Conjugate convolution operators and inner amenability

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Abstract

Let $G$ be a group and $L^\infty(G)$ be the $C^*$-algebra of bounded complex-valued functions on $G$. $G$ is called inner amenable if there exists a positive norm 1 functional $m$ on $L^\infty(G)$ such that $m(\rho(y)f) = m(f)$ for each $y \in G$, $f \in L^\infty(G)$ (where $\rho(y)f(x) = f(yxy^{-1})$); the functional $m$ is called an inner invariant mean.

In this paper, among the other things, we prove a variety of characterizations of inner amenable groups. We also give sufficient conditions on an inner invariant mean to be a topologically inner invariant mean.

1 Introduction

There are a lot of results in abstract harmonic analysis on amenability of a locally compact group. A good deal of attention was paid to the study of inner amenable groups. The study of inner invariant means was initiated by Effros [5] and pursued by Akemann [1], Yuan [25] for discrete groups, Lau and Patterson [13] and Yuan [26] for locally compact groups, and Ling [15] and Mohammadzadeh and Nasr-Isfahani [18] for semigroups. Amenable locally compact groups and [IN]-groups are inner amenable. Furthermore when $G$ is connected, then $G$ is amenable if and only if $G$ is inner amenable [16]. Amenability and inner amenability of Lau algebras is studied in [12] and [19]. For terminologies regarding invariant means on locally compact groups, the reader is
referred to [20]. Let \( \pi_\infty \) be the isometric representation of \( G \) on \( L^\infty(G) \) given by 
\[ \pi_\infty(x)f(t) = f(x^{-1}tx) \]
It is shown that \( L^\infty(G) \) has an inner invariant mean if and only if the commutant \( \pi_\infty(G)' \) of \( \pi_\infty(G) \) contains a nonzero compact operator [14]. The literature on inner amenability has grown substantially in recent years, see [9], [11] and [17].

In this paper, we investigate inner invariant means on \( L^\infty(G) \) and its closed subalgebra \( U^\infty(G) \) of all \( f \in L^\infty(G) \) for which the mapping \( y \mapsto \rho(y)f \) is continuous [7]. We also study topologically inner invariant means on certain closed subspaces \( X \) of \( U^\infty(G) \) and their relation with inner invariant means on \( X \). We show that every topologically inner invariant mean on \( L^\infty(G) \) is also inner invariant. The converse remains open. Sufficient conditions on an inner invariant mean to be a topologically inner invariant mean are given. We characterize inner amenable groups by introducing the so-called conjugate convolution operators which develop the techniques of the usual convolution operators. We give sufficient conditions and some necessary conditions for \( G \) to have an inner invariant mean.

\section{Preliminaries and notations}
Throughout this paper \( G \) will denote a locally compact group with left Haar measure \( dx \), modular function \( \Delta \), and identity \( e \). For \( 1 \leq p < \infty \), \( L^p(G) \) is the space of complex-valued measurable functions \( \varphi \) on \( G \) such that \( \int |\varphi(x)|^p dx < \infty \). Let \( L^\infty(G) \) be the algebra of essentially bounded measurable complex-valued functions on \( G \). For \( y \in G \) and \( f \) a function on \( G \) we use the notation
\[ yf(x) = f(y^{-1}x), \quad \rho(y)f(x) = f(xy^{-1}) \quad (x \in G). \]
If \( \varphi \in L^1(G) \), \( \psi \in L^p(G) \) \((1 \leq p < \infty)\) and \( f \in L^\infty(G) \), then \( \varphi \oplus \psi \) as member of \( L^p(G) \) is given
\[ \varphi \oplus \psi(x) = \int \varphi(y)\psi(y^{-1}x)\Delta(y)^{\frac{1}{p}} dy \quad (x \in G) \]
while \( \varphi \odot f \) as member of \( L^\infty(G) \) is given by
\[ \varphi \odot f(x) = \int \varphi(y)f(xy^{-1}) dy \quad (x \in G). \]
We have \( \| \varphi \oplus \psi \|_p \leq \| \varphi \|_1 \| \psi \|_p \) and \( \| \varphi \odot f \| \leq \| \varphi \|_1 \| f \| \). More information on this product can be found in [23] and [24]. More generally, for \( 1 \leq p \leq \infty \), let \( \pi_p \) be the isometric representation of \( G \) on \( L^p(G) \) given by
\[ \pi_p(y)\varphi(x) = \varphi(y^{-1}x)\Delta(y)^{\frac{1}{p}} \quad (x, y \in G, \ \varphi \in L^p(G)). \]
Thus for all \( y \in G \), we have \( \| \varphi \|_p = \| \pi_p(y)\varphi \|_p \). We denote by \( P^p(G) \) the convex set of all nonnegative functions \( \varphi \) in \( L^p(G) \) such that \( \| \varphi \|_p = 1 \). If \( A \) is measurable subset of \( G \), then \( |A| \) is the measure of \( A \). For any subset \( A \) of \( G \), \( 1_A \) denotes the characteristic function of \( A \). If \( 0 < |A| < \infty \), we also consider the mapping
\[ \xi_A(x) = \frac{1_A(x)}{|A|} \quad \text{defined on } G. \]
Duality between Banach spaces is denoted by \( \langle \cdot, \cdot \rangle \); thus for \( f \in L^\infty(G) \) and \( \varphi \in L^1(G) \), we have \( \langle f, \varphi \rangle = \int f(x) \varphi(x) \, dx \). As far as possible, we follow [7] in our notation and refer to [22] for basic functional analysis and to [10] for basic harmonic analysis.

