Abstract Sectional Category*

F.J. Díaz   J.M. García Calcines   P.R. García Díaz

A. Murillo Mas   J. Remedios Gómez

Abstract

We study, in an abstract axiomatic setting, the notion of sectional category of a morphism. From this, we unify and generalize known results about this invariant in different settings as well as we deduce new applications.

Introduction

The sectional category secat (p) of a fibration $p : E \twoheadrightarrow B$, originally introduced by A. Schwarz [20], is defined as the least integer n such that B admits a cover constituted by $n + 1$ open subsets, on each of which p has a local section. It is a lower bound of the Lusternik-Schnirelmann category of the base space and it is also a generalization of this invariant since secat $(p) = \text{cat}(B)$ when E is contractible. Apart from the original applications of the sectional category in the classification of bundles or the embedding problem [20], this numerical invariant has proved to be useful in different settings. For instance, Smale [21] showed that the sectional category of a certain fibration provides a lower bound for the complexity of algorithms computing the roots of a complex polynomial. We can also mention the work of M. Farber [8, 9] who introduced the topological complexity of a given space X as the sectional category of the path fibration $X^I \to X \times X$, $\alpha \mapsto (\alpha(0), \alpha(1))$. In robotics, when X is thought to be the configuration space associated to the motion of a given mechanical system, this invariant measures,

roughly speaking, the minimum amount of instructions of any algorithm control-ling the given system.

In general, the sectional category of a fibration is hard to compute. The no-tion of Lusternik-Schnirelmann category (L.-S. category, for short) has the same disadvantage. In order to face this problem for L.-S. category there have been several attempts to describe it in a more functorial and therefore manageable form; among the most successful ones we can mention the Whitehead and Ganea characterizations. Many other approximations of L.-S. category have been in-troduced. One of them relies in an important algebraic technique for obtaining lower bounds. It consists of taking models of spaces in an algebraic category where a notion of L.-S.-category type invariant is given. Such algebraic category must posses an abstract notion of homotopy, usually established in an axiomatic homotopy setting, such as a Quillen model category. Then the algebraic L.-S. category of the model of X is a lower bound of the original L.-S. category of X. During the progress of this technique, several algebraic notions of L.-S. category have been appearing. In 1993, in order to give a common point for all of them, Doeraene [6] introduced the notion of L.-S. category in a Quillen model category. Actually, in his work Doeraene develops two different notions of L.-S. category, which are the analogous to the Ganea and Whitehead characterizations in the topological case and proves that, under the crucial cube axiom, these notions agree, as expected. As far as the sectional category is concerned, not much has been done in this direction. In the work of A. Schwarz [20] it was established a Ganea-type characterization of sectional category. Namely, if $p : E \twoheadrightarrow B$ is a fibration we can consider $j_n : *_B^n E \to B$, which is the n-th fold join of p. If the base space B is paracompact, then A. Schwarz proved that $\mathrm{secat}(p) \leq n$ if and only if j_n admits a (homotopy) section. Clapp and Puppe [5, Cor. 4.9] also ob-tained a Whitehead-type characterization of sectional category; more precisely, for a given map $p : E \to B$ with associated cofibration $\hat{p} : E \to \hat{B}$, $\mathrm{secat}(p) \leq n$ if and only if the diagonal map $\Delta_{n+1} : \hat{B} \to \hat{B}^{n+1}$ factors, up to homotopy, through the n-th fat wedge $T^n(\hat{p}) = \{(b_0, b_1, ..., b_n) \in \hat{B}^{n+1} : x_i \in E, \text{ for some } i\}$. With this characterization Fassò [10] studied the sectional category of the corresponding algebraic model of p in rational homotopy. These functorial characterizations in the topological case open a door through an axiomatization of sectional category. In this direction an initial advance has been made by T. Kahl in [15]. In his work he gives the notion of abstract sectional category through a certain variation of inductive L.-S. category in the sense of Hess-Lemaire [12].

Our aim in this paper is to develop, in the same spirit as Doeraene did in [6] with the L.-S. category, the notion of abstract sectional category through a Whitehead-Ganea approach, and to deduce some applications. In the first section we recall some background to set the axiomatic framework in which we shall work as well as the main tools that will be used. In §2 we introduce, under such approach, the concept of sectional category of a given morphism. Then, in §3 we present the main properties of this invariant and finally, in the fourth section, we give some applications.

1 Preliminaries: *J*-category and main notions.

In this paper we shall work in a *J*-category [6], which includes the cases of a pointed cofibration and fibration category in the sense of Baues [2] or a pointed proper model category [18, 19] as long as they satisfy the "cube lemma". The aim of this section is to provide some of the most important notions and properties given in such a homotopy setting. For proofs and more details the reader is referred to Doeraene's paper [6] or his thesis [7].

Explicitly, a *J*-category $\mathcal{C}$ is a category with a zero object 0 and endowed with three classes of morphisms called fibrations ($\twoheadrightarrow$), cofibrations ($\rightarrowtail$) and weak equivalences ($\xrightarrow{\sim}$), satisfying the following set of axioms (J1)-(J5) below. Recall that a morphism which is both a fibration (resp. cofibration) and a weak equivalence is called *trivial fibration* (resp. *trivial cofibration*). An object B is called *cofibrant model* if every trivial fibration $p : E \xrightarrow{\sim}\twoheadrightarrow B$ admits a section.

(J1) Isomorphisms are trivial cofibrations and also trivial fibrations. Fibrations and cofibrations are closed by composition. If any two of f, g, gf are weak equivalences, then so is the third.

(J2) The pullback of a fibration $p : E \twoheadrightarrow B$ and any morphism $f : B' \to B$

$$\begin{array}{ccc} E' & \xrightarrow{\overline{f}} & E \\ \downarrow{\scriptstyle \overline{p}} & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{f} & B \end{array}$$

always exists and $\overline{p}$ is a fibration. Moreover, if f (resp. p) is a weak equivalence, then so is $\overline{f}$ (resp. $\overline{p}$). The dual assertion is also required.

(J3) For any map $f : X \to Y$ there exist an *F*-factorization (i.e., $f = p\tau$ where τ is a weak equivalence and p is a fibration) and a *C*-factorization (i.e., $f = \sigma i$, where i is a cofibration and σ is a weak equivalence).

(J4) For any object X in $\mathcal{C}$, there exists a trivial fibration $p_X : \overline{X} \xrightarrow{\sim}\twoheadrightarrow X$, in which $\overline{X}$ is a cofibrant model. The morphism $p_X : \overline{X} \xrightarrow{\sim}\twoheadrightarrow X$ is called *cofibrant replacement* for X.

