m-infrabarrelledness and m-convexity

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Abstract

m-infrabarrelledness, in the context of locally convex algebras, is considered to prove results previously obtained for barrelled algebras. Thus, any unital commutative m-infrabarrelled advertibly complete and pseudo-complete locally m-convex algebra with bounded elements has the Q-property; hence, it is functionally continuous (: all characters are continuous). In the framework of commutative GB^*-algebras with jointly continuous multiplication and bounded elements, the notions *m-infrabarrelled algebra* and *C^*-algebra* coincide. In unital uniform locally m-convex algebras, m-infrabarrelledness is equivalent to the Banach algebra structure, modulo pseudo-completeness. Moreover, m-infrabarrelledness for locally A-convex algebras (in particular, A-normed ones) is also examined.

1 Introduction

Infrabarrelledness has been introduced in the framework of locally convex spaces (see e.g. [12, p. 217, Definition 2]). A. Mallios considered m-infrabarrelledness in the setting of topological algebras ([16, pp. 306–307]). We first examine this notion in locally A-convex algebras. A barrelled locally A-convex algebra is actually a locally m-convex algebra ([6, p. 74]). We obtain that m-infrabarrelledness is sufficient modulo additional conditions (Proposition 3.1). Relative results can be found in [24, Proposition 2, Proposition 3]. Besides, it has been shown that any unital commutative Fréchet m-convex algebra every element of which is bounded is necessarily a Q-algebra ([17, p. 59, Theorem 13.6]). From Corollary 3 in [27, p. 296], it appears in particular, that any unital commutative barrelled,

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complete m-convex algebra every element of which is bounded is necessarily a Q-algebra. The latter result enlightens the first. So, one wonders which conditions are really behind the fact. Actually, advertible completeness, pseudo-completeness and m-infrabarrelledness are sufficient to get the Q-property (Proposition 4.1). In the class of GB^*-algebras, m-infrabarrelledness appears also to be strong enough. Indeed, we obtain, under an additional property, an analogue of an Allan's result (Proposition 5.1). Similarly for uniform locally m-convex algebras (Proposition 5.6).

A notion of "m-infrabarrelledness" has been introduced by A. K. Chilana and S. Sharma [5]. This is different from that given by Mallios [16, p. 307, Definition 9.4]. The latter being more general (see at the end of Preliminaries).

2 Preliminaries

A topological algebra is an algebra E over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) endowed with a topological vector space topology τ for which multiplication is separately continuous. Let (E, τ) be a locally convex algebra with separately continuous multiplication, whose topology τ is given by a family $(p_\lambda)_{\lambda\in\Lambda}$ of seminorms; we also write $(E, (p_\lambda)_\lambda)$. The algebra (E, τ) is said to be locally A-convex (see [8, p. 18, Definition 2.5]; see also [6], [7]) if, for every x and every λ, there is $M(x, \lambda) > 0$ such that

$$\max\left[p_\lambda(xy), p_\lambda(yx)\right] \leq M(x, \lambda)p_\lambda(y); \forall y \in E.$$

In the case of a single space norm, $(E, \|.\|)$, it is called an A-normed algebra. If $M(x, \lambda) = M(x)$ depends only on x, we say that (E, τ) is a locally uniformly A-convex algebra ([7, p. 477, Definition 3.1]). If

$$p_\lambda(xy) \leq p_\lambda(x)p_\lambda(y); \forall x, y \in E, \text{ and } \forall \lambda \in \Lambda,$$

then (E, τ) is named a locally m-convex algebra ([17], see also [16]). Recall that a locally convex algebra has a continuous multiplication if, for every λ, there is a λ' such that

$$p_\lambda(xy) \leq p_{\lambda'}(x)p_{\lambda'}(y); \forall x, y \in E.$$

A unital topological algebra is said to be a Q-algebra (or it has the Q-property) if the group $G(E)$ of its invertible elements is open. The spectrum of an element x, denoted by Spx, is the set $\{z \in \mathbb{C} : x - ze \notin G(E)\}$. The spectral radius $\rho(x)$ of x is $\rho(x) = \sup\{|z| : z \in Spx\}$.