3 Main results

We start by recalling the following definition.

**Definition 3.1.** Let \( X \) be a subspace of \( L^\infty(G) \) with \( 1_G \in X \) that is closed under complex conjugation:

(i) We say that \( X \) is invariant (topologically invariant), if \( \rho(y)f \in X \) \( (\varphi \circ f \in X) \) whenever \( y \in G \), \( f \in X \) and \( \varphi \in P^1(G) \);

(ii) A mean on \( X \) is a norm one nonnegative functional \( m \) on \( X \) such that \( m(1_G) = 1 \);

(iii) Let \( X \) be an invariant subspace of \( L^\infty(G) \). A mean \( m \) on \( X \) is called inner invariant mean if \( \langle m, \rho(y)f \rangle = \langle m, f \rangle \) for all \( f \in X \) and \( y \in G \);

(iv) Let \( X \) be a topologically invariant subspace of \( L^\infty(G) \). A mean \( m \) on \( X \) is called topologically inner invariant mean if

\[
\langle m, \varphi \circ f \rangle = \langle m, f \rangle
\]

for all \( \varphi \in P^1(G) \) and \( f \in X \);

(v) A locally compact group \( G \) is called inner amenable group if it admits an inner invariant mean on \( L^\infty(G) \).

We denote by \( U^\infty(G) \) the Banach space consisting of the complex-valued functions \( f \) in \( L^\infty(G) \) that are uniformly continuous, that is, the mapping \( y \mapsto \rho(y)f \) from \( G \) into \( L^\infty(G) \) is continuous [7]. The present author has proved that \( U^\infty(G) \) is a Banach algebra and \( \varphi \circ f \in U^\infty(G) \) for every \( \varphi \in L^1(G) \) and \( f \in L^\infty(G) \) (see Lemma 2.3 in [7]). Clearly \( U^\infty(G) \) is an invariant subspace of \( L^\infty(G) \).

**Lemma 3.2.** Let \( G \) be a locally compact group. Then the following statements hold:

(i) Let \( X \) be a closed subspace of \( U^\infty(G) \). Then \( X \) is invariant if and only if it is topologically invariant;

(ii) Let \( X \) be a closed subspace of \( U^\infty(G) \) with \( 1_G \in X \) that is closed under complex conjugation and topologically invariant. A mean \( m \) on \( X \) is inner invariant if and only if it is topologically inner invariant.

**Proof.** (i): By the same argument as used at the proof of Lemma 2.5 in [7], we see that \( X \) is invariant if and only if it is topologically invariant.
(ii): Let $m$ be an inner invariant mean on $X$, and let $f \in X$ and $\varphi \in P^1(G)$. Since the measures in $P^1(G)$ with compact supports are norm dense in $P^1(G)$, without loss of generality we may assume that $\varphi$ has a compact support. By Theorem 3.27 in [22],

$$\langle m, \varphi \odot f \rangle = \int \langle m, \rho(y)f \rangle \varphi(y)dy = \int \langle m, f \rangle \varphi(y)dy = \langle m, f \rangle.$$ 

This shows that $m$ is topologically inner invariant mean.

To prove the converse, let $m$ be a topologically inner invariant mean on $X$ and fix $\varphi \in P^1(G)$. For $f \in X$ and $y \in G$,

$$\langle m, \rho(y)f \rangle = \langle m, \varphi \odot \rho(y)f \rangle = \langle m, y \varphi \odot f \rangle = \langle m, f \rangle.$$

Thus, $m$ is an inner invariant mean on $X$.