A commutative square

$$\begin{array}{ccc} D & \xrightarrow{f'} & C \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & B \end{array}$$

is said to be a *homotopy pullback* if for some (equivalently *any*) *F*-factorization of g (equivalently f or both), the induced map from D to the pullback $E' = A \times_B E$ is

a weak equivalence
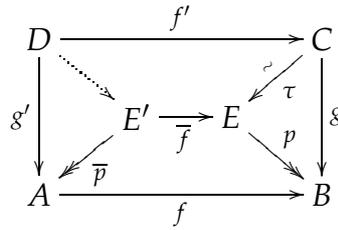


The notion of *homotopy pushout* is dually defined.

(J5) *The cube axiom.* Given any commutative cube where the bottom face is a homotopy pushout and the vertical faces are homotopy pullbacks, then the top face is a homotopy pushout.

Remark 1. *As pointed out by Doeraene, (J1)-(J4) axioms allow us to replace 'some' by 'any' in the definition of homotopy pullback, or to use an F-factorization of f instead of g.*

We are particularly interested in knowledge of objects and morphisms *up to weak equivalence*. Two objects A and A' in $\mathcal{C}$ are said to be *weakly equivalent* if there exists a finite chain of weak equivalences joining A and A'

$$A \overset{\sim}{\rule{1.5em}{0.4pt}} \bullet \overset{\sim}{\rule{1.5em}{0.4pt}} \bullet \cdots\cdots \bullet \overset{\sim}{\rule{1.5em}{0.4pt}} A'$$

where the symbol $\bullet \rule{1.5em}{0.4pt} \bullet$ means an arrow with either left or right orientation. One can analogously define the notion of *weakly equivalent morphisms* by considering a finite chain of weak equivalences in the category $\mathrm{Pair}(\mathcal{C})$ of morphisms in $\mathcal{C}$ ([2] Def. II.1.3)
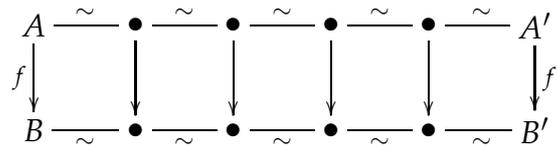
Definition 2. Given two morphisms $f : A \to B$ and $g : C \to B$, consider any *F*-factorization of $g = p\tau$ and the pullback of f and p. Let $\overline{f}$ and $\overline{p}$ the base extensions of f and p respectively. Then, take any *C*-factorization of $\overline{f} = \sigma i$ and the pushout of $\overline{p}$ and i. This pushout object is denoted by $A *_B C$ and is called the *join* of A and C over B. The dotted induced map from $A *_B C$ to B is called the *join morphism* of f and g.
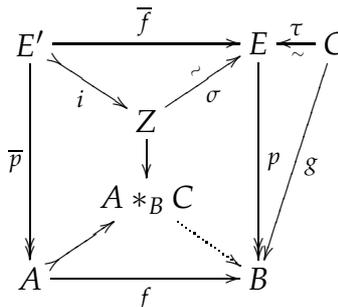
The object $A *_B C$ and the join map are well defined and they are symmetrical up to weak equivalence [6, 7].

An important result that allows us to see that if a property holds for some F-factorization, then it also holds for any F-factorization is the following lemma. Recall from [2] that in a fibration category a *relative cocylinder* of a fibration $p : E \twoheadrightarrow B$ is just an F-factorization of the morphism $(id_E, id_E) : E \to E \times_B E$, where $E \times_B E$ denotes the pullback of p with itself

$$E \xrightarrow{(id_E, id_E)} E \times_B E$$

$$\sim \searrow \quad \nearrow (d_0, d_1)$$

$$Z_p$$

Then, given $f, g : X \to E$ such that $pf = pg$, it is said that *f is homotopic to g relative to p* ($f \simeq g$ rel. p) if there exists a morphism $F : X \to Z_p$ such that $d_0 F = f$ and $d_1 F = g$. When $p = 0 : E \twoheadrightarrow 0$ is the zero morphism we obtain the notion of non relative homotopy (and write $f \simeq g$). In this case, the cocylinder of $0 : E \twoheadrightarrow 0$ will be denoted by E^I.

Lemma 3. [2, II.1.11] Consider a commutative diagram of unbroken arrows:

$$\begin{array}{ccc} D & \xrightarrow{g} & E \\ \tau \downarrow \sim & \nearrow{}_l & \downarrow p \\ A & \xrightarrow{f} & B \end{array}$$

(a) If A is a cofibrant model, then there is a morphism $l : A \to E$ such that $pl = f$.

(b) If A and D are cofibrant models, then there is a morphism $l : A \to E$ for which $pl = f$ and $l\tau \simeq g$ rel. p. Moreover, if g is a weak equivalence, then so is l.

We also recall the notion of weak lifting.

Definition 4. Let $f : A \to B$ and $g : C \to B$ be morphisms in $\mathcal{C}$. We say that f admits a *weak lifting along g* if for some F-factorization $g = p\tau$ of g and for some cofibrant replacement $p_A : \overline{A} \xrightarrow{\sim} A$ of A there exists a commutative diagram

$$\begin{array}{c} C \\ \tau \nwarrow \quad \downarrow g \\ E \\ s \nearrow \quad \searrow p \\ \overline{A} \xrightarrow{f p_A} B \end{array}$$

In the particular case $f = id_B$ we say that $g : C \to B$ admits a *weak section*.

This notion does not depend on the choice of the F-factorization nor on the cofibrant replacement. In order to check this fact one has to use Lemma 3 above and the following result. The details are left to the reader.

Lemma 5. [2, II.1.6] Let $p : X \xrightarrow{\sim} Y$ be a trivial fibration and $f : A \to Y$ any morphism, with A a cofibrant model. Then there exists a lift of f with respect to p, i.e. a morphism $\tilde{f} : A \to X$ such that $p\tilde{f} = f$

$$
\begin{array}{ccc}
 & & X \\
 & {}^{\tilde{f}} \nearrow & {}^{\sim}\downarrow p \\
A & \xrightarrow[f]{} & Y
\end{array}
$$

Another important notion that will be used in this paper is the one of weak pullback.