An element x of a topological algebra E is said to be bounded (i-bounded in the sense of S. Warner ([26]; see also [1, p. 400, (2.1) Definition]) if there is $\alpha > 0$ such that $\{(\alpha x)^n : n = 1, 2, ...\}$ is bounded (the term regular is also used). Let (E, τ) be a locally convex space. The bounded structure (bornology) of (E, τ) is the collection, denoted by $\mathbb{B}$, of all subsets B of E which are bounded in the sense of Kolmogorov-von Neumann, that is B is absorbed by every neighborhood of zero. If $\tau_{\|.\|}$ is the topology induced by a norm $\|.\|$, we write $\mathbb{B}_{\|.\|}$. We say that a locally convex space (E, τ) is Mackey complete (M-complete) if its bounded structure $\mathbb{B}$ admits a fundamental system $\mathcal{B}$ of Banach discs ("completant" discs) that is, for every B in $\mathcal{B}$, the vector space generated by B is a Banach space when endowed with the gauge $\|.\|_B$ of B. A locally convex algebra is said to be pseudo-complete if every closed bounded and idempotent (alias multiplicative) disc is completant (see [1, p. 400, (2.2) Notation and p. 401, (2.5) Definition]). For the bornological notions, see [11]. An *m*-barrel is an idempotent barrel. An absolutely convex subset of (E, τ) is said to be bornivorous (resp. *m*-bornivorous) if it absorbs every bounded (resp. *m*-bounded) subset of (E, τ). A locally convex algebra (E, τ) is said to be *m*-barrelled if every *m*-barrel is a neighborhood of zero (this class of locally convex algebras was introduced in [15]). Further, a locally convex algebra is *m*-infrabarrelled in the sense of A. K. Chilana and S. Sharma if any bornivorous *m*-barrel is a neighborhood of zero [5]. In this paper, we use the following definition, due to A. Mallios, where it is only required that an *m*-barrel to be *m*-bornivorous. See [16, p. 307, Definition 9.4].

Definition 2.1. [Mallios]. A locally convex algebra (E, τ) is said to be *m*-infrabarrelled if every *m*-bornivorous *m*-barrel in E is a neighborhood of zero.

In the sequel, we will need the following simple observation.

Remark 2.2. A normed algebra is *m*-infrabarrelled. Indeed its unit ball is an idempotent bounded set. So, actually, any other subset which absorbs it, is automatically a neighborhood of zero.

3 A-convex algebras

Any locally *m*-convex algebra is A-convex (see [6, p. 74, Example (2.4)]) while a locally A-convex algebra being moreover barrelled is actually a locally *m*-convex algebra [ibid. p. 74]. In fact, *m*-barrelledness is sufficient. On the other hand, pseudo-completeness is the least completion requirement when dealing with spectral theory. With the latter condition barrelledness can be weakened to *m*-infrabarrelledness, modulo the sequential continuity of multiplication. For convenience, we recall some facts from [18], [19] and [20], needed in the sequel. If $(E, (p_\lambda)_\lambda)$ is a unital locally-A-convex algebra, then it can be endowed with a stronger *m*-convex topology $M(\tau)$, where τ is the topology of E. This is determined by the family $(q_\lambda)_\lambda$ of seminorms given by

$$q_\lambda(x) = \sup\{p_\lambda(xu) : p_\lambda(u) \leq 1\}.$$

If $(E, (p_\lambda)_\lambda)$ is not unital, consider its topological unitization (E_1, τ_1) and then take the restriction of $M(\tau_1)$ to E.

If $(E, (p_\lambda)_{\lambda\in\Lambda})$ is locally uniformly A-convex, then there is also an algebra norm $\|\cdot\|_0$ on E, that yields a topology stronger than $M(\tau)$, given by

$$\|x\|_0 = \sup\{q_\lambda(x) : \lambda \in \Lambda\} \tag{3.1}$$

(viz. $p_\lambda(x) \leq \|x\|_0, \forall\lambda$, and $\forall x \in E$; see [7, p. 477, (**) and Lemma 3.2]).

Notice that here, the existence of a unit is necessary (see Remark 3.4 below).