Let $G$ be a locally compact group. For $\varphi, \psi \in L^1(G)$, $f \in L^\infty(G)$ and $m, n \in L^\infty(G)^*$, the elements $f.\varphi$ and $n.f$ of $L^\infty(G)$ and $m.n \in L^\infty(G)^*$ are defined by

$$\langle f.\varphi, \psi \rangle = \langle f, \varphi \odot \psi \rangle, \quad \langle n.f, \varphi \rangle = \langle n, f.\varphi \rangle, \quad \langle m.n, f \rangle = \langle m, n.f \rangle,$$

respectively. Clearly $\|f.\varphi\| \leq \|f\|\|\varphi\|_{1,1}$, $\|n.f\| \leq \|n\|\|f\|$ and $\|m.n\| \leq \|m\|\|n\|$.

Elementary calculations shows that $\varphi \odot f = f.\varphi$ for every $f \in L^\infty(G)$ and $\varphi \in L^1(G)$.

For each $\varphi \in L^1(G)$, define a seminorm $\rho_\varphi$ on the linear space $L^\infty(G)$ by $\rho_\varphi(f) = \|f.\varphi\|$, $f \in L^\infty(G)$. Note that $\mathcal{P} = \{\rho_\varphi; \varphi \in L^1(G)\}$ separates the points of $L^\infty(G)$. The locally convex topology on $L^\infty(G)$ determined by these seminorms is denoted by $\tau_c$. We first remark that the $\tau_c$-topology may be characterized in another manner. Indeed, it is a standard device to embed $L^\infty(G)$ into $B(L^1(G), L^\infty(G))$ by an operator $T$ so that $T(f)(\varphi) = f.\varphi$, $f \in L^\infty(G)$, $\varphi \in L^1(G)$. Then $T$ is one-to-one and linear. On the other hand, $B(L^1(G), L^\infty(G))$ naturally carries the strong operator topology. So $T$ allows us to consider the induced topology on $L^\infty(G)$ which is the same as the $\tau_c$-topology. In [8] the author studied the $\tau_c$-topology on the dual $M_p(S)^*$ of the semigroup algebra $M_p(S)$ of a locally compact foundation semigroup $S$. From these observations we immediately deduce the following Lemma.

**Lemma 3.3.** Let $G$ be a locally compact group. For each $\varphi \in L^1(G)$, the mapping $f \mapsto \varphi \odot f$ from $(L^\infty(G), \tau_c)$ into $(L^\infty(G), \|\|)$ is continuous.

We are now in a position to establish one of the main results of this section.

**Theorem 3.4.** Let $G$ be a locally compact group, $X$ a subspace of $L^\infty(G)$ with $1_G \in X$ that is closed under complex conjugation, invariant and topologically invariant. Then the following properties hold:

(i) Every topologically inner invariant mean $m$ on $X$ is $\tau_c$-continuous;

(ii) An inner invariant mean on $X$ is topologically inner invariant mean if and only if it is $\tau_c$-continuous;
(iii) Let \( m \) be an inner invariant mean on \( X \). Suppose there is some \( \varphi_0 \in P^1(G) \) such that \( \langle m, \varphi_0 \circ f \rangle = \langle m, f \rangle \) for all \( f \in X \). Then \( m \) is topologically inner invariant mean.

Note that an analogue of statement (ii) for topological left invariant means has proved by Crombez, see Lemma 2.1 in [3]. Also, there is an argument similar to statement (iii) for topological left invariant means, see Proposition 22.2 in [21].

**Proof.** (i): Let \( m \) be a topologically inner invariant mean on \( X \), and let \( f_\alpha \to f \) in the \( \tau_c \)-topology of \( X \). By Lemma 3.3, for \( \varphi \in P^1(G) \), \( f_\alpha . \varphi \to f . \varphi \) in the norm topology. We conclude that

\[
\lim_{\alpha} \langle m, f_\alpha \rangle = \lim_{\alpha} \langle m, \varphi \circ f_\alpha \rangle = \lim_{\alpha} \langle m, f_\alpha . \varphi \rangle = \langle m, f . \varphi \rangle = \langle m, \varphi \circ f \rangle = \langle m, f \rangle.
\]

This shows that \( m \) is \( \tau_c \)-continuous.

(ii): Let \( m \) be an inner invariant mean on \( X \). If \( m \) is topologically inner invariant, then \( m \) is \( \tau \)-continuous; see (i).

