Stability of solutions to integro-differential equations in Hilbert spaces*

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Dedicated to Jan Van Casteren on the occasion of his 65th birthday

Abstract

In this paper, we investigate the uniform exponential stability of solutions to abstract integro-differential equations in Hilbert spaces by the theory of operator semigroups and Laplace transforms of vector-valued functions. New criterions are given based on the growth property of associated vector-valued functions on the right half plane. Examples are presented to illustrate our results.

1 Introduction

We are concerned with the following abstract integro-differential equations

$$u'(t) = Au(t) + \int_0^t a(t-s)Au(s)\,\mathrm{d}s \quad (t \geq 0), \quad u(0) = x. \tag{1.1}$$

Here, A is the infinitesimal generator of a C_0-semigroup $T(t)$ defined on a Hilbert space H, the scalar function $a(\cdot) \in L^2[0,\infty)$, which is called *kernel function*. These equations provide useful and important mathematical models for engineering

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problems, for instance, the transient velocity field in an *isotropic viscoelastic fluid*, *Timoshenko beam*, and *heat conduction with memory* [9]. So, the integro-differential equations and systems in abstract spaces have received much attention [1, 3, 4, 6, 7, 8, 9, 14]. Recently, uniform exponential stability of the solutions to abstract Volterra equations was studied in [9] and [2]. It is worth noting that, as shown in [2, Example 1.2], this problem is nontrivial even in the special case of $a(t) = \alpha e^{-\beta t}$ with $\alpha \neq 0$, $\beta > 0$. In this paper, we study the problem in the setting of Hilbert spaces, and improve essentially the previous results in the setting of Banach spaces by dropping the assumption that the associated semigroup $T(t)$ is immediately norm continuous. Moreover, the new criterions are given in terms of the growth of the related vector-valued functions on the right half plane.

Definition 1.1. The solutions to (1.1) are called *uniformly exponentially stable* if there exist some constants $\omega < 0$ and $M > 0$ such that for each $x \in D(A)$, the corresponding solution $u(t)$ satisfies

$$\|u(t)\| \leq Me^{\omega t}\|x\|, \quad t \geq 0. \tag{1.2}$$

It is well-known (see, e.g., [7, Sect. 1]) that (1.1) is *well-posed* if and only if the *resolvent* $\mathbb{S}(t)$ of (1.1) exists. In this situation, $u(t) = \mathbb{S}(t)x$ $(t \geq 0)$ with $x \in H$ is a mild solution of (1.1), which gives the unique classical solution if $x \in D(A)$. Moreover, the *growth bound* of the resolvent $\mathbb{S}(t)$ is

$$\omega_0 := \inf\{\omega : \text{ there exists } M > 0 \text{ such that (1.2) holds}\} = \limsup_{t\to\infty} \frac{\log\|\mathbb{S}(t)\|}{t}. \tag{1.3}$$

However, generally speaking, $\mathbb{S}(t)$ is subtle so that (1.3) is inconvenient in use for obtaining the uniform exponential stability of the solutions.

The rest of this paper is organized as follows. Section 2 is devoted to proving our results. In Section 3, we give two examples to illustrate the abstract results.

2 Results and proofs

It is known that the integro-differential equation (1.1) as well as its inhomogeneous Cauchy problem can be converted to an abstract Cauchy problem on a product space (see, e.g., [5, VI. 7]). This technique has been widely used (see, e.g., [1, 2, 4, 7, 13]). In the following, we restate some notations and related results for the sake of convenience. First, we introduce the product Hilbert space $\mathscr{H} := H \times L^2(\mathbb{R}_+, H)$ with the inner product $\left\langle \binom{x_1}{f_1}, \quad \binom{x_2}{f_2} \right\rangle_{\mathscr{H}} := (x_1, x_2)_H + (f_1, f_2)_{L^2(\mathbb{R}_+,H)}$. Next, define on $\mathscr{H}$ the operator matrices

$$\mathscr{T}(t) := \begin{pmatrix} T(t) & R(t) \\ 0 & S(t) \end{pmatrix}, \quad t \geq 0.$$

Here, $S(t)$ (the left shift semigroup) and $R(t)$ are defined on $L^2(\mathbb{R}_+, H)$,

$$(S(t)f)(\tau) := f(\tau + t), \quad R(t)f := \int_0^t T(t-s)f(s)\,\mathrm{d}s, \quad f \in L^2(\mathbb{R}_+, H).$$