Proposition 3.1. *If $(E, (p_\lambda)_{\lambda\in\Lambda})$ is an m-infrabarrelled pseudo-complete locally A-convex algebra with continuous multiplication, then it is a locally m-convex algebra.*

Proof. We already have $\tau \subset M(\tau)$. For the inverse inclusion, a zero neighborhood basis for $M(\tau)$ consists of sets of the form $V = \bigcap_{1\le i\le n}\{x \in E : q_{\lambda_i}(x) \le \epsilon_i\}$, with $\epsilon_i \le 1$. Thus V is an m-barrel for τ. It is also m-bornivorous. Indeed, let B be an idempotent bounded disc. It is contained in the closure $\overline{B}$ which is idempotent, since multiplication is continuous. By pseudo-completeness, $\overline{B}$ is completant. So it is absorbed by the barrel V. Then, by hypothesis, V is a neighborhood of zero for τ. Whence $M(\tau) \subset \tau$. ∎

Pseudo-completeness does not imply the (sequential) continuity of multiplication; take e.g., the algebra $(C[0,1], \|.\|_1)$ (here, $\|.\|_1$ stands for the norm of $L^1[0,1]$). But if (E,τ) is Mackey-complete, then multiplication is automatically sequentially continuous [21, p. 398, Proposition 2.2]. Actually, we do not need any additional condition on multiplication.

Proposition 3.2. *If $(E,(p_\lambda)_{\lambda\in\Lambda})$ is an m-infrabarrelled Mackey-complete locally A-convex algebra, then it is a locally m-convex algebra.*

Proof. We argue as in the proof of the previous proposition. By Mackey-completeness, B is contained in a completant bounded disc B_1. Now, again by Mackey-completeness, B_1 is absorbed by any barrel, and so by V. ∎

In the uniformly A-convex case, the conclusions are stronger.

Proposition 3.3. *Let $(E,(p_\lambda)_{\lambda\in\Lambda})$ be a unital locally uniformly A-convex algebra. Then the following hold.*

(i) If $(E,(p_\lambda)_{\lambda\in\Lambda})$ is pseudo-complete, then it is m-infrabarrelled with continuous multiplication if and only if it is a Banach algebra.

(ii) If $(E,(p_\lambda)_{\lambda\in\Lambda})$ is Mackey-complete, then it is m-infrabarrelled if and only if it is a Banach algebra.

Proof. (i) It is known that $(E,(p_\lambda)_{\lambda\in\Lambda})$ and $(E,\|\cdot\|_0)$ (see (3.1)) have the same m-bounded subsets (see [24, p. 398, the comments before Proposition 2.1]). So the unit ball of $\|\cdot\|_0$ is an m-bornivorous m-barrel in $(E,(p_\lambda)_{\lambda\in\Lambda})$.

(ii) In that case, $(E,(p_\lambda)_{\lambda\in\Lambda})$ and $(E,\|\cdot\|_0)$ have the same bounded sets [23]. One then argues as in (i). ∎

Remark 3.4. The existence of a unit is essential in Proposition 3.3. Let X be a non compact, locally compact and metrizable space such that $X = \cup K_n$ where $(K_n)_n$ is an exhaustive sequence of compact subsets of X. Take the complex algebra $\mathcal{K}(X)$ of continuous functions with compact support and $E_n = \mathcal{K}(X, K_n)$ the subalgebra of functions with support in K_n. It is known that $\mathcal{K}(X)$ is algebraically the inductive limit of the E_n's (see for instance, [16, p. 128, (4.6); see also p. 127, 4.(1)]). Take the strict inductive limit topology τ of the Banach algebras $(E_n, \|\cdot\|_n)$, where $\|\cdot\|_n$ is the supremum norm on E_n. Then (E,τ) is a locally m-convex algebra which is also locally uniformly A-convex. Moreover, it is complete and barrelled, but neither unital or a normed algebra.

It is known that a barrelled A-normed algebra is a normed one. In fact m-barrelledness is sufficient. But not every normed algebra is m-barrelled: Take a complex non barrelled normed space E and endow it with the null multiplication (i.e., $xy = 0$, for every x, y); then every barrel is in fact an m-barrel. The unitization E_1 of E is also non m-barrelled. To have more examples, consider the topological product algebra of E_1 or E with any Banach algebra, commutative or not, unital or not. However, a normed algebra is always m-infrabarrelled (see Remark 2.2). Actually, we have the following characterization of Banach algebras among pseudo-complete A-normed ones.