To prove the converse, let \( m \) be an inner invariant mean on \( X \). Let \( f \in X \), \( \varphi \in P^1(G) \) and \( \varepsilon > 0 \) be given. We further assume that \( \varphi \) has a compact support, say \( K \). If \( \| f \| = 0 \), we have trivially \( \langle m, \varphi \circ f \rangle = \langle m, f \rangle \). We now consider the case \( \| f \| > 0 \). The sets

\[
V(\varphi \circ f, \varphi_1, ..., \varphi_n, \delta) = \{ h \in X; \| h . \varphi - (\varphi \circ f) . \varphi_i \| < \delta, i = 1, ..., n \}
\]

where \( \delta > 0 \) and \( \{ \varphi_1, ..., \varphi_n \} \) is a finite subset of \( L^1(G) \), form a basis of open neighborhoods of \( \varphi \circ f \) in the \( \tau_c \)-topology of \( X \). Now, we choose a neighborhood \( V(\varphi \circ f, \varphi_1, ..., \varphi_n, \delta) \) of \( \varphi \circ f \) in \( X \) such that \( |\langle m, h \rangle - \langle m, \varphi \circ f \rangle| < \varepsilon \) whenever \( h \in V(\varphi \circ f, \varphi_1, ..., \varphi_n, \delta) \).

Since the mapping \( y \mapsto y . \varphi_i \) is continuous [6], for every \( y \in K \), there exists a relatively compact neighbourhood \( U_y \) of \( y \) in \( G \) such that \( \| y . \varphi_i - x . \varphi_i \| < \frac{\delta}{\| f \|} \) whenever \( x \in U_y \) and \( i \in \{1, ..., n\} \). Now cover \( K \) by \( \{ U_y; y \in K \} \). By compactness we may extract a finite subcover \( U_{y_1}, ..., U_{y_l} \) of \( K \). We can find \( l \) Borel subsets \( A_1, ..., A_l \) of \( K \) such that

\[
K = \bigcup_{j=1}^{l} A_j, \ A_j \cap A_r = \emptyset (j \neq r), \ \| y . \varphi_i - x . \varphi_i \|_1 < \frac{\delta}{\| f \|}
\]

whenever \( y \in A_j \) and \( i \in \{1, ..., n\} \). If \( j \in \{1, ..., l\} \), we also put \( a_j = \int_{A_j} \varphi(y)dy \).

Then \( \sum_{j=1}^{l} a_j = 1 \). For every \( i \in \{1, ..., n\} \),

\[
| \sum_{j=1}^{l} a_j \rho(y_j) f . \varphi_i - (\varphi \circ f) . \varphi_i | = | \sum_{j=1}^{l} a_j y_j . \varphi_i \circ f - \varphi_i \circ (\varphi \circ f) | \\
\leq \sum_{j=1}^{l} \int_{A_j} \varphi(z) | y_j . \varphi_i \circ f - \varphi_i \circ f | dz \\
\leq \sum_{j=1}^{l} \int_{A_j} \varphi(z) \| y_j . \varphi_i - \varphi_i \|_1 \| f \| dz < \delta.
\]
This shows that \( \sum_{j=1}^{l} \alpha_i \rho(y_j)f \in V(\varphi \odot f, \varphi_1, ..., \varphi_n, \delta) \), and so

\[
|\langle m, f \rangle - \langle m, \varphi \odot f \rangle| = \left| \langle m, \sum_{j=1}^{l} \alpha_i \rho(y_j)f \rangle - \langle m, \varphi \odot f \rangle \right| < \epsilon.
\]

As \( \epsilon > 0 \) may be chosen arbitrarily, we have \( \langle m, f \rangle = \langle m, \varphi \odot f \rangle \). Finally, if \( \varphi \) is any element in \( P^1(G) \), let \( \{\varphi_n\} \subseteq P^1(G) \) be a sequence of elements with compact support such that \( \varphi_n \to \varphi \). Then from the above special case, we conclude that \( \langle m, \varphi \odot f \rangle = \langle m, f \rangle \).

(iii): Let \( m \) be an inner invariant mean and \( \langle m, \varphi_0 \odot f \rangle = \langle m, f \rangle \) for all \( f \in X \). To show that \( m \) is topologically inner invariant mean, it is sufficient to prove that \( m \) is \( \tau_c \)-continuous. But suppose \( f_\alpha \to f \) in the \( \tau_c \)-topology. Since \( \varphi_0 \odot f_\alpha = f_\alpha \cdot \varphi_0 \to f \cdot \varphi_0 = \varphi_0 \odot f \) in the norm topology, we see that

\[
\lim_{\alpha} \langle m, f_\alpha \rangle = \lim_{\alpha} \langle m, \varphi_0 \odot f_\alpha \rangle = \langle m, \varphi_0 \odot f \rangle = \langle m, f \rangle.
\]

Hence \( m \) is topologically inner invariant mean.