Then $\mathscr{T}(t)$ forms a C_0-semigroup on $\mathscr{H}$ with the generator given by

$$\mathscr{A} := \begin{pmatrix} A & \delta_0 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}s} \end{pmatrix}, \quad D(\mathscr{A}) := D(A) \times H^1(\mathbb{R}_+, H),$$

where, $H^1(\mathbb{R}_+, H)$ denotes the vector-valued Sobolev space and δ_0 the Dirac distribution, i.e., $\delta_0(f) = f(0)$ for each $f \in H^1(\mathbb{R}_+, H)$. Finally, define

$$Bx := a(\cdot)Ax, \quad x \in D(A),$$

$$\mathscr{B}\begin{pmatrix} x \\ f \end{pmatrix} := \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}\begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ Bx \end{pmatrix}, \quad \begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathscr{B}) := D(\mathscr{A})$$

and denote

$$\mathscr{A}_p := \mathscr{A} + \mathscr{B}, \quad D(\mathscr{A}_p) := D(\mathscr{A}).$$

Then it follows that $\mathscr{A}_p$ also generates a C_0-semigroup denoted by $\mathscr{S}(t)$. In view of [2, Proposition 2.8], we have

Proposition 2.1. *Let M be a closed subspace of $L^2(\mathbb{R}_+, H)$ such that M is $S(t)$-invariant and $a(\cdot)Ax \in M$ for all $x \in D(A)$. Then the following two assertions hold.*

(a) The operator $\mathscr{A}_p|_{\mathcal{D}}$ generates the C_0-semigroup $\mathscr{S}(t)|_{\mathcal{M}}$ defined on the Hilbert space $\mathcal{M} := H \times M$. Here $\mathcal{D} := D(A) \times \{f \in H^1(\mathbb{R}_+, H) \cap M : f' \in M\} := D(A) \times M_1$.

(b) For each $x \in D(A)$ and $f \in M_1$, if we write

$$\mathscr{S}(t)|_{\mathcal{M}}\begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} u(t) \\ F(t,\cdot) \end{pmatrix}, \tag{2.1}$$

then $u(t)$ is the unique classical solution to the Volterra equation

$$u'(t) = Au(t) + \int_0^t a(t-s)Au(s)\,\mathrm{d}s + f(t) \ \ (t \geq 0), \quad u(0) = x. \tag{2.2}$$

Remark 2.2. Let the function $t \mapsto u(t)$ be a solution to the equation (2.2) and put

$$F(t,\tau) = f(t+\tau) + \int_{t+\tau}^{\tau} a(s)Au(t+\tau-s)\,\mathrm{d}s, \ \ t \geq 0, \ \tau \geq 0.$$

Then, the semigroup generated by the operator $\mathscr{A}_p$ is “essentially” given by the mapping

$$\begin{pmatrix} x \\ f(\cdot) \end{pmatrix} \mapsto \begin{pmatrix} u(t) \\ F(t,\cdot) \end{pmatrix}, \quad t \geq 0,$$

where $u(0) = x$ and $F(0,\tau) = f(\tau)$ are given.

The following two lemmas are quite useful in the paper. The first one is a partial extension of the Gearhart theorem [5, Theorem V.1.11] and comes from [12, Lemma 3.11.7]. The second one gives an explicit expression of $R(\lambda, \mathscr{A}_p)$.

Lemma 2.3. **([12, Lemma 3.11.7])** *Let A be the generator of a C_0-semigroup $T(t)$ on a Hilbert space with $\rho(A) \supset \mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Assume that there exist some constants $K > 0$ and $\gamma \in [0, 1)$ such that*

$$\|(\lambda - A)^{-1}\| \leq \frac{K}{(\operatorname{Re}\lambda)^{\gamma}}, \quad \lambda \in \mathbb{C}_+. \tag{2.3}$$

Then $\omega(A) \leq -\delta$. Here, $\omega(A)$ denotes the growth bound of $T(t)$ and

$$\delta := \begin{cases} \frac{1}{K}, & \gamma = 0, \\ (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}} K^{-\frac{1}{1-\gamma}}, & 0 < \gamma < 1. \end{cases} \tag{2.4}$$

