Singular behavior of the solution of the Cauchy-Dirichlet heat equation in weighted L^p-Sobolev spaces

Colette De Coster Serge Nicaise

Dedicated to Jan Van Casteren on the occasion of his 65th birthday

Abstract

We consider the heat equation on a polygonal domain Ω of the plane in weighted L^p-Sobolev spaces

$$\begin{aligned} \partial_t u - \Delta u = h, &\quad \text{in } \Omega \times]0,T[, \\ u = 0, &\quad \text{on } \partial\Omega \times [0,T], \\ u(\cdot,0) = 0, &\quad \text{in } \Omega. \end{aligned} \tag{0.1}$$

Here h belongs to $L^p(0,T;L^p_\mu(\Omega))$, where $L^p_\mu(\Omega) = \{v \in L^p_{loc}(\Omega) : r^\mu v \in L^p(\Omega)\}$, with a real parameter μ and $r(x)$ the distance from x to the set of corners of Ω. We give sufficient conditions on μ, p and Ω that guarantee that problem (0.1) has a unique solution $u \in L^p(0,T;L^p_\mu(\Omega))$ that admits a decomposition into a regular part in weighted L^p-Sobolev spaces and an explicit singular part.

1 Introduction

In this work we consider the Cauchy-Dirichlet problem for the heat equation (0.1) on a polygonal domain Ω of the plane. We give the singular decomposition of the solution of (0.1) in weighted L^p-Sobolev spaces with precise regularity information on the regular and singular parts. The classical Fourier transform techniques

Received by the editors August 2010.
Communicated by M. Sioen.
2000 *Mathematics Subject Classification* : 35K15, 35B65.
Key words and phrases : heat equation, singular behavior, nonsmooth domains.

do not allow to handle such a general case. Hence we use the theory of sums of operators as in G. Da Prato and P. Grisvard [4] and G. Dore and A. Venni [7]. These results have been fruitfully used to prove the singular behavior of elliptic problems in non-Hilbertian Sobolev spaces in [10].

Although the analysis of the heat equation is well developed in weighted L^2-Sobolev spaces [9, 12, 11, 2] or in L^p-Sobolev spaces [10], to the best of our knowledge such a singularity result does not exist in the framework of weighted L^p-Sobolev spaces. For maximal regularity type results in weighted L^p-Sobolev spaces, we refer to [4, 13, 16, 14, 15].

In [6], we have considered the same kind of results for the periodic-Dirichlet problem

$$\begin{aligned} \partial_t u - \Delta u &= g, && \text{in } \Omega \times]-\pi, \pi[, \\ u &= 0, && \text{on } \partial\Omega \times [-\pi, \pi], \\ u(\cdot, -\pi) &= u(\cdot, \pi), && \text{in } \Omega. \end{aligned}$$

Some of the results presented there are useful in our context too.

The first step, which consists in the study of the Helmholtz equation

$$-\Delta u + zu = g, \text{ in } \Omega, \quad u = 0, \text{ on } \partial\Omega, \tag{1.1}$$

where z is a complex number, was performed in [5].

The paper is organized as follows: In section 2 we apply the approach of Da Prato-Grisvard [4] to obtain a decomposition but with non-optimal regularity informations. Section 3 is devoted to the proof of the regularity of $(\partial_t - \Delta)S$, where S is the singular part of the solution obtained before. The use of the approach of Dore-Venni [7] and the results from section 3 allows to get the optimal regularity result in section 4.

In the whole paper the notation $a \lesssim b$ means the existence of a positive constant C, which is independent of the quantities a, b (and eventually of the above parameter z) under consideration such that $a \leq Cb$.