Definition 6. Let $f : A \to B$, $f' : A' \to B'$ and $b : B \to B'$ be morphisms in $\mathcal{C}$. It is said that A-A'-B'-B is a *weak pullback* if for some F-factorization $f' = p\tau$ and some cofibrant replacement $p_A : \overline{A} \xrightarrow{\sim} A$ of A there exists a homotopy pullback

$$
\begin{array}{ccc}
\overline{A} & \xdashrightarrow{x} X \xleftarrow[\sim]{\tau} & A' \\
{}^{fp_A}\downarrow & \text{h.p.b.} \,\,{}_p\searrow & \downarrow f' \\
B & \xrightarrow[b]{} & B'
\end{array}
$$

Remark 7. *Any homotopy pullback is a weak pullback. Again, Lemma 3, axiom (J4) and Lemma 5 allow us to replace the word 'some' by 'any' in the above definition. We also have to take into account that the composition of homotopy pullbacks is a homotopy pullback (in fact there is a Prism Lemma for homotopy pullbacks [6, Prop. 1.1]) and that the weak equivalences in the category $Pair(\mathcal{C})$ of morphisms in $\mathcal{C}$ are homotopy pullbacks.*

2   Sectional category. Ganea and Whitehead approaches.

As in Doeraene's work, from now on we will assume that $\mathcal{C}$ *is a J-category in which all objects are cofibrant models*. Therefore we will take as cofibrant replacements the corresponding identities. It is important to remark that in a general *J*-category we will also obtain the same results. However, the exposition and/or the arguments in this general case would be affected by unessential technical complications. So just for the sake of simplicity and comfort we admit this assumption without loss of generality. Essentially, the key to pass from our assumption to the general case is established by considering cofibrant replacements:

- *Any object X in $\mathcal{C}$ has a cofibrant replacement, that is, a trivial fibration $p_X : \overline{X} \xrightarrow{\sim}$ X, in which $\overline{X}$ is a cofibrant model.* ((J4) axiom)

- *Any morphism $f : X \to Y$ in $\mathcal{C}$ has a cofibrant replacement, that is, given cofibrant replacements p_X, p_Y of X and Y, there exists an induced morphism $\overline{f} : \overline{X} \to \overline{Y}$ making commutative the following square*

$$
\begin{array}{ccc}
\overline{X} & \xrightarrow{\overline{f}} & \overline{Y} \\
{}^{p_X}\downarrow{}^{\sim} & & {}^{\sim}\downarrow{}^{p_Y} \\
X & \xrightarrow[f]{} & Y
\end{array}
$$

Observe that the second statement holds thanks to Lemma 5. Using these simple facts when necessary and working a little bit harder the reader should be able to prove our results when not all objects are cofibrant models.

We are now prepared for the definition of sectional category of a morphism in $\mathcal{C}$ under two different approaches. In the following definition, only axioms (J1)-(J4) are needed.

Definition 8. Let $p : E \to B$ be any morphism in $\mathcal{C}$ (not necessarily a fibration). We consider for each n a morphism $h_n : *_B^n E \to B$ inductively defined as follows:

1. $h_0 = p : E \to B$ (so $*_B^0 E = E$)

2. Assume that $h_{n-1} : *_B^{n-1} E \to B$ is already constructed. Then h_n is the join morphism of p and h_{n-1} :
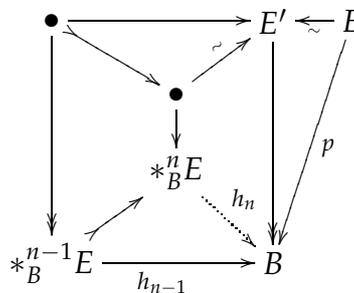
Then, the *Ganea sectional category of* p, Gsecat(p), is the least integer $n \leq \infty$ such that h_n admits a weak section
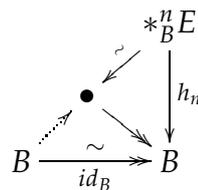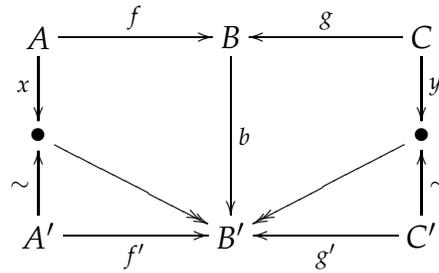

Remark 9. *Observe that Gsecat$(p) = 0$ if and only if p has a weak section. Moreover, in the topological setting this invariant coincides with secat(p), the classical sectional category of a given fibration $p : E \twoheadrightarrow B$, with B paracompact. In fact, the n-th iterated join of p over B, $h_n : *_B^n E \to B$ has a homotopy section if and only if B can be covered by $n + 1$ open subsets, each of them having a local homotopy section [14, 20].*

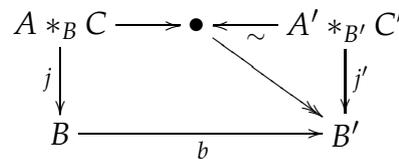Now we show that Gsecat is invariant up to weak equivalence.

Proposition 10. *If $p : E \to B$ and $p' : E' \to B'$ are weakly equivalent morphisms, then* Gsecat$(p) = $ Gsecat(p').

For the proof we shall use the following result.

**Lemma 11.** [6, Lemma 3.5] Consider the following commutative diagram in $\mathcal{C}$

$$
\begin{array}{ccccc}
A & \xrightarrow{f} & B & \xleftarrow{g} & C \\
\downarrow{x} & & \downarrow{b} & & \downarrow{y} \\
\bullet & & & & \bullet \\
\uparrow{\sim} & \searrow & \downarrow & \swarrow & \uparrow{\sim} \\
A' & \xrightarrow{f'} & B' & \xleftarrow{g'} & C'
\end{array}
$$

That is, bf admits a weak lifting along f' and bg admits a weak lifting along g'.
Let $j : A *_B C \to B$ and $j' : A' *_{B'} C' \to B'$ denote the corresponding join maps.
Then bj admits a weak lifting along j'

$$
\begin{array}{ccc}
A *_B C \longrightarrow \bullet \xleftarrow{\sim} A' *_{B'} C' \\
\downarrow{j} \qquad\qquad\qquad\qquad \downarrow{j'} \\
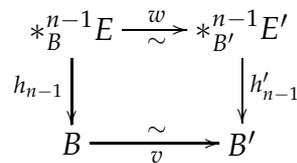B \xrightarrow{\quad b \quad} B'
\end{array}
$$