Proposition 3.5. *Let $(E, \|\cdot\|)$ be a pseudo-complete A-normed algebra. The following assertions are equivalent.*
(1) $(E, \|\cdot\|)$ is a Banach algebra.
(2) $(E, \|\cdot\|)$ is m-infrabarrelled.

Proof. (1) $\Rightarrow$ (2) : See Remark 2.2.
(2) $\Rightarrow$ (1) : It is known that $(E, \|\cdot\|)$ and $(E, \|\cdot\|_0)$ have the same m-bounded subsets [24] (see also (3.1)). But the unit ball B_0, with respect to $\|\cdot\|_0$, is a $\|\cdot\| - m$-barrel. It is also m-bornivorous. So, by m-infrabarrelledness, it is a neighborhood of zero. Hence $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. Now pseudo-completeness of the normed algebra $(E, \|\cdot\|_0)$ implies that it is Banach. ∎

Though it is contained in the previous proposition, the following characterization of Banach algebras is worthwhile to be stated.

Proposition 3.6. *An A-normed algebra is a Banach algebra if and only if it is pseudo-complete and m-infrabarrelled.*

Remark 3.7. It is known that $(C[0,1], \|\cdot\|_1)$ is not barrelled. By the previous proposition, it is not m-infrabarrelled, as well. Actually, one can consider the algebras $(E, \tau_p) = (C[0,1], \|\cdot\|_p)$, $p \in \mathbb{N}^*$ with $\|f\|_p = (\int_0^1 |f(t)|\, dt)^{\frac{1}{p}}$. They are pseudo-complete A-normed algebras. None of them is m-infrabarrelled. Relative to this, we observe that they have the same m-bounded subsets, which are exactly the bounded subsets for the uniform norm $\|\cdot\|_\infty$. But for $p \neq q$, one has $\mathbb{B}_p \neq \mathbb{B}_q$.

Example 3.8. Let $C_b(\mathbb{R})$ be the algebra of complex continuous bounded functions on the real field $\mathbb{R}$ with the usual pointwise operations. Denote by $C_0^+(\mathbb{R})$ the set of all strictly positive real-valued continuous functions on $\mathbb{R}$ vanishing at infinity (as elements of $C_b(\mathbb{R})$). Consider the family $\{p_\varphi : \varphi \in C_0^+(\mathbb{R})\}$ of seminorms given by

$$p_\varphi(f) = \sup\{|f(x)\varphi(x)| : x \in \mathbb{R}\}; f \in C_b(\mathbb{R}),$$

that determine a locally convex topology, say β. The space $(C_b(\mathbb{R}), \beta)$ is actually, a complete locally convex algebra. It is not a locally m-convex algebra ([8, p. 20, Examples 3]), nor a Q-algebra. Otherwise, it should be strongly sequential hence sequential, which is not the case (see [13, p. 417, Example 7]). But, it is a locally uniformly A-convex algebra with continuous multiplication. One has $\mathbb{B}_\beta = \mathbb{B}_{\tau_{\|\cdot\|_\infty}}$. But $\beta \subset M(\beta) \subset \tau_{\|\cdot\|_\infty}$ (for the symbol $M(\beta)$, see the comments at the beginning of Section 3). Hence $\mathbb{B}_\beta = \mathbb{B}_{M(\beta)} = \mathbb{B}_{\tau_{\|\cdot\|_\infty}}$. Now, $B_{q_\lambda} = \{x : q_\lambda(x) \leq 1\}$ is an m-bornivorous m-barrel. But it is not a β-neighborhood of zero. So, $(C_b(\mathbb{R}), \beta)$ is not m-infrabarrelled.

Example 3.9. Let $(E, \|\cdot\|)$ be an infinite-dimensional commutative semisimple self-adjoint Banach algebra and $\sigma = \sigma(E, E')$ its weak topology. Then (E, σ) is a locally convex algebra. It is Mackey-complete, since $\mathbb{B}_{\tau_{\|\cdot\|}} = \mathbb{B}$. It is not a locally m-convex algebra. Otherwise, by [26, p. 314, Theorem 1], it would be a topological algebra and by [ibid. p. 315, Theorem 2], finite-dimensional, that is a contradiction. Now, one has $\|x\| = \sup\{|\langle x, x'\rangle| : x' \in B'\}$, where B' is the closed unit ball of the topological dual E'. So $B = \{x : \|x\| \leq 1\}$ is an m-σ-barrel. It is also m-bornivorous. But, it is not a σ-neighborhood of zero. Therefore (E, σ) is not m-infrabarrelled.