Lemma 2.4. *Let A be the generator of $T(t)$ with $\rho(A) \supset \mathbb{C}_+$. Assume that $\lambda \left[1 + \hat{a}(\lambda)\right]^{-1} \in \rho(A)$ for each $\lambda \in \mathbb{C}_+$. Then $\rho(\mathscr{A}_p) \supset \mathbb{C}_+$ and for $\lambda \in \mathbb{C}_+$, $R\left(\lambda, \mathscr{A}_p\right)$ is given by*

$$R\left(\lambda, \mathscr{A}_p\right)\begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} H(\lambda)\left[x + \hat{f}(\lambda)\right] \\ \tilde{a}_\lambda(\cdot)\tilde{H}(\lambda)\left[x + \hat{f}(\lambda)\right] + \tilde{f}_\lambda(\cdot) \end{pmatrix}. \tag{2.5}$$

Here, the hat indicates the Laplace transform (cf., e.g., [15, Chap. 1]) and

$$H(\lambda) := \left[\lambda - (1 + \hat{a}(\lambda))\, A\right]^{-1}, \quad \tilde{H}(\lambda) := AH(\lambda) = -\frac{1}{1 + \hat{a}(\lambda)}\left[I - \lambda H(\lambda)\right], \tag{2.6}$$

$$\tilde{f}_\lambda(s) := \left(R\left(\lambda, \frac{\mathrm{d}}{\mathrm{d}s}\right) f\right)(s) = e^{\lambda s}\int_s^\infty e^{-\lambda t} f(t)\,\mathrm{d}t, \quad f \in L^2(\mathbb{R}_+, H), \tag{2.7}$$

$$\tilde{a}_\lambda(s) := e^{\lambda s}\int_s^\infty e^{-\lambda t} a(t)\,\mathrm{d}t. \tag{2.8}$$

Proof. For each $\lambda \in \rho(A)$ with $\operatorname{Re}\lambda > 0$, by [5, Proposition VI.7.25] we easily see that $\lambda \in \rho(\mathscr{A}_p)$ if and only if

$$\lambda\left[1 + \hat{a}(\lambda)\right]^{-1} \in \rho(A).$$

Moreover, for each $\lambda \in \rho(A)$ with $\operatorname{Re}\lambda > 0$, from [5, Lemma VI.7.23, Lemma VI.7.24] we know

$$R(\lambda, \mathscr{A}) = \begin{pmatrix} R(\lambda, A) & R(\lambda, A)\delta_0 R\left(\lambda, \frac{\mathrm{d}}{\mathrm{d}s}\right) \\ 0 & R\left(\lambda, \frac{\mathrm{d}}{\mathrm{d}s}\right) \end{pmatrix}.$$

Consequently, defining $H(\lambda)$ as in (2.6) we calculate

$$\begin{aligned} R\left(\lambda, \mathscr{A}_p\right) &= \left[I - R(\lambda, \mathscr{A})\mathscr{B})\right]^{-1} R\left(\lambda, \mathscr{A}\right) \\ &= \begin{pmatrix} I - R(\lambda, A)\delta_0 R\left(\lambda, \frac{\mathrm{d}}{\mathrm{d}s}\right) B & 0 \\ -R\left(\lambda, \frac{\mathrm{d}}{\mathrm{d}s}\right) B & I \end{pmatrix}^{-1} R(\lambda, \mathscr{A}) \\ &= \begin{pmatrix} \left[I - \hat{a}(\lambda)R(\lambda, A)A\right]^{-1} & 0 \\ R\left(\lambda, \frac{\mathrm{d}}{\mathrm{d}s}\right) B\left[I - \hat{a}(\lambda)R(\lambda, A)A\right]^{-1} & I \end{pmatrix}\begin{pmatrix} R(\lambda, A) & R(\lambda, A)\delta_0 R\left(\lambda, \frac{\mathrm{d}}{\mathrm{d}s}\right) \\ 0 & R\left(\lambda, \frac{\mathrm{d}}{\mathrm{d}s}\right) \end{pmatrix} \\ &= \begin{pmatrix} H(\lambda) & H(\lambda)\delta_0 R\left(\lambda, \frac{\mathrm{d}}{\mathrm{d}s}\right) \\ R\left(\lambda, \frac{\mathrm{d}}{\mathrm{d}s}\right) BH(\lambda) & R\left(\lambda, \frac{\mathrm{d}}{\mathrm{d}s}\right) BH(\lambda)\delta_0 R\left(\lambda, \frac{\mathrm{d}}{\mathrm{d}s}\right) + R\left(\lambda, \frac{\mathrm{d}}{\mathrm{d}s}\right) \end{pmatrix}. \end{aligned}$$

By (2.7) (cf. [5, II.2.10]), a simple computation yields (2.5). ■

The following result provides a criterion to judge uniform exponential stability of the solutions to (1.1).