Let \( G \) be a compact nondiscrete abelian group. By Proposition 22.3 in [21], there exists a left invariant mean \( m \) on \( L^\infty(G) \) such that \( \langle m, \varphi \odot f \rangle \neq \langle m, f \rangle \) for some \( f \in L^\infty(G) \) and \( \varphi \in P^1(G) \). This shows that \( m \) can not be a topologically left invariant mean. It is easy to see that every topologically inner invariant mean on \( L^\infty(G) \) is inner invariant mean on \( L^\infty(G) \). We do not know whether or not the converse holds. The next theorem of this section exhibits a number of assertions which are equivalent to inner amenability of a locally compact group \( G \).

**Theorem 3.5.** A locally compact group \( G \) is inner amenable if and only if there exists a net \( \{\varphi_\alpha\} \) in \( P^1(G) \) satisfying any one of the following conditions:

(i) For every \( \varphi, \psi \in P^1(G) \), \( \lim_{\alpha} \| \psi \odot (\varphi \odot \varphi_\alpha) - \psi \odot \varphi_\alpha \|_1 = 0 \);

(ii) For every \( \varphi \in P^1(G) \) and \( f \in U^\infty(G) \), \( \lim_{\alpha} \langle f, \varphi \odot \varphi_\alpha - \varphi_\alpha \rangle = 0 \);

(iii) For every compact subset \( K \) of \( G \) and every \( f \in U^\infty(G) \),

\[
\lim \sup \{ |\langle f, \pi_1(y)\varphi_\alpha - \varphi_\alpha \rangle| ; \ y \in K \} = 0.
\]

**Proof.** Let \( G \) be inner amenable. By Theorem 2 in [24], there exists a net \( \{\varphi_\alpha\} \) in \( P^1(G) \) such that \( \lim_{\alpha} \| \varphi \odot \varphi_\alpha - \varphi_\alpha \|_1 = 0 \) for every \( \varphi \in P^1(G) \). For every \( \varphi, \psi \in P^1(G) \),

\[
\lim_{\alpha} \| \psi \odot (\varphi \odot \varphi_\alpha) - \psi \odot \varphi_\alpha \|_1 \leq \lim_{\alpha} \| \varphi \odot \varphi_\alpha - \varphi_\alpha \|_1 = 0.
\]

(i) implies (ii): Let \( f \in U^\infty(G) \) and \( \varphi \in P^1(G) \). By Cohen’s factorization theorem, \( U^\infty(G) = L^1(G) \odot L^\infty(G) \) [23]. Therefore \( f \) is of the form \( f = \psi_0 \odot f_0 \) for some \( \psi_0 \in L^1(G) \) and \( f_0 \in L^\infty(G) \). By considering Jordan decomposition, it is clear that statement (i) holds for any \( \psi \in L^1(G) \). Hence

\[
\lim_{\alpha} \langle f, \varphi \odot \varphi_\alpha - \varphi_\alpha \rangle = \lim_{\alpha} \langle f_0, \psi_0 \odot (\varphi \odot \varphi_\alpha) - \psi_0 \odot \varphi_\alpha \rangle = 0.
\]
(ii) implies $G$ is inner amenable: It suffices to show that $U^\infty(G)$ has a topologically inner invariant mean. By Proposition 3.3 in [21], the net $\{\varphi_\alpha\}$ admits a subnet $\{\varphi_\beta\}$ converging to a mean $m$ in the weak* topology of $L^\infty(G)$. For all $f \in U^\infty(G)$ and $\varphi \in P^1(G)$,

$$\langle m, \varphi \circ f - f \rangle = \lim_{\beta} \langle f, \varphi \otimes \varphi_\beta - \varphi_\beta \rangle = 0.$$  

(iii) implies $G$ is inner amenable: This is similar to the last implication. Let $\{\varphi_\alpha\}$ be as in statement (iii) and define $m$ as above. Then for $f \in U^\infty(G)$ and $x \in G$,

$$\langle m, \rho(x)f - f \rangle = \lim_{\beta} \langle f, \pi_1(x)\varphi_\beta - \varphi_\beta \rangle = 0.$$  

Inner amenable implies (iii): This is an immediate consequence of Theorem 1 of [24].

**Theorem 3.6.** Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. A locally compact group $G$ is inner amenable if and only if

$$\inf\{\sup\{\inf\{\langle \pi_p(y)\varphi + \varphi, \varphi \rangle; y \in K\}; \varphi \in P^p(G), \varphi \in P^q(G)\}; K \in \mathcal{K}\} = 2,$$

where $\mathcal{K}$ is the family of compact subsets of $G$.