Furthermore, if b, x and y are weak equivalences, then $A *_B C$ is weakly equivalent to $A' *_{B'} C'$ via the above diagram.
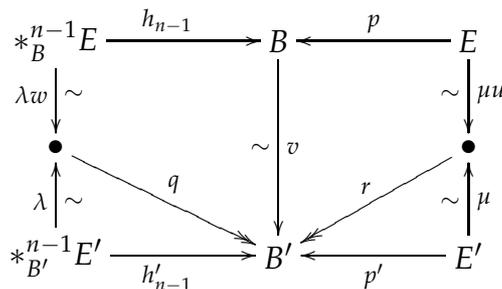
Proof of Proposition 10. We can suppose without loss of generality that there is a commutative diagram of the following form

$$
\begin{array}{ccc}
E & \xrightarrow{\ \sim\ u\ } & E' \\
\downarrow{p} & & \downarrow{p'} \\
B & \xrightarrow{\ \sim\ v\ } & B'
\end{array}
$$
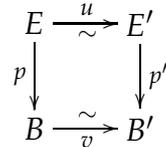
Let us see by induction on n that $h_n : *_B^n E \to B$ and $h'_n : *_{B'}^n E' \to B'$ are weakly equivalent morphisms. Indeed, for $n = 0$ it is certainly true. Now suppose that h_{n-1} and h'_{n-1} are weakly equivalent. Again we can assume, without loss of generality, that there is a commutative square

$$
\begin{array}{ccc}
*_B^{n-1}E & \xrightarrow{\ \sim\ w\ } & *_{B'}^{n-1}E' \\
\downarrow{h_{n-1}} & & \downarrow{h'_{n-1}} \\
B & \xrightarrow{\ \sim\ v\ } & B'
\end{array}
$$

Now take $h'_{n-1} = q\lambda$ and $p' = r\mu$ F-factorizations. Then we have a commutative diagram

$$
\begin{array}{ccccc}
*_B^{n-1}E & \xrightarrow{h_{n-1}} & B & \xleftarrow{p} & E \\
\downarrow{\lambda w}{\sim} & & \downarrow{\sim}{v} & & \downarrow{\sim}{\mu u} \\
\bullet & & & & \bullet \\
\uparrow{\lambda}{\sim} \searrow^{q} & & & \swarrow^{r} & \uparrow{\mu}{\sim} \\
*_{B'}^{n-1}E' & \xrightarrow{h'_{n-1}} & B' & \xleftarrow{p'} & E'
\end{array}
$$

which, applying Lemma 11, gives rise to this one

$$\begin{array}{ccc}
*^n_B E & \xrightarrow{\sim} \bullet \xleftarrow{\sim} & *^n_{B'} E' \\
\downarrow{\scriptstyle h_n} & \searrow & \downarrow{\scriptstyle h'_n} \\
B & \xrightarrow[v]{\sim} & B'
\end{array}$$

Now we have that h_n admits a weak section if and only if h'_n admits a weak section. In order to check this assertion, one has just to take into account Lemma 3 and the fact that the pullback of $\bullet \twoheadrightarrow B'$ and $v : B \xrightarrow{\sim} B'$ gives rise to an F-factorization of h_n in a natural way. ∎

Now we give a Whitehead-type definition of sectional category.

Definition 12. Let $p : E \to B$ be any morphism in $\mathcal{C}$ where B is e-fibrant, that is, the zero morphism $B \to 0$ is a fibration. We define $j_n : T^n(p) \to B^{n+1}$ inductively as follows:

1. $j_0 = p : E \to B$ (so $T^0(p) = E$)

2. If $j_{n-1} : T^{n-1}(p) \to B^n$ is constructed, then j_n is the following join construction:

$$\begin{array}{ccc}
\bullet \longrightarrow \bullet \xleftarrow{\sim} B^n \times E \\
\downarrow \quad \nearrow \quad \downarrow \quad \diagdown \\
\bullet \\
\downarrow \\
T^n(p) \dashrightarrow{\scriptstyle j_n} \\
\nearrow \qquad \downarrow \\
T^{n-1}(p) \times B \xrightarrow[j_{n-1} \times id_B]{} B^{n+1}
\end{array}$$

Then the *Whitehead sectional category of p*, $\mathrm{Wsecat}(p)$, is the least integer $n \le \infty$ such that the diagonal morphism $\Delta_{n+1} : B \to B^{n+1}$ admits a weak section along $j_n : T^n(p) \to B^{n+1}$:

$$\begin{array}{ccc}
& T^n(p) \\
& \diagup \quad \downarrow{\scriptstyle j_n} \\
\bullet \\
\diagup \quad \searrow \\
B \xrightarrow[\Delta_{n+1}]{} B^{n+1}
\end{array}$$

Observe that, in order to define $\mathrm{Wsecat}(p)$, B must be e-fibrant to ensure that all products B^n, $T^n(p) \times B$ and $B^n \times E$ exist ($n \ge 0$). Now we extend $\mathrm{Wsecat}(p)$ to the general case, in which B need not be e-fibrant. For this, consider an F-factorization $B \xrightarrow[\sim]{\tau} F \longrightarrow\!\!\!\!\!\rightarrow 0$ of the zero morphism. Then we define

$$\mathrm{Wsecat}(p) := \mathrm{Wsecat}(\tau p)$$

Lemma 13. If $p : E \to B$ is any morphism, then $\mathrm{Wsecat}(p)$ does not depend on the choice of the F-factorization for $B \to 0$.

Proof. Consider two such factorizations $B \xrightarrow{\tau}_{\sim} F \twoheadrightarrow 0$ and $B \xrightarrow{\tau'}_{\sim} F' \twoheadrightarrow 0$. Then, by Lemma 3(b) applied to the following commutative diagram

$$
\begin{array}{ccc}
B & \xrightarrow{\tau}_{\sim} & F \\
{\scriptstyle \tau'}\big\downarrow{\scriptstyle \sim} & {\scriptstyle h} \nearrow & \big\downarrow \\
F' & \twoheadrightarrow & 0
\end{array}
$$

there exists a weak equivalence $h : F' \xrightarrow{\sim} F$ such that $h\tau' \simeq \tau$. Take a homotopy $H : B \to F^I$ satisfying $d_0 H = h\tau'$ and $d_1 H = \tau$ and consider the commutative diagram, where the codomain of each vertical arrow is an e-fibrant object

$$
\begin{array}{ccccccc}
E & = & E & = & E & = & E \\
{\scriptstyle \tau' p}\big\downarrow & & {\scriptstyle h\tau' p}\big\downarrow & & {\scriptstyle Hp}\big\downarrow & & {\scriptstyle \tau p}\big\downarrow \\
F' & \xrightarrow[h]{\sim} & F & \xleftarrow[d_0]{\sim} & F^I & \xrightarrow[d_1]{\sim} & F
\end{array}
$$

This diagram shows that τp and $\tau' p$ are weakly equivalent morphisms. Observe that, since $F \times F$ is e-fibrant and (by definition) there is a fibration $(d_0, d_1) : F^I \twoheadrightarrow F \times F$, we have that the cocylinder object F^I is also e-fibrant. Finally, considering a similar argument to that given in the proof of Proposition 10 we obtain the identity $\mathrm{Wsecat}(\tau p) = \mathrm{Wsecat}(\tau' p)$. ∎

Proposition 14. If $p : E \to B$ and $p' : E' \to B'$ are weakly equivalent morphisms, then $\mathrm{Wsecat}(p) = \mathrm{Wsecat}(p')$.