Theorem 2.5. *Assume that*

(i) *A generates a C_0-semigroup $T(t)$ on a Hilbert space H with $\rho(A) \supset \mathbb{C}_+$.*

(ii) $\lambda\,[1+\hat{a}(\lambda)]^{-1} \in \rho(A)$ *for each* $\lambda \in \mathbb{C}_+$.

(iii) *There exists a closed subspace M of $L^2(\mathbb{R}_+, H)$ such that M is $S(t)$-invariant and $a(\cdot)Ax \in M$ for all $x \in D(A)$.*

(iv) *The estimates*

$$\|\hat{f}(\lambda)\| \le P(\lambda)\|f\|_{L^2}, \quad \lambda \in \mathbb{C}_+, \quad f \in M \tag{2.9}$$

and

$$\left\| e^{\lambda\cdot} \int_{\cdot}^{\infty} e^{-\lambda t} f(t)\,\mathrm{d}t \right\|_{L^2} \le Q(\lambda)\|f\|_{L^2}, \quad \lambda \in \mathbb{C}_+, \quad f \in M \tag{2.10}$$

hold for some functions $P(\lambda)$ and $Q(\lambda)$.

Then the solutions to (1.1) *are uniformly exponentially stable if there exist two constants $C > 0$ and $\gamma \in [0,1)$ such that*

$$\begin{aligned} \xi(\lambda) \;&:=\; \left[\|H(\lambda)\|^2 + \frac{2\|\tilde{a}_\lambda(\cdot)\|_{L^2}^2}{|1+\hat{a}(\lambda)|^2} \cdot \|I - \lambda H(\lambda)\|^2\right] \cdot \left[|P(\lambda)|^2 + 1\right] + |Q(\lambda)|^2 \\ &\le\; \frac{C}{(\mathrm{Re}\lambda)^{2\gamma}}, \quad \lambda \in \mathbb{C}_+. \end{aligned} \tag{2.11}$$

Here, $H(\lambda)$ and $\tilde{a}_\lambda(s)$ are defined by (2.6) *and* (2.8).

Proof. By Proposition 2.1, for each $x \in D(A)$, the first coordinate of (2.1) with $f = 0$ is just the unique classical solution to (1.1). Thus, to show that the solutions to (1.1) are uniformly exponentially stable, it is sufficient to prove that $\mathscr{S}(t)|_{\mathcal{M}}$ is exponentially stable. To reach the purpose, by Lemma 2.3, we only need to show that $\rho\left(\mathscr{A}_p|_{\mathcal{D}}\right) \supset \mathbb{C}_+$ and that the estimate (2.3) holds for $\mathscr{A}_p|_{\mathcal{D}}$. Actually, it is evident that $\rho\left(\mathscr{A}_p|_{\mathcal{D}}\right) \supset \rho\left(\mathscr{A}_p\right)$. Therefore, by Lemma 2.4, if $\rho(A) \supset \mathbb{C}_+$ and

$$\lambda\,[1+\hat{a}(\lambda)]^{-1} \in \rho(A)$$

for each $\lambda \in \mathbb{C}_+$, then

$$\rho\left(\mathscr{A}_p|_{\mathcal{D}}\right) \supset \rho\left(\mathscr{A}_p\right) \supset \mathbb{C}_+.$$

Further, for each $\lambda \in \mathbb{C}_+$ and $x \in H$, $f \in M$, by (2.5) we estimate

$$\begin{aligned} \left\| R\left(\lambda, \mathscr{A}_p|_{\mathcal{D}}\right) \begin{pmatrix} x \\ f \end{pmatrix} \right\|^2 \;&=\; \left\| \begin{pmatrix} H(\lambda)\left[x + \hat{f}(\lambda)\right] \\ \tilde{a}_\lambda(\cdot)\tilde{H}(\lambda)\left[x + \hat{f}(\lambda)\right] + \tilde{f}_\lambda(\cdot) \end{pmatrix} \right\|^2 \\ &\le\; \|H(\lambda)\|^2 \cdot \left\|x + \hat{f}(\lambda)\right\|^2 \\ &\quad + 2\|\tilde{a}_\lambda(\cdot)\|_{L^2}^2 \cdot \|\tilde{H}(\lambda)\|^2 \cdot \left\|x + \hat{f}(\lambda)\right\|^2 + 2\|\tilde{f}_\lambda(\cdot)\|_{L^2}^2. \end{aligned}$$