**Proof.** Suppose that $G$ is inner amenable. Let $K$ be a compact subset of $G$ and $\varepsilon > 0$. By Theorem 1 in [26], there exists $\varphi \in P^1(G)$ such that, for every $y \in K$,

$$\|\pi_1(y)\varphi - \varphi\|_1 < \varepsilon^p.$$

For $a \geq 0$, the map $x \mapsto x^p - a^p - (x - a)^p$ is increasing from $\mathbb{R}^+ \to \mathbb{R}$. So that $(b - a)^p \leq b^p - a^p$ for all $b \geq a$. Let $\varphi = \varphi^\frac{b}{p}$. For every $y \in K$, we obtain

$$\|\pi_p(y)\varphi - \varphi\|_p^p = \int |\varphi^\frac{1}{p}(y^{-1}xy)\Delta(y)^\frac{1}{p} - \varphi^\frac{1}{p}(x)|^p dx \leq \int |\varphi(y^{-1}xy)\Delta(y) - \varphi(x)| dx \leq \|\pi_1(y)\varphi - \varphi\|_1 < \varepsilon^p.$$

Now let $\varphi = \varphi^\frac{q}{p}$. For every $y \in K$,

$$\langle \pi_p(y)\varphi + \varphi, \varphi \rangle = \langle \pi_p(y)\varphi - \varphi, \varphi \rangle + 2\langle \varphi, \varphi \rangle > 2 - \varepsilon.$$  

As $\varepsilon > 0$ and $\varepsilon \in \mathcal{K}$ are arbitrary, we have

$$\inf\{\sup\{\inf\{\langle \pi_p(y)\varphi + \varphi, \varphi \rangle; y \in K\}; \varphi \in P^p(G), \varphi \in P^q(G)\}; K \in \mathcal{K}\} = 2.$$

Conversely if the condition holds, let $K$ be a compact subset of $G$ and $\varepsilon > 0$. Then there exist $\varphi \in P^p(G)$ and $\psi \in P^q(G)$ such that $\langle \pi_p(y)\varphi + \varphi, \psi \rangle > 2 - \varepsilon$ for every $y \in K$. It follows that $\|\pi_p(y)\varphi + \varphi\|_p > 2 - \varepsilon$ for every $y \in K$. For every $y \in K$, by the Clarkson’s inequalities, we obtain

$$\|\pi_p(y)\varphi + \varphi\|_p^p + \|\pi_p(y)\varphi - \varphi\|_p^p \leq 2^{p-1}(\|\pi_p(y)\varphi\|_p^p + \|\varphi\|_p^p) = 2^p.$$  

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in case \( p \geq 2 \), and so \( \|\pi_p(y)\varphi - \varphi\|_p^p < 2^p - (2 - \epsilon)^p \). We have

\[
\|\pi_p(y)\varphi + \varphi\|_p^p + \|\pi_p(y)\varphi - \varphi\|_p^p \leq 2^{q+1-p}(\|\pi_p(y)\varphi\|_p^p + \|\varphi\|_p^p)^{p-1} = 2^q
\]

in case \( 1 < p < 2 \), and so \( \|\pi_p(y)\varphi - \varphi\|_p^q < 2^q - (2 - \epsilon)^q \). Since this holds for all \( y \in K \), we conclude that \( G \) is inner amenable [24].

**Corollary 3.7.** Let \( 1 < p, q < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). The following conditions are equivalent:

(i) \( G \) is inner amenable;

(ii) \( \inf\{\sup\{\langle \phi \otimes \varphi + \varphi, \psi \rangle ; \varphi \in P^p(G), \psi \in P^q(G)\}, \phi \in P^1(G)\} = 2. \)

**Proof.** (i) implies (ii): Let \( \phi \in P^1(G) \) and \( \epsilon \in (0, 1) \). Choose \( \phi_1 \in C_c(G)^+ \) with compact support \( K \) such that \( \|\phi - \phi_1\|_1 < \epsilon \), hence \( \|\phi_1\|_1 > 1 - \epsilon \) [10]. By Theorem 3.6, we may determine \( \varphi \in P^p(G) \) and \( \psi \in P^q(G) \) such that \( \langle \pi_p(y)\varphi + \varphi, \psi \rangle > 2 - \epsilon \) for all \( y \in K \). By integration, we obtain \( \langle \phi_1 \otimes \varphi + \varphi, \psi \rangle \geq (2 - \epsilon)\|\phi_1\|_1 > (2 - \epsilon)(1 - \epsilon) \). We have

\[
\langle \phi \otimes \varphi + \varphi, \psi \rangle \geq \langle \phi_1 \otimes \varphi + \varphi, \psi \rangle \geq (2 - \epsilon)(1 - \epsilon).
\]