Remark 3.10. Let $(E, (p_\lambda)_\lambda)$ be a locally A-convex algebra and the associated A-normed algebras $E_\lambda = E/N_\lambda$. Sufficient conditions have been given on the factors E_λ to make $(E, (p_\lambda)_\lambda)$ a locally m-convex algebra (see [24] for details). This has to do with Γ-completeness [16, p. 280, Definition 6.1] i.e., E_λ is a normed algebra for every λ. Also P-completeness [8, p. 19] is another notion i.e, E_λ is a Banach algebra for every λ. In view of Proposition 3.1, we are led to the following characterization of complete m-convex algebras among locally A-convex ones.

Proposition 3.11. *Let $(E, (p_\lambda)_\lambda)$ be a locally A-convex algebra. Then it is a complete locally m-convex algebra if and only if each factor E_λ is pseudo-complete and m-infrabarrelled.*

Remark 3.12. None of the mentioned sufficient conditions of the previous proposition is necessary; while m-infrabarrelledness is.

Remark 3.13. Proposition 3.11 is applied to the very classical spaces $\mathcal{K}(\mathbb{R})$, $\mathcal{D}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ of distribution theory. One has just to observe that every element is bounded (cf. [3] or [9], for a detailed presentation of these spaces).

4 The Q-property

It is known that a commutative Fréchet locally m-convex algebra every element of which is bounded (viz. the local spectrum is bounded) is a Q-algebra (see [17, p. 59, Theorem 13.6]). The same conclusion has been obtained assuming the boundedness of every element, completeness and barrelledness (see [27, p. 296, Corollary 3]). Completeness implies M-completeness (hence pseudo-completeness) and advertible completeness. In the unital case, the conditions mentioned above are weakened, as the next proposition shows.

Proposition 4.1. *Let (E, τ) be a unital commutative advertibly complete and pseudo-complete locally m-convex algebra every element of which is bounded. If (E, τ) is m-infrabarrelled, then it is a Q-algebra.*

Proof. It is known [16, p. 74, Corollary 7.4] that the spectral radius is given by

$$\rho(x) = \sup\{|f(x)| : f \in \mathcal{M}(E)\}$$

where $\mathcal{M}(E)$ is the set of continuous characters (the Gel'fand spectrum) of (E, τ). Hence ρ is lower semi-continuous and therefore the ball $B_\rho = \{x : \rho(x) \leq 1\}$

is closed. Thus it is an m-barrel. It is also m-bornivorous. Indeed, let B be an idempotent bounded disc. It is contained in the closure $\overline{B}$ which, by the continuity of multiplication, is idempotent. By pseudo-completeness, $\overline{B}$ is completant. So $(E_B, \|.\|_B)$ is a Banach space. Now, the restriction of ρ to $(E_B, \|.\|_B)$ remains lower semi-continuous. It is then continuous, hence bounded. Then, by hypothesis, B_ρ is a neighborhood of zero for τ and hence E is a Q-algebra (see [16, p. 103, Proposition 6.3]). ∎

A Q'*-algebra* is a topological algebra in which every maximal regular left or right ideal is closed (see [10, p. 148, Definition 1.1]). Thus, *a topological algebra, as in the previous result, being also m-infrabarrelled is a* Q'*-algebra* (see [17, p. 80, Lemma E.4]).

Remark 4.2. Since a Q-algebra is functionally continuous, the previous proposition shows that algebras known as non barrelled are in fact, not m-infrabarrelled. This is the case for the algebra $C\left([0,\Omega[\right)$ exhibited in [17, p. 16, Example 3.7]; see also Example 4.6 below.

Remark 4.3. A unital commutative and complete locally m-convex algebra every element of which is bounded is not necessarily m-infrabarrelled. It should be a Q-algebra. Indeed, $B_\rho = \{x : \rho(x) \leq 1\}$ is an m-barrel. It is also bornivorous (see [23]), hence m-bornivorous. So it must be a neighborhood of zero.