Proof. We can suppose, without loss of generality, that there is a commutative square

$$
\begin{array}{ccc}
E & \xrightarrow{u}_{\sim} & E' \\
{\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p'} \\
B & \xrightarrow[v]{\sim} & B'
\end{array}
$$

Now, if $B' \xrightarrow{\tau'}_{\sim} F' \twoheadrightarrow 0$ is an F-factorization of the zero morphism, then an F-factorization $B \xrightarrow{\tau}_{\sim} F \xrightarrow{w} F'$ of $\tau' v$ gives rise to $B \xrightarrow{\tau}_{\sim} F \twoheadrightarrow 0$, another F-factorization, and a commutative square

$$
\begin{array}{ccc}
E & \xrightarrow{u}_{\sim} & E' \\
{\scriptstyle \tau p}\big\downarrow & & \big\downarrow{\scriptstyle \tau' p'} \\
F & \xrightarrow[w]{\sim} & F'
\end{array}
$$

Again, the result follows considering a similar argument to that given in the proof of Proposition 10. ∎

We now see that Gsecat and Wsecat coincide in a *J*-category.

Theorem 15. If $p : E \to B$ is any morphism, then

$$\mathrm{Gsecat}(p) = \mathrm{Wsecat}(p).$$

We first recall some useful properties about weak pullbacks. Again we refer the reader to [6].

Lemma 16 (Prism Lemma for weak pullbacks). [6, Prop. 2.5] Consider the following diagram

$$
\begin{array}{ccccc}
A & & B & & C \\
\downarrow & & \downarrow & & \downarrow \\
X & \longrightarrow & Y & \longrightarrow & Z
\end{array}
$$

If *B-C-Z-Y* is a weak pullback, then *A-B-Y-X* is a weak pullback if and only if *A-C-Z-X* is a weak pullback.

Lemma 17. [6, Lemma 3.5] Consider a weak pullback

$$
\begin{array}{ccc}
D \longrightarrow \bullet \xleftarrow{\ \sim\ } C \\
{\scriptstyle g}\big\downarrow \quad \text{h.p.b.} \searrow \quad \big\downarrow {\scriptstyle g'} \\
A \xrightarrow[\ f\ ] B
\end{array}
$$

and let $h : X \to A$ be any morphism. Then h admits a weak lifting along g if and only if fh admits a weak lifting along g'.

And now we come to the Join Theorem. This result relies strongly on the cube axiom (J5 axiom) and therefore it does not admit a dual version.

Lemma 18 (Join Theorem). [6, Th. 2.7] If there are weak pullbacks

$$
\begin{array}{ccc}
A \longrightarrow X \xleftarrow{\ \sim\ } A' & \qquad & C \longrightarrow Y \xleftarrow{\ \sim\ } C' \\
{\scriptstyle f}\big\downarrow \ \ \text{h.p.b.} {\scriptstyle p} \searrow \ \big\downarrow & \qquad & {\scriptstyle g}\big\downarrow \ \ \text{h.p.b.} {\scriptstyle q} \searrow \ \big\downarrow \\
B \xrightarrow[\ b\ ] B' & \qquad & B \xrightarrow[\ b\ ] B',
\end{array}
$$

then there is a weak pullback

$$
\begin{array}{ccc}
A *_B C \longrightarrow \bullet \xleftarrow{\ \sim\ } A' *_{B'} C' \\
\big\downarrow \qquad \text{h.p.b.} \searrow \qquad \big\downarrow \\
B \xrightarrow[\ b\ ] B'
\end{array}
$$

Proof of Theorem 15. First suppose that B is e-fibrant. We will see by induction on $n \geq 0$ that for any map $p \colon E \to B$, there is weak pullback:

$$
\begin{array}{ccc}
*_B^n E \longrightarrow \bullet \xleftarrow{\ \sim\ } T^n(p) \\
{\scriptstyle h_n}\big\downarrow \quad \text{h.p.b.} \searrow \quad \big\downarrow {\scriptstyle j_n} \\
B \xrightarrow[\ \Delta_{n+1}\ ] B^{n+1}
\end{array}
$$

For $n = 0$ it is trivially true. Suppose the statement true for $n - 1$ and consider the diagram

$$
\begin{array}{ccccc}
*_B^{n-1}E & & T^{n-1}(p) \times B & \xrightarrow{\ pr\ } & T^{n-1}(p) \\
h_{n-1}\downarrow & \textcircled{1} & \downarrow j_{n-1}\times id_B & & \downarrow j_{n-1} \\
B & \xrightarrow{\ \Delta_{n+1}\ } & B^n \times B & \xrightarrow{\ pr\ } & B^n
\end{array}
$$

where the right square is a pullback in which $pr : B^n \times B \twoheadrightarrow B^n$ is a fibration (observe that B is e-fibrant and use (J2) axiom). Therefore this pullback is also a homotopy pullback and a weak pullback. Now, applying the Prism Lemma together with the induction hypothesis we deduce that diagram $\textcircled{1}$ is also a weak pullback.

The same argument applied to the diagram

$$
\begin{array}{ccccc}
E & \xrightarrow{(p,p,...,p,id_E)} & B^n \times E & \xrightarrow{\ pr\ } & E \\
p\downarrow & \textcircled{2} & \downarrow id_{B^n}\times p & & \downarrow p \\
B & \xrightarrow{\ \Delta_{n+1}\ } & B^n \times B & \xrightarrow{\ pr\ } & B
\end{array}
$$

implies that $\textcircled{2}$ is a weak pullback. We obtain the expected result by applying the Join Theorem to the weak pullbacks $\textcircled{1}$ and $\textcircled{2}$. The theorem easily follows now from this fact together with Lemma 17.

When B is not e-fibrant, consider $B \xrightarrow[\sim]{\tau} F \longrightarrow 0$ an F-factorization. Then we have that $Gsecat(p) = Gsecat(\tau p)$ by Proposition 10. But we have already proved that $Gsecat(\tau p) = Wsecat(\tau p)$ (=$Wsecat(p)$). $\blacksquare$

Remark 19. *When our category $\mathcal{C}$ does not satisfy the cube axiom (J5), the most we can say is that $Wsecat(p) \leq Gsecat(p)$. Indeed, an argument similar to the one used in Theorem 15 using Lemma 11 instead of Lemma 18, proves that for each $n \geq 0$, $\Delta_{n+1}h_n$ admits a weak lifting along j_n, i.e., there is a commutative diagram*

$$
\begin{array}{ccc}
*_B^n E \longrightarrow \bullet \xleftarrow{\ \sim\ } T^n(p) \\
h_n\downarrow \qquad\qquad\quad \searrow\ \ \downarrow j_n \\
B \xrightarrow{\ \Delta_{n+1}\ } B^{n+1}
\end{array}
$$

The general case, in which B is not necessarily e-fibrant, follows easily. Now, if id_B admits a weak lifting along h_n, then it is easy to check that $\Delta_{n+1} = \Delta_{n+1}id_B$ admits a weak lifting along $\Delta_{n+1}h_n$. Using Lemma 21 below we obtain that Δ_{n+1} admits a weak lifting along j_n.