This shows that \( \inf\{\sup\{\langle \phi \otimes \varphi + \varphi, \psi \rangle ; \varphi \in P^p(G), \psi \in P^q(G)\}, \phi \in P^1(G)\} = 2. \)

(ii) implies (i): Let \( \phi \in P^1(G) \). By assumption, given \( \epsilon \in (0, 1) \), there exist \( \varphi \in P^p(G) \) and \( \psi \in P^q(G) \) such that \( \langle \phi \otimes \varphi + \varphi, \psi \rangle > 2 - \epsilon \). It follows that \( \langle \phi \otimes \varphi, \psi \rangle > 1 - \epsilon \). We consider \( L_\varphi : L^p(G) \to L^p(G) \) by \( L_\varphi(\psi) = \phi \otimes \varphi \). Clearly \( \|L_\varphi\| > 1 - \epsilon \), and so \( \|L_\varphi\| = 1 \). Since this holds for all \( \phi \in P^1(G) \), by a form of the Riesz-Thorin Convexity Theorem ([4], VI.10.11), \( L_\varphi : L^2(G) \to L^2(G) \) has norm 1. Define \( \omega_1 : \{L_\varphi; \phi \in L^1(G)\} \to \mathbb{C} \) by \( \omega_1(L_\varphi) = \int \varphi(x)dx \). By the Hahn Banach theorem for states (see Proposition 2.3.24 in [2]), we can extend \( \omega_1 \) to a state \( \omega \) on the algebra \( B(L^2(G)) \) of bounded operators on \( L^2(G) \). Therefore \( G \) is inner amenable by Theorem 2 in [26].

Lau and Paterson [13] gave a necessary condition on a locally compact group \( G \) to have an inner invariant mean \( m \) such that \( \langle m, 1_V \rangle = 0 \) for some compact neighborhood \( V \) of \( G \) invariant under the inner automorphisms. Let \( A \) be a Borel subset of \( G \). In the following theorem, we provide a necessary and sufficient condition for \( G \) to have an inner invariant mean \( m \) with \( \langle m, 1_A \rangle = 1 \).

**Theorem 3.8.** Let \( G \) be an inner amenable group and let \( A \) be a Borel subset of \( G \). Then the following statements are equivalent:

(i) There is a topologically inner invariant mean on \( L^\infty(G) \) such that \( \langle m, 1_A \rangle = 1 \);

(ii) \( \inf\{\sup\{\langle \pi_1(y)\varphi, 1_A \rangle ; y \in K \}; \varphi \in P^1(G)\}; K \in K \} = 1. \)
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Proof. (i) implies (ii): Assume that there is a topologically inner invariant mean $m$ on $L^\infty(G)$ such that $\langle m, 1_A \rangle = 1$. As $P^1(G)$ is weak* dense in the convex set of all means on $L^\infty(G)$ (see Proposition 3.3 in [21]), there exists a net $\{\varphi_\alpha\}$ in $P^1(G)$ such that, for every $\varphi \in P^1(G)$, $\{\varphi \otimes \varphi_\alpha - \varphi_\alpha\}$ converges to 0 in the weak topology of $L^1(G)$. Let $\varphi_0 \in P^1(G)$ be fixed and put $\varphi_\alpha = \varphi_0 \otimes \varphi_\alpha$. It is easy to see that $\{\psi_\alpha\}$ converging to $\varphi_0,m$ in the weak* topology of $L^\infty(G)$, and also $\langle \varphi_0,m,1_A \rangle = 1$. Let $\epsilon > 0$ and $K \subseteq G$ compact be given. As $\varphi_0 \in L^1(G)$, the mapping $y \mapsto y\varphi_0$ is continuous [10], so there exists an open neighbourhood $V$ of $e$ in $G$ such that, for all $y \in V$, $\|y\varphi_0 - \varphi_0\|_1 < \frac{\epsilon}{2}$ [6]. We may determine a subset $\{y_1, ..., y_n\}$ in $K$ such that $K \subseteq \bigcup_{i=1}^n y_i V$ and $\|y\varphi_0 - y_i\varphi_0\|_1 < \frac{\epsilon}{2}$ whenever $y \in y_i V \cap K$ and $i \in \{1, ..., n\}$. There exists $a_0 \in I$ such that, for every $\alpha \in I$ with $\alpha \geq a_0$ and every $i \in \{1, ..., n\}$

$$\left|\langle \sigma_1(y_i)\varphi_\alpha - \varphi_\alpha, 1_A \rangle\right| \leq \left|\langle \sigma_1(y_i)\varphi_\alpha - \varphi_\alpha, 1_A \rangle\right| + \left|\langle \varphi_\alpha - \varphi_\alpha, 1_A \rangle\right|$$

$$\leq \left|\langle \sigma_1(y_i)\varphi_\alpha - \varphi_\alpha, 1_A \rangle\right| + \left|\langle \varphi_\alpha - \varphi_\alpha, 1_A \rangle\right|$$

$$\leq \left|\langle \sigma_1(y_i)\varphi_\alpha - \varphi_\alpha, 1_A \rangle\right| + \begin{cases} \frac{\epsilon}{2} & \text{if } \varphi_\alpha \geq a_0 \text{ on } K \end{cases}$$