Remark 4.4. A normed algebra is m-infrabarrelled (see Remark 2.2). But this is not the case for a metrizable locally m-convex algebra. Let $C_c(\mathbb{R}_+)$ be the complex algebra of continuous functions on $\mathbb{R}_+$ which are constant from a positive real number on. Endow it with the topology τ of uniform convergence on compacta. It becomes a commutative non complete metrizable locally m-convex algebra. It is advertibly complete. Since

$$\|f\|_\infty = \sup\{|f|_K : K \text{ a compact subset of } \mathbb{R}_+\},$$

where $|f|_K = \sup\{|f(x)| : x \in K\}$, the subset

$$B_1 = \{f : |f|_K \leq 1, \forall K\} = \{f : \|f\|_\infty \leq 1\}$$

is an m-barrel. Moreover, it is the greatest closed bounded idempotent disc. Hence it is m-bornivorous. But it is not a neighborhood of zero, otherwise the topology τ should be equivalent to $\tau_{\|.\|_\infty}$. Observe also that $\mathbb{B}\tau \neq \mathbb{B}\tau_{\|.\|_\infty}$ otherwise, the identity map $Id : (E,\tau) \longrightarrow (E,\tau_{\|.\|_\infty})$ should be bounded, hence continuous, since (E,τ) is bornological (metrizable). However, for the bounded structures one has $\mathbb{B}_r\tau = \mathbb{B}\tau_{\|.\|_\infty}$ where $\mathbb{B}_r\tau$ designates the collection of all regular bounded subsets for τ. So (E,τ) is pseudo-complete. But it is not a Q-algebra. Thus m-infrabarrelledness appears to be a necessary condition in Proposition 4.1. This example shows also that a bornological algebra (as a space) is not necessarily m-infrabarrelled.

Example 4.5. m-barrelledness is necessary in Proposition 4.1 even if the regularity of elements is strengthened to that of all bounded sets, as well as pseudo-completeness to completeness. Let $C[0,1]$ be the algebra of complex continuous functions on the interval $[0,1]$. Endowed with the topology τ of uniform convergence on denumerable compact subsets of $[0,1]$, it is a complete locally m-convex algebra. For the bounded structures, one has $\mathbb{B}\tau = \mathbb{B}_{\tau_{\|\cdot\|_\infty}}$. It is not m-infrabarrelled, since $B_\infty = \{f : \|f\|_\infty \leq 1\}$ is a bornivorous (hence m-bornivorous) m-barrel which is not a neighborhood of zero for τ.

Example 4.6. Let Ω be the first non countable ordinal and endow the set $[0,\Omega[$ with the order topology. Consider $C([0,\Omega[)$ the complex algebra of continuous functions, on $[0,\Omega[$, endowed with the topology of uniform convergence on compacta. It is a commutative complete locally m-convex algebra. It is not a Q-algebra, so it is not m-infrabarrelled.

5 GB^*-algebras

m-infrabarrelledness appears to be a strong notion in GB^*-algebras and uniform ones. Indeed, one obtains an improvement of an Allan's result. In [2], G.R. Allan introduces GB^*-algebras and shows that a (unital) barrelled GB^*-algebra every element of which is bounded, is actually a C^*-algebra [ibid. p. 95, (2.8) Corollary]. An analogous result has been shown in the context of uniform locally m-convex algebras ([22, p. 110, Proposition 5.4]). Here, there are similar results where barrelledness is weakened to m-barrelledness. Recall that if (E,τ) is a unital locally convex algebra, one denotes by $\mathcal{B}$ the collection of all subsets B of E which are closed bounded and idempotent discs containing the unit element e. If (E,τ) has a continuous involution, then we put $\mathcal{B}^* = \{B \in \mathcal{B} : B^* = B\}$. In the latter case, we say that (E,τ) is a GB^*-algebra ([2, p. 94, (2.5) Definition]) if it is pseudo-complete and (i) (E,τ) is symmetric i.e., $e + x^*x$ has a bounded inverse, for every x in E, (ii) $\mathcal{B}^*$ has a greatest element. Recall also that a locally m-convex algebra is said to be uniform if there is a family $(p_\lambda)_{\lambda\in\Lambda}$ of seminorms defining its topology such that $p_\lambda(x^2) = [p_\lambda(x)]^2$, for every λ and every x.