This combined with (2.6), (2.9) and (2.10) yields

$$
\begin{aligned}
&\left\| R\left(\lambda, \mathscr{A}_p|_{\mathcal{D}}\right) \begin{pmatrix} x \\ f \end{pmatrix} \right\|^2 \\
\leq\ & \left(2\|H(\lambda)\|^2 + \frac{4\|\tilde{a}_\lambda(\cdot)\|^2_{L^2}}{|1+\hat{a}(\lambda)|^2} \cdot \|I - \lambda H(\lambda)\|^2 \right) \|x\|^2 \\
& + \left(2\|H(\lambda)\|^2 + \frac{4\|\tilde{a}_\lambda(\cdot)\|^2_{L^2}}{|1+\hat{a}(\lambda)|^2} \cdot \|I - \lambda H(\lambda)\|^2 \right) |P(\lambda)|^2 \cdot \|f\|^2_{L^2} \\
& + 2|Q(\lambda)|^2 \cdot \|f\|^2_{L^2} \\
\leq\ & 2\xi(\lambda) \left[\|x\|^2 + \|f\|^2_{L^2} \right],
\end{aligned}
\tag{2.12}
$$

where, $\xi(\lambda)$ is the same as in (2.11). Consequently, applying Lemma 2.3 we conclude that $\mathscr{S}(t)|_{\mathcal{M}}$ is exponentially stable if (2.11) holds for some constants $C > 0$ and $\gamma \in [0, 1)$. The proof is complete. ■

In the special case of $a(t) = \alpha e^{-\beta t}$ with $\beta > 0$, $\alpha \in \mathbb{C} \setminus \{0\}$, Theorem 2.5 takes a concise form owing to the following

Proposition 2.6. *$M := \{e^{-\beta s} x : x \in H\}$ is a closed subspace of $L^2(\mathbb{R}_+, H)$ such that M is $S(t)$-invariant and $a(\cdot)Ax \in M$ for all $x \in D(A)$. Moreover,* (2.9) *and* (2.10) *hold for $P(\lambda) = \frac{\sqrt{2\beta}}{|\lambda+\beta|}$ and $Q(\lambda) = \frac{1}{|\lambda+\beta|}$.*

Proof. The first part of this statement, which is easy to be checked, can be found in [2, Lemma 3.1]. Also, it is not hard to compute

$$
\hat{a}(\lambda) = \frac{\alpha}{\lambda + \beta}, \quad \tilde{a}_\lambda(s) = \frac{\alpha}{\lambda+\beta} e^{-\beta s}.
$$

So (2.9) and (2.10) hold for

$$
P(\lambda) = \frac{\sqrt{2\beta}}{|\lambda + \beta|}, \quad Q(\lambda) = \frac{1}{|\lambda+\beta|}, \quad \lambda \in \mathbb{C}_+.
$$

This completes the proof. ■

Theorem 2.7. *Assume that*

(i) *A generates a C_0-semigroup on a Hilbert space H with $\rho(A) \supset \mathbb{C}_+$.*

(ii) *$a(t) = \alpha e^{-\beta t} (\beta > 0, \alpha \neq 0)$ with $\mathrm{Re}(\alpha + \beta) > 0$.*

(iii) *$\frac{\lambda(\lambda+\beta)}{\lambda+\alpha+\beta} \in \rho(A)$ for each $\lambda \in \mathbb{C}_+$.*

Then the solutions to (1.1) *are uniformly exponentially stable if there exist two constants $C > 0$ and $\gamma \in [0, 1)$ such that*

$$
\left\| \left[\frac{\lambda(\lambda+\beta)}{\lambda+\alpha+\beta} - A \right]^{-1} \right\| \leq \frac{C}{(\mathrm{Re}\lambda)^\gamma}, \quad \lambda \in \mathbb{C}_+. \tag{2.13}
$$

Proof. First, putting

$$P(\lambda) = \frac{\sqrt{2\beta}}{|\lambda+\beta|}, \quad Q(\lambda) = \frac{1}{|\lambda+\beta|}$$

and calculating accordingly the corresponding terms in (2.11), we obtain

$$H(\lambda) := \frac{\lambda+\beta}{\lambda+\alpha+\beta}\left[\frac{\lambda(\lambda+\beta)}{\lambda+\alpha+\beta} - A\right]^{-1} \tag{2.14}$$

and

$$\begin{aligned} \xi(\lambda) &:= \left[\|H(\lambda)\|^2 + \frac{|\alpha|^2}{\beta\cdot|\lambda+\alpha+\beta|^2}\cdot\|I-\lambda H(\lambda)\|^2\right]\cdot\left[\frac{2\beta}{|\lambda+\beta|^2}+1\right] + \frac{1}{|\lambda+\beta|^2} \\ &:= \xi_0(\lambda) + \frac{1}{|\lambda+\beta|^2}. \end{aligned} \tag{2.15}$$