For any $y \in K$, there exist $i \in \{1, ..., n\}$ and $v \in V$ such that $y = y_i v$. Then we have

$$\left|\langle \sigma_1(y)\varphi_\alpha - \varphi_\alpha, 1_A \rangle\right| = \left|\langle \sigma_1(y)\varphi_\alpha - \sigma_1(y_i)\varphi_\alpha + \sigma_1(y_i)\varphi_\alpha - \varphi_\alpha, 1_A \rangle\right|$$

$$\leq \left|\langle \sigma_1(y)\varphi_\alpha - \sigma_1(y_i)\varphi_\alpha, 1_A \rangle\right| + \begin{cases} \frac{\epsilon}{2} & \text{if } \varphi_\alpha \geq a_0 \text{ on } K \end{cases}$$

for every $\alpha \geq a_0$. This shows that $\lim_\alpha \langle \sigma_1(y)\varphi_\alpha - \varphi_\alpha, 1_A \rangle = 0$ uniformly on compacta.

Now let $K$ be a compact subset of $G$ and $\epsilon > 0$. Then there is some $a_0 \in I$ such that

$$\left|1 - \langle \psi_0, 1_A \rangle\right| = \left|\langle \varphi_0,m, 1_A \rangle - \langle \psi_0, 1_A \rangle\right| < \frac{\epsilon}{2}$$

and $\left|\langle \sigma_1(y)\psi_0 - \psi_0, 1_A \rangle\right| < \frac{\epsilon}{2}$ for all $y \in K$. Clearly $\langle \sigma_1(y)\psi_0, 1_A \rangle > 1 - \epsilon$ for all $y \in K$. We conclude that

$$\inf\{\sup\{\inf\{\langle \sigma_1(y)\varphi, 1_A \rangle; y \in K\}; \varphi \in P^1(G)\}; K \in \mathcal{K}\} = 1.$$

(ii) implies (i): We consider the directed set $I = \mathcal{K} \times (0, 1)$ where, for $\alpha = (K, \epsilon) \in I$, $\alpha' = (K', \epsilon') \in I$, $\alpha' \geq \alpha$ in case $K \subseteq K'$ and $\epsilon' \leq \epsilon$. By assumption, given $\alpha = (K, \epsilon)$, there exist $\varphi_\alpha \in P^1(G)$ such that $\langle \sigma_1(y)\varphi_\alpha, 1_A \rangle > 1 - \epsilon$ for all $y \in K$. Let $\varphi \in P^1(G)$ be such that $\varphi$ is supported on $K$. We have

$$\langle \varphi \otimes \varphi_\alpha, 1_A \rangle = \int \langle \sigma_1(y)\varphi_\alpha, 1_A \rangle \varphi(y)dy \geq 1 - \epsilon.$$
if \( m_0 \) is an inner invariant mean on \( L^\infty(G) \), then \( m_0|_{U^\infty(G)} \) is an inner invariant mean on \( U^\infty(G) \). By Lemma 3.2, \( m_0|_{U^\infty(G)} \) is a topologically inner invariant mean. On the other hand, any topologically inner invariant mean on \( U^\infty(G) \) may be extended to a topologically inner invariant mean on \( L^\infty(G) \). Thus we can find a topologically inner invariant mean \( m_1 \) on \( L^\infty(G) \). Clearly \( m = m_1.n \) is a mean on \( L^\infty(G) \). Let \( \{\psi_\gamma\} \) be a net in \( P^1(G) \) converging to \( m_1 \) in the weak* topology of \( L^\infty(G) \). We have

\[
|\langle m, 1_A \rangle| = |\langle m_1.n, 1_A \rangle| = |\langle m_1, n.1_A \rangle| = \lim_\gamma |\langle \psi_\gamma, n.1_A \rangle|
= \lim_\gamma |\langle n, 1_A, \psi_\gamma \rangle| = 1.
\]

It is straightforward to verify that \( m \) is a topologically inner invariant mean (since \( m_1 \) is) on \( L^\infty(G) \). This completes our proof.  

Acknowledgements I would like to thank the referee for his/her careful reading of my paper and many valuable suggestions.

References

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