From now on we will denote $Gsecat(p) = Wsecat(p)$ by $secat(p)$ and call it the *sectional category* of p.

3 Main properties of the sectional category

We begin by observing that the Lusternik Schnirelmann category of an object B in $\mathcal{C}$ is the sectional category of the zero morphism $0 \to B$. Indeed (see [6]) the n-th Ganea map $p^n \colon G^n B \to B$ is precisely the n-th join over B, $h_n \colon *_B^n E \to B$, of $0 \to B$ and therefore,

$$\mathrm{cat}(B) = \mathrm{secat}(0 \to B).$$

On the other hand, given $b : B \to B'$ any morphism, we define $\mathrm{cat}(b)$ as the least integer $n \leq \infty$ such that b admits a weak lifting along $p'^n \colon G^n B' \to B'$. Compare the next result with [15].

Theorem 20. Let $p : E \to B$, $p' : E' \to B'$ and $b \colon B \to B'$ be morphisms in $\mathcal{C}$ defining a weak pullback. Then,

$$\mathrm{secat}(p) \leq \min\{\mathrm{cat}(b), \mathrm{secat}(p')\}.$$

For its proof we need the following lemma.

Lemma 21. [6, Lemma 3.4] Let $f : A \to B$, $g : C \to B$ and $h : D \to B$ be morphisms. If f admits a weak lifting along g and g admits a weak lifting along h, then f admits a weak lifting along h.
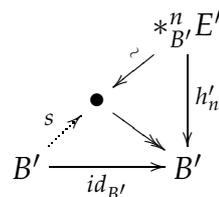
Proof of 20. By induction, using the Join Theorem (Lemma 18) repeatedly on the given weak pullback
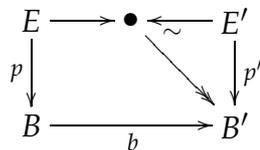
we obtain, for every $n \geq 0$, a weak pullback of the form
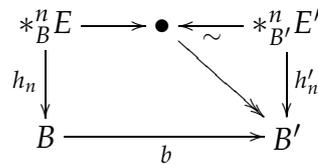
Hence, if $\mathrm{secat}(p') \leq n$, h'_n admits a weak section:

In particular, $b : B \to B'$ admits a weak lifting along h'_n through the morphism $sb : B \to \bullet$. By Lemma 17, h_n admits a weak section and $\mathrm{secat}(p) \leq n$.

Now suppose that $\mathrm{cat}(b) \leq n$, that is, b admits a weak lifting along $p'^n : G^n B' \to B'$. Consider the following diagram obtained by simply choosing any F-factorization of p':

$$
\begin{array}{ccc}
0 \longrightarrow \bullet \xleftarrow{\sim} E' \\
\downarrow \quad \searrow \quad \downarrow p' \\
B' \xrightarrow{\ id\ } B'
\end{array}
$$

As this is not, in general, a weak pullback, apply Lemma 11 inductively to obtain that $p'^n : G^n B' \to B'$ admits a weak lifting along $h'_n : *_{B'}^n E' \to B'$. Finally, by Lemma 21 we conclude that b admits a weak lifting along $h'_n : *_{B'}^n E' \to B'$, which by Lemma 17, is equivalent to the fact that $h_n : *_B^n E \to B$ admits a weak section. ∎

Even if our data is not a weak pullback, we can prove a similar result, which should be compared with [15]:

Theorem 22. Let $p : E \to B$ and $p' : E' \to B$ be morphisms in $\mathcal{C}$. If p admits a weak lifting along p', then $\mathrm{secat}(p') \leq \mathrm{secat}(p)$. In particular,

$$\mathrm{secat}(p) \leq \mathrm{cat}(B).$$

Moreover, if $p : E \to B$ admits a weak lifting along the zero morphism $0 \to B$ (in particular, when E is weakly contractible, i.e., E and 0 are weakly equivalent) then $\mathrm{secat}(p) = \mathrm{cat}(B)$.

Proof. For the first assertion, apply Lemma 11 inductively to the diagram

$$
\begin{array}{ccc}
E \longrightarrow \bullet \xleftarrow{\sim} E' \\
p\downarrow \quad \searrow \quad \downarrow p' \\
B \xrightarrow{\ id_B\ } B
\end{array}
$$

to conclude that, for every $n \geq 0$, h_n admits a weak lifting along h'_n. If $\mathrm{secat}(p) \leq n$, id_B admits a weak lifting along h_n and, by Lemma 21, id_B admits a weak lifting along h'_n. Hence, $\mathrm{secat}(p') \leq n$.

On the other hand, recall that $\mathrm{cat}(B) = \mathrm{secat}(0 \to B)$ and observe that the zero morphism admits a weak lifting along any morphism. Thus, $\mathrm{secat}(p) \leq \mathrm{cat}(B)$. Finally note that, if E is a weakly trivial object, by Lemma 3, p admits a weak lifting along $0 \to B$. ∎

3.1 Modelization functors.

We now study the behaviour of secat through a *modelization functor*. Recall from [6] that a covariant functor $\mu : \mathcal{C} \to \mathcal{D}$ between categories satisfying (J1)-(J4) axioms is called a modelization functor if it preserves weak equivalences, homotopy pullbacks and homotopy pushouts. We say that μ is *pointed* if $\mu(0) = 0$. If $\mu : \mathcal{C} \to \mathcal{D}$ is contravariant, it is said to be a modelization functor if the corresponding covariant functor $\mu : \mathcal{C}^{op} \to \mathcal{D}$ is a modelization functor. Here we prove:

Theorem 23. If $\mu : \mathcal{C} \to \mathcal{D}$ is a modelization functor between J-categories, then for any morphism $p : E \to B$ of $\mathcal{C}$

$$\mathrm{secat}(\mu(p)) \leq \mathrm{secat}(p)$$

For it we shall need the following

Lemma 24. [6, Prop. 6.7] Let $\mu : \mathcal{C} \to \mathcal{D}$ be a modelization functor and let $j : A *_B C \to B$ denote the join map of $f : A \to B$ and $g : C \to B$. Then, there is a commutative diagram

where j' denotes the join morphism of $\mu(f)$ and $\mu(g)$.