Next, since $\mathrm{Re}(\alpha+\beta) > 0$, we can find a constant $C_1 > 0$ satisfying

$$\begin{aligned} \frac{\|I-\lambda H(\lambda)\|^2}{|\lambda+\alpha+\beta|^2} &\le \left[\frac{1+|\lambda|\cdot\|H(\lambda)\|}{|\lambda+\alpha+\beta|}\right]^2 \\ &\le \left[\|H(\lambda)\| + \frac{1}{|\lambda+\alpha+\beta|}\right]^2 \\ &\le C_1\cdot\|H(\lambda)\|^2 + \frac{1}{|\lambda+\alpha+\beta|^2}, \quad \lambda\in\mathbb{C}_+ \end{aligned}$$

which implies that

$$\xi_0(\lambda) \le C_2\cdot\left[\|H(\lambda)\|^2 + \frac{1}{|\lambda+\alpha+\beta|^2}\right], \quad \lambda\in\mathbb{C}_+ \tag{2.16}$$

holds for some constant $C_2 > 0$. In addition, it is clear that there exists a constant $C_3 > 0$ satisfying

$$\left|\frac{\lambda+\beta}{\lambda+\alpha+\beta}\right| \le C_3, \quad \lambda\in\mathbb{C}_+.$$

Thus, if (2.13) is assumed, then from (2.14) and (2.16) we infer that there exists a constant $C_4 > 0$ such that

$$\xi_0(\lambda) \le \frac{C_4}{(\mathrm{Re}\lambda)^{2\gamma}} + \frac{C_2}{|\lambda+\alpha+\beta|^2}, \quad \lambda\in\mathbb{C}_+. \tag{2.17}$$

Next, noting that $0 \le \gamma < 1$, one can check that there exist constants $C_5, C_6 > 0$ such that

$$\frac{1}{|\lambda+\beta|^2} \le \frac{C_5}{(\mathrm{Re}\lambda)^{2\gamma}}, \quad \frac{1}{|\lambda+\alpha+\beta|^2} \le \frac{C_6}{(\mathrm{Re}\lambda)^{2\gamma}}, \quad \lambda\in\mathbb{C}_+. \tag{2.18}$$

By combining (2.18) with (2.17), and applying Theorem 2.5 we complete the proof. ∎

3 Examples

In this section, we give two examples to illustrate our results. Note that we have estimates for the associated growth bounds, i.e., (3.6) and (3.16).

Example 3.1. Take $H = \mathbb{C}$ and $A = -I$, $a(t) = -e^{-2t}$ in (1.1). Then it is not hard to calculate

$$u(t) = \left(\frac{5+\sqrt{5}}{10} e^{\frac{-3+\sqrt{5}}{2}t} + \frac{5-\sqrt{5}}{10} e^{\frac{-3-\sqrt{5}}{2}t} \right) x. \tag{3.1}$$

Thus, the solutions are uniformly exponentially stable. In the following, we shall verify this again by using Theorem 2.7. First, it is easy to check that all the conditions in Theorem 2.7 are satisfied. Next, substituting $\alpha = -1$, $\beta = 2$ and $A = -I$ into (2.14) we obtain

$$H(\lambda) = \frac{\lambda+2}{\lambda^2+3\lambda+1}, \quad I - \lambda H(\lambda) = \frac{\lambda+1}{\lambda^2+3\lambda+1}.$$

Writing $\lambda = \zeta + i\,\eta$ with $\zeta > 0$, we have

$$\begin{aligned} (\zeta^2+3\zeta+1-\eta^2)^2 + (2\zeta+3)^2\eta^2 &= (\zeta^2+3\zeta+1)^2 + \eta^4 - 2(\zeta^2+3\zeta+1)\eta^2 \\ &\quad +(2\zeta+3)^2\eta^2 \\ &= (\zeta^2+3\zeta+1)^2 + \eta^4 + (2\zeta^2+6\zeta+7)\eta^2 \\ &\geq \frac{1}{4}\left[(\zeta+2)^2+\eta^2\right]. \end{aligned} \tag{3.2}$$

It follows immediately from (3.2) that

$$\|H(\lambda)\|^2 = \frac{|\lambda+2|^2}{|\lambda^2+3\lambda+1|^2} \leq 4, \quad \lambda \in \mathbb{C}_+. \tag{3.3}$$

