_0}^k}, \\ z_3 &= \frac{3\,({y_1}^k - y_2)^m - 2^m{y_0}^{m-3}{y_3}^2 + 2^{m+1}{y_0}^{m-3}{y_1}^m y_3 - 2^m{y_0}^{m-3}{y_1}^{2m}}{3\cdot 2^m{y_0}^m}. \end{aligned}$$

We see that the complexity of the corresponding “formal” involutive rational mapping depends essentially on the degree of polynomials y_2 and y_3. Thus, we can construct examples of involutive rational mappings of arbitrary complexity.

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