Proof of Theorem 23. In view of Lemma 21 it is sufficient to prove that, for each n, $h_n^{\mu(p)}$ admits a weak lifting along $\mu(h_n^p)$

where h_n^p and $h_n^{\mu(p)}$ are the n-th join morphisms p and $\mu(p)$ respectively. For $n = 0$ is trivially true. By assuming the assertion true for $n - 1$, and choosing any F-factorization of $\mu(p)$ we obtain a commutative diagram of the form

By Lemma 11 $h_n^{\mu(p)}$ admits a weak section along the join morphism of $\mu(h_{n-1}^p)$ and $\mu(p)$:

$$(3)$$

On the other hand, applying Proposition 24 above to the morphisms $h_{n-1}^p : *_B^{n-1}E \to B$ and $p : E \to B$, we obtain a commutative diagram

$$
\begin{array}{ccc}
\bullet & \xrightarrow{\ \sim\ } & \mu(*_B^n E) \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\mu(h_n^p)} \\
\mu(*_B^{n-1}E) *_{\mu(B)} \mu(E) & \longrightarrow & \mu(B)
\end{array}
$$

Taking any F-factorization of $\mu(h_n^p)$ and applying Lemma 3 we deduce that the join morphism $\mu(*_B^{n-1}E) *_{\mu(B)} \mu(E) \to \mu(B)$ admits a weak lifting along $\mu(h_n^p)$. Finally, by Lemma 21 applied to (3), we conclude the inductive step. ∎

Remark 25. *Observe that, for the proof of Theorem 23 we have used the Ganea-type version of sectional category. If (J5) axiom is not satisfied, then using similar arguments we can also obtain the same result for the Whitehead-type version of sectional category. The same remark also applies to the remaining results of this section.*

Corollary 26. Consider $\mu : \mathcal{C} \to \mathcal{D}$ and $\nu : \mathcal{D} \to \mathcal{C}$ modelization functors between *J*-categories and let $p : E \to B$ be a morphism in $\mathcal{C}$ such that $\nu(\mu(p))$ is weakly equivalent to p. Then

$$\operatorname{secat}(\mu(p)) = \operatorname{secat}(p)$$

As an example we apply the theorem above to the *abstract topological complexity* of a given object. For any e-fibrant object B we define its topological complexity, $TC(B)$, as the sectional category of the diagonal morphism $\Delta_B : B \to B \times B$. If B is not e-fibrant consider any F-factorization $B \xrightarrow{\ \sim\ } F \longrightarrow 0$ and set

$$TC(B) := TC(F).$$

Then, $TC(B)$ does not depend on the e-fibrant object F; indeed, if we take another F-factorization $B \xrightarrow{\ \sim\ } F' \longrightarrow 0$, then there exists a weak equivalence $h : F' \xrightarrow{\sim} F$ (see the proof of Lemma 13). The naturality of the diagonal morphism applied to h together with the fact that $h \times h : F' \times F' \xrightarrow{\sim} F \times F$ is a weak equivalence (by the dual of the Gluing Lemma [2, II.1.2]) prove that $\Delta_F : F \to F \times F$ and $\Delta_{F'} : F' \to F' \times F'$ are weakly equivalent morphisms. Therefore

$$TC(F) = \operatorname{secat}(\Delta_F) = \operatorname{secat}(\Delta_{F'}) = TC(F').$$

The invariant $TC(B)$ only depends on the weak type of B; given $f : B \xrightarrow{\sim} B'$ a weak equivalence, if we consider an F-factorization $B' \xrightarrow[\sim]{\tau'} F' \longrightarrow 0$, then any F-factorization of the composite $\tau'f : B \to F'$

$$
\begin{array}{ccc}
B & \xrightarrow[\sim]{\tau'f} & F' \\
{\scriptstyle\tau}\searrow{\scriptstyle\sim} & & \nearrow{\scriptstyle g} \\
& F &
\end{array}
$$

gives rise to a trivial fibration $g : F \xrightarrow{\sim} F'$, which shows that

$$TC(B) = TC(F) = TC(F') = TC(B').$$

Theorem 27. For any pointed modelization functor $\mu : \mathcal{C} \to \mathcal{D}$ and any object B,

$$\text{TC}(\mu(B)) \leq \text{TC}(B)$$

Proof. Taking into account that μ preserves weak equivalences and TC only depends on the weak type, we can suppose without loss of generality that B is an e-fibrant object. Since $\mu(B)$ need not be e-fibrant we consider any F-factorization

so that $\text{TC}(\mu(B)) = \text{TC}(F)$. Now consider the following commutative cube:

Here pr_1 and pr_2 denote the projection morphisms. As μ is a pointed modelization functor, the top face is a homotopy pullback. On the other hand, the bottom face is a strict pullback (and a homotopy pullback) and $\omega = (\tau\mu(pr_1), \tau\mu(pr_2))$ is the induced morphism from the universal property of the pullback. Since the top and bottom faces are homotopy pullbacks and the unbroken vertical morphisms are weak equivalences, by [6, Cor. 1.12] (or the dual of the Gluing Lemma [2, II.1.2]) we see that ω is also a weak equivalence. From the following commutative diagram

we deduce that $\mu(\Delta_B)$ and Δ_F are weakly equivalent morphisms. Then, by Proposition 10 we have that $\text{TC}(\mu(B)) = \text{secat}(\Delta_F) = \text{secat}(\mu(\Delta_B))$ while, by Theorem 23, $\text{secat}(\mu(\Delta_B)) \leq \text{secat}(\Delta_B) = \text{TC}(B)$. $\blacksquare$

Corollary 28. Consider two pointed modelization functors $\mu : \mathcal{C} \to \mathcal{D}$ and $\nu : \mathcal{D} \to \mathcal{C}$ and let B be an object in $\mathcal{C}$ such that $\nu(\mu(B))$ is weakly equivalent to B. Then

$$\text{TC}(\mu(B)) = \text{TC}(B)$$

4   Some applications.

We start by an immediate application in rational homotopy theory. A classical fact [3, §8] assures the existence of an adjunction

$$\textbf{CDGA}^{\varepsilon} \underset{\langle \cdot \rangle}{\overset{A_{PL}}{\rightleftarrows}} \textbf{SSet}^{*}$$