Analogously, we can get

$$\|I - \lambda H(\lambda)\|^2 = \frac{|\lambda+1|^2}{|\lambda^2+3\lambda+1|^2} \leq 1, \quad \lambda \in \mathbb{C}_+. \tag{3.4}$$

Thus, substituting (3.3) and (3.4) into (2.15) we estimate

$$\begin{aligned} \xi(\lambda) \leq \left(4 + \frac{1}{2|\lambda+1|^2}\right) \cdot \left(\frac{4}{|\lambda+2|^2} + 1\right) + \frac{1}{|\lambda+2|^2} \\ \leq \left(4+\frac{1}{2}\right) \times 2 + \frac{1}{4} = \frac{37}{4}, \quad \lambda \in \mathbb{C}_+. \end{aligned} \tag{3.5}$$

That is to say, (2.15) holds for $\gamma = 0$ and $C = \frac{37}{4}$. By Theorem 2.7, we conclude that the corresponding solutions are uniformly exponentially stable. Moreover, denoting by ω_0 the associated growth bound, and combining (3.5) with (2.12) and (2.4) we have the estimate

$$\omega_0 \leq -\frac{1}{\sqrt{2 \times \frac{37}{4}}} = -\frac{\sqrt{74}}{37}. \tag{3.6}$$

On the other hand, by (3.1), the real growth bound is $\omega_0 = -\frac{3-\sqrt{5}}{2}$. This leads to the comparison

$$\frac{\sqrt{74}}{37} < 0.24 < 0.35 < \frac{3-\sqrt{5}}{2}$$

which shows that (3.6) is reasonable. ∎

Example 3.2. Consider the initial-boundary value problem for the Volterra equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = i\frac{\partial^2 u}{\partial x^2}(t,x) - u(t,x) + \alpha \int_0^t e^{-\beta(t-s)} \left[i\frac{\partial^2 u}{\partial x^2}(s,x) - u(s,x) \right] \mathrm{d}s, \quad t \geq 0, \\ u(0,t) = u(\pi,t) = 0, \\ u(0) = u_0, \end{cases} \tag{3.7}$$

here, as usual, i denotes the imaginary unit, the constants $\alpha \in \mathbb{R}$ and $\beta > 0$ satisfy

$$\alpha + \beta > 0, \quad \alpha < 0. \tag{3.8}$$

First, let $H = L^2[0,\pi]$ and define

$$\tilde{A} := \frac{\mathrm{d}^2}{\mathrm{d}x^2}, \quad A := i\tilde{A} - I, \quad D(A) = D(\tilde{A}) := \left\{ f \in H^2[0,\pi] : f(0) = f(\pi) = 0 \right\}.$$

Then (3.7) can be formulated into an abstract form like (1.1). Indeed, it is well known that $\tilde{A}$ is self-adjoint (see, e.g., [11, pp. 280, (b) of Example 3]) and hence by Stone's theorem ([10, Theorem 1.10.8]), $i\tilde{A}$ generates a unitary C_0-group $e^{i\tilde{A}t}$. Thus, A generates the C_0-group

$$T(t) = e^{-t}e^{i\tilde{A}t}, \quad t \in \mathbb{R}.$$

Note that the self-adjointness of $\tilde{A}$ also implies

$$\|(\lambda - \tilde{A})^{-1}\| = \frac{1}{\operatorname{dist}\left(\lambda, \sigma(\tilde{A})\right)}, \quad \lambda \in \rho(\tilde{A}).$$

Moreover, we can compute directly

$$\sigma(\tilde{A}) = \sigma_p(\tilde{A}) = \{-n^2 : n = 1,2,\ldots\}$$

and hence

$$\sigma(A) = \sigma_p(A) = \{-1 - i\,n^2 : n = 1,2,\ldots\}. \tag{3.9}$$

It follows immediately from (3.9) that $\rho(A) \supset \mathbb{C}_+$. Next, write $\lambda = \zeta + i\,\eta$ with $\zeta > 0$. Since $\alpha < 0$ and $\alpha + \beta > 0$, a simple calculation yields

$$\operatorname{Re}\left(\frac{\lambda(\lambda+\beta)}{\lambda+\alpha+\beta}\right) = \frac{\zeta(\zeta+\beta)(\zeta+\alpha+\beta) + (\zeta-\alpha)\eta^2}{(\zeta+\alpha+\beta)^2 + \eta^2} > 0.$$