Proposition 5.1. *Let (E,τ) be a commutative GB^*-algebra with jointly continuous multiplication every element of which is bounded. Then E is m-infrabarrelled if and only if it is a C^*-algebra.*

Proof. Let B_0 be the greatest element of $\mathcal{B}^*$ and denote by $\|.\|_0$ its gauge on the algebra E_{B_0} generated by B_0. Then $(E_{B_0}, \|.\|_0)$ is a C^*-algebra [2, p. 94, (2.6) Theorem]. By hypothesis, we actually have $E = E_{B_0}$. Observe that one already has $\tau \subset \tau_{\|\cdot\|_0}$. Now, take any bounded idempotent subset B of (E,τ). Without any loss of generality, we consider that it contains the unit of E. The subset $B \cup B^*$ is bounded and self-adjoint. Due to commutativity, its idempotent hull is $B \cup B^* \cup BB^*$ which is equal to BB^*, since B and B^* both contain the unit. Now the closure of the absolutely convex hull of BB^* is also bounded and idempotent. Hence it is contained in B_0, which is then m-bornivorous. So it is a neighborhood of zero, since (E,τ) is supposed to be m-infrabarrelled. Whence $\tau_{\|\cdot\|_0} \subset \tau$. ■

The following particular case is worthwhile to be mentioned. We recall that a *locally C^*-algebra* is an involutive complete locally (m-)convex algebra $(E,(p_\lambda)_{\lambda\in\Lambda})$, such that each p_λ, $\lambda \in \Lambda$ is a C^*-seminorm (viz. $p_\lambda(x^*x) = p_\lambda(x)^2$ for every $x \in E$, and all $\lambda \in \Lambda$ (see [14, p. 198, Definition 2.2]).

Proposition 5.2. *Any commutative locally C^*-algebra $(E,(p_\lambda)_{\lambda\in\Lambda})$ every element of which is bounded is m-infrabarrelled if and only if it is a C^*-algebra.*

Proof. Obviously, $B_1 = \{x : p_\lambda(x) \leq 1; \forall\lambda\}$ is an m-barrel. It is also the greatest closed bounded and idempotent disc. So it is m-bornivorous. It is then a neighborhood of zero, by hypothesis. ∎

Remark 5.3. In the unital case, an immediate proof of Proposition 5.2 follows from Proposition 4.1 and [9, p. 111, Corollary 8.2].

Remark 5.4. The algebra $C[0,1]$ of complex continuous functions on the interval $[0,1]$, endowed with the topology τ of uniform convergence on denumerable compact subsets of $[0,1]$ (see Example 4.5) is a locally C^*-algebra every element of which is bounded. It is not a Q-algebra, hence neither a C^*-algebra nor an m-infrabarrelled one. Thus m-infrabarrelledness is necessary in the previous proposition.

By Proposition 5.2 and [9, p. 111, Corollary 8.2] we get the next.

Corollary 5.5. *Let E be a commutative locally C^*-algebra, every element of which is bounded. The following are equivalent.*
(1) E is m-infrabarrelled.
(2) E is a Q-algebra.
(3) E is a C^*-algebra.

Arguing in an analogous way, one obtains the following result. Here, we remind that any uniform locally m-convex algebra is commutative and semisimple (see [16, p. 275, Lemma 5.1]).

Proposition 5.6. *Let (E,τ) be a unital pseudo-complete uniform locally m-convex algebra. Then it is m-infrabarrelled if and only if it is a Banach algebra.*

According to [4, p. 499, Theorem 2], any complete uniform locally m-convex algebra, which is a Q-algebra is a uniform Banach algebra. The converse is also true. Thus, in connection with Proposition 5.6, we get an analogue of Corollary 5.5, in case the C^*-property is replaced by the "uniform" property. Namely, we have the next.

Corollary 5.7. *Let E be a unital complete uniform locally m-convex algebra. The following are equivalent.*
(1) E is m-infrabarrelled.
(2) E is a Q-algebra.
(3) E is a Banach algebra.

Remark 5.8. Again the algebra $C[0,1]$ in Remark 5.4 shows the necessity of m-infrabarrelledness in the previous proposition.

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