between the categories of augmented commutative differential graded algebras over a field $\mathbb{K}$ of characteristic zero, and pointed simplicial sets. The category $\textbf{SSet}^{*}$ is known to be a *J*-category endowed with Kan fibrations, injective maps and maps realizing to homotopy equivalences [18, Chap.III§.3], [6, Prop.A.8]. The category $\textbf{CDGA}^{\varepsilon}$ is also a (proper) closed model category [3, §4] (and thus J1-J4 are satisfied) in which fibrations are surjective morphisms, weak equivalences are morphisms inducing homology isomorphisms (the so called "quasi-isomorphisms") and cofibrations are "relative Sullivan algebras" [11, §14], i.e., inclusions $A \to A \otimes \Lambda V$ in which ΛV denotes the free commutative algebra generated by the graded vector space V and the differential on $A \otimes \Lambda V$ satisfies a certain "minimality" condition. However, this is *not* a *J*-category and the Eckmann-Hilton dual of a partial version of the cube axiom is satisfied when restricting to 1-connected algebras [6, A.18]. The functors $\langle \cdot \rangle$ and A_{PL} do not in general respect weak equivalences although $\langle \cdot \rangle$ sends cofibrations to fibrations and $\langle \cdot \rangle$ can be slightly modified to send fibrations to cofibrations [3, §8]. Therefore, as they stand, they are not modelization functors. However, it is also known [3, §8,9] that, restricting those functors to the categories

$$\textbf{CDGA}^{1}_{cf\mathbb{Q}} \rightleftarrows \textbf{Kan-Complexes}^{1}_{\mathbb{Q}}$$

of cofibrant 1-connected commutative differential graded algebras of finite type over $\mathbb{Q}$ (known as Sullivan algebras [11, §12]) and 1-connected rational Kan complexes of finite type, then they do preserve weak equivalences and via [6, Prop.6.5] they are modelization functors.

On the other hand, in [10, Ch.8], Fassò introduced, for a map of finite type 1-connected CW-complexes, or equivalently for a simplicial map of finite type 1-connected Kan complexes $E \xrightarrow{p} B$, the *rational sectional category* of p, $\text{secat}_0(p)$ which can be seen as the sectional category in the opposite category of $\textbf{CDGA}^{1}_{cf\mathbb{Q}}$ of $A_{PL}(p_{\mathbb{Q}})$, being $p_{\mathbb{Q}}$ the map in $\textbf{Kan-Complexes}^{1}_{\mathbb{Q}}$ obtained by rationalization [3, §11]. Thus, by Corollary 26,

$$\text{secat}_0(p) = \text{secat}(p_{\mathbb{Q}})$$

Our second application concerns localization functors. Let P be a (possibly empty) set of primes and

$$(-)_P \colon \textbf{CW}_{\mathcal{N}} \longrightarrow \textbf{CW}_{\mathcal{N}}$$

denotes the P-localization functor (see [13, §2] or [1, Chap.III] where it is shown that localization can chosen to be a functor as it stands, not just in the homotopy category) in the pointed category of spaces of the homotopy type of nilpotent CW-complexes. Then, this functor sends homotopy pushouts to homotopy pushouts and homotopy pullbacks (if the chosen homotopy pullback stays in this category) to homotopy pullbacks [13, §7]. (Note that, considering closed cofibrations, Hurewicz fibrations and homotopy equivalences, the category of well pointed topological spaces **Top*** has the structure of a J-category; see [22, Thm.11] for axioms (J1)-(J4) plus [16, Thm.25] for (J5)). Thus, even though strictly speaking this is not a modelization functor as it is defined on a certain subcategory of **Top***, the arguments in Theorem 23 could be followed mutatis mutandi as long as all constructions there remain within our category. But this is in fact the case as the homotopy pullback (or pushout) of two maps in **CW**$_\mathcal{N}$ can be chosen to live also in this category [13, §7]. Hence,

$$\mathrm{secat}(f_P) \leq \mathrm{secat}\, f.$$

However, the situation is drastically different in the general case as all sort of possible P-localizations (extending the one on nilpotent complexes) do not, in general, preserve homotopy pullbacks and homotopy pushouts.

Here, we consider the Casacuberta-Peschke localization functor on **Top*** [4] and start by setting some notation. Given a group G we denote by P[G] the ring localization of the group ring $\mathbb{Z}_P G$ obtained by inverting all of the elements $1 + g + \cdots + g^{n-1}$, where $g \in G$, and $(n, p) = 1$ for any $p \in P$ (see [4, §2]).

Following [17] we say that a P-torsion group G is an *acting group* for a space X if there is an epimorphism $f : \pi_1 X \twoheadrightarrow G$ such that, for each $m \geq 2$, the action $\pi_1 X \to Aut(\pi_m X)$ factors through G.

Proposition 29. Let $f : X \to Y$ a map for which:

  (i) $\pi_1(*_Y^n f) : \pi_1(*_Y^n X) \xrightarrow{\cong} \pi_1 Y$ is an isomorphism of P-local groups for any $n \geq 0$.

  (ii) $\pi_1(*_Y^n X)$ and $\pi_1 Y$ have a common acting group G for any $n \geq 0$.

  (iii) If we denote $\pi_1 Y$ by π, the morphism $\mathbb{Z}_P \pi \to P[\pi]$ induces isomorphisms on homology with local coefficients $H_*(-; \mathbb{Z}_P \pi) \to H_*(-; P[\pi])$.

Then,

$$\mathrm{secat}(f_P) \leq \mathrm{secat}(f).$$

Proof. Again, note that the argument in Theorem 23 could be applied if, for any $n \geq 1$, there is a homotopy commutative diagram of the form:

To this end, an inductive process, as in [6, Prop.6.7] will work as long as the following two conditions hold:

(1) The localization of the homotopy pullback

$$
\begin{array}{ccc}
Q_n & \longrightarrow & X \\
\downarrow & & \downarrow f \\
*_Y^{n-1}X & \xrightarrow{h_{n-1}} & Y
\end{array}
$$

$$
\begin{array}{ccc}
(Q_n)_P & \longrightarrow & X_P \\
\downarrow & & \downarrow f_P \\
(*_Y^{n-1}X)_P & \xrightarrow{h_{n-1P}} & Y_P
\end{array}
$$

is again a homotopy pullback.

(2) The localization of the homotopy pushout

$$
\begin{array}{ccc}
Q_n & \longrightarrow & X \\
\downarrow & & \downarrow \\
*_Y^{n-1}X & \longrightarrow & *_Y^n X
\end{array}
$$

$$
\begin{array}{ccc}
(Q_n)_P & \longrightarrow & X_P \\
\downarrow & & \downarrow \\
(*_Y^{n-1}X)_P & \longrightarrow & (*_Y^n X)_P
\end{array}
$$

is again a homotopy pushout.

However, by hypothesis, we may apply [17, Thm.4.3] to prove statement (1) (respec. [17, Thm.2.1] to prove (2)).   ■

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Departamento de Matemática Fundamental
Universidad de La Laguna
38271 La Laguna, Spain.
email:fradiaz@ull.es,jmgarcal@ull.es,prgdiaz@ull.es,jremed@ull.es

Departamento de Algebra, Geometría y Topología
Universidad de Málaga
Ap.59, 29080 Málaga, Spain.
email:aniceto@agt.cie.uma.es