As a consequence,

$$\frac{\lambda(\lambda+\beta)}{\lambda+\alpha+\beta} \in \mathbb{C}_+ \subset \rho(\tilde{A}), \quad \lambda \in \mathbb{C}_+$$

and

$$\frac{\lambda(\lambda+\beta)}{\lambda+\alpha+\beta} \in \mathbb{C}_+ \subset \rho(A), \quad \lambda \in \mathbb{C}_+ \tag{3.10}$$

which means that condition (iii) of Theorem 2.7 is satisfied. Furthermore, the constraint $\alpha+\beta>0$ implies the existence of a constant $K>0$ satisfying

$$\frac{|\lambda+\beta|}{|\lambda+\alpha+\beta|} \le K, \quad \lambda \in \mathbb{C}_+. \tag{3.11}$$

On the other hand, for each $\lambda = \zeta + i\eta \in \mathbb{C}_+$ we see easily

$$\operatorname{dist}\big(-(\lambda+1)\,i, \sigma(\tilde{A})\big) \ge \mathrm{Re}\lambda + 1, \quad \lambda \in \mathbb{C}_+ \subset \rho(\tilde{A}). \tag{3.12}$$

By (3.12) we estimate

$$\begin{aligned} \|(\lambda - A)^{-1}\| &= \left\| \left[-(\lambda+1)\,i - \tilde{A}\right]^{-1} \right\|, \quad \lambda \in \rho(A) \\ &= \frac{1}{\operatorname{dist}\big(-(\lambda+1)\,i, \sigma(\tilde{A})\big)}, \quad \lambda \in \rho(A) \\ &\le \frac{1}{\mathrm{Re}\lambda+1} \le 1, \quad \lambda \in \mathbb{C}_+. \end{aligned} \tag{3.13}$$

Combining (3.13) with (3.10) and (3.11) we deduce

$$\begin{aligned} \|H(\lambda)\| &= \frac{|\lambda+\beta|}{|\lambda+\alpha+\beta|} \left\| \left[\frac{\lambda(\lambda+\beta)}{\lambda+\alpha+\beta} - A \right]^{-1} \right\| \\ &\le K \cdot \frac{1}{\mathrm{Re}\left(\frac{\lambda(\lambda+\beta)}{\lambda+\alpha+\beta}\right)+1} \\ &\le K, \quad \lambda \in \mathbb{C}_+ \end{aligned} \tag{3.14}$$

and hence

$$\|I - \lambda H(\lambda)\| \le 1 + |\lambda| \cdot \|H(\lambda)\| \le 1 + K|\lambda|, \quad \lambda \in \mathbb{C}_+. \tag{3.15}$$

Now, from (3.14) and (3.15) we derive

$$\begin{aligned} \xi(\lambda) &= \left[\|H(\lambda)\|^2 + \frac{|\alpha|^2}{\beta \cdot |\lambda+\alpha+\beta|^2} \cdot \|I - \lambda H(\lambda)\|^2 \right] \cdot \left[\frac{2\beta}{|\lambda+\beta|^2} + 1 \right] + \frac{1}{|\lambda+\beta|^2} \\ &\le \left[K^2 + \frac{\alpha^2}{\beta} \left(\frac{K|\lambda|+1}{|\lambda+\alpha+\beta|} \right)^2 \right] \cdot \frac{2+\beta}{\beta} + \frac{1}{\beta^2} \\ &\le \left[K^2 + \frac{\alpha^2}{\beta} \left(K + \frac{1}{\alpha+\beta} \right)^2 \right] \cdot \frac{2+\beta}{\beta} + \frac{1}{\beta^2}, \quad \lambda \in \mathbb{C}_+. \end{aligned}$$

That is to say, (2.11) holds for

$$\gamma = 0, \quad C := \left[K^2 + \frac{\alpha^2}{\beta} \left(K + \frac{1}{\alpha+\beta} \right)^2 \right] \cdot \frac{2+\beta}{\beta} + \frac{1}{\beta^2}.$$

Therefore, the solutions to (3.7) are uniformly exponentially stable. In addition, by (2.4) we have an estimate for the associated growth bound ω_0, namely,

$$\omega_0 \leq \omega_0(\mathscr{A}_p|_{\mathcal{D}}) \leq -\frac{1}{\sqrt{2C}}. \tag{3.16}$$

■

Remark 3.3. Since $T(t)$ is a C_0-group, it is not eventually norm continuous. This implies that [2, Corollary 3.5] is not applicable to Example 3.2.

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