2 Application of Da Prato-Grisvard's approach [4]

Let us assume in the future that the assumptions of [6, Theorem 2.3] are satisfied, i.e.,

(H) Let $p \geq 2$ and Ω be a bounded polygonal domain of $\mathbb{R}^2$, i.e., its boundary is the union of a finite number of line segments. Denote by $S_j, j = 1, \ldots, J$, the vertices of $\partial\Omega$ enumerated clockwise and, for $j \in \{1, 2, \ldots, J\}$, let ψ_j be the interior angle of Ω at the vertex S_j and $\lambda_j = \frac{\pi}{\psi_j}$.
For all $j = 1, \ldots, J$, let $\mu_j > -\lambda_j$ satisfy $2 - \frac{2}{p} - \mu_j \neq k\lambda_j$, for all $k \in \mathbb{Z}^*$, and

$$\mu_j < \frac{2p-2}{p} \text{ if } p > 2; \quad \mu_j \leq 1 \text{ if } p = 2; \qquad |\mu_j| < \frac{2\sqrt{p-1}}{p}\lambda_j. \tag{2.1}$$

We shall apply the results from [4] (see also [6, Theorem 2.1]) on the space

$$E = L^p(I; L^p_{\vec{\mu}}(\Omega)) \text{ with } L^p_{\vec{\mu}}(\Omega) = \{f \in L^p_{loc}(\Omega) \mid wf \in L^p(\Omega)\},$$

where $I = [0, T]$, $w(x) \simeq r(x)^{\mu_j}$ near S_j, $w(x) \simeq 1$ far from the corners and with the operators

$$A : D(A) \subset E \to E : u \mapsto -\Delta u, \quad \text{with}$$
$$D(A) = L^p(I; D(\Delta_{p,\vec{\mu}})) \text{ where } D(\Delta_{p,\vec{\mu}}) = \{u \in H_0^1(\Omega) \,|\, \Delta u \in L^p_{\vec{\mu}}(\Omega)\},$$

and

$$B_T : D(B_T) \subset E \to E : u \mapsto \partial_t u, \quad \text{with}$$
$$D(B_T) = W^{1,p}_{\text{left}}(I; L^p_{\vec{\mu}}(\Omega)) = \{u \in E \mid \partial_t u \in E, u(\cdot, 0) = 0\}.$$

Proposition 2.1. *Under assumptions (H), the operator $A + B_T$ has an inverse closure i.e., for all $g \in L^p(I; L^p_{\vec{\mu}}(\Omega))$, there exists a unique strong solution $u \in L^p(I; L^p_{\vec{\mu}}(\Omega))$ of $(A + B_T)u = g$ i.e. there exists $(u_n)_n \subset D(A) \cap D(B_T)$ such that $u_n \to u$ and $Au_n + B_T u_n \to g$. Moreover we have*

$$u = \frac{1}{2\pi i} \int_\gamma (A + z\,I)^{-1}(z\,I - B_T)^{-1} g \, dz, \tag{2.2}$$

with $\gamma : \mathbb{R} \to \mathbb{C}$ defined for example by $\gamma(s) = |s|\, e^{-i(\frac{\pi}{2}+\delta)}$ for $s \leq 0$, $\gamma(s) = |s|\, e^{i(\frac{\pi}{2}+\delta)}$ for $s > 0$, with $\delta \in]0, \theta_A - \frac{\pi}{2}[$ and $\theta_A \in]\frac{\pi}{2}, \pi[$ given by [6, Theorem 2.3].

Proof. The proof follows the lines of [6, Proposition 3.1] with minor changes concerning B_T: a simple calculation proves that $\rho(B_T) = \mathbb{C}$ and, in the verification that, for all $\theta_B < \frac{\pi}{2}$, there exists $M \geq 0$ such that, for all $\mu \in S_{B_T} = \{\mu \in \mathbb{C} \mid |\arg(\mu)| \leq \theta_B\}$, $\|(B_T + \mu\,I)^{-1}\| \leq M|\mu|^{-1}$, denoting $v = w^p |u|^{p-2}\bar{u}$, we have to replace

$$\frac{p}{2}\left(\int_\Omega \int_{-\pi}^{\pi} v \partial_t u \, dt dx + \overline{\int_\Omega \int_{-\pi}^{\pi} v \partial_t u \, dt dx}\right) = 0,$$

valid in the periodic case, by

$$\frac{p}{2}\left(\int_\Omega \int_0^T v \partial_t u \, dt dx + \overline{\int_\Omega \int_0^T v \partial_t u \, dt dx}\right) = \int_\Omega |u(x,T)|^p w(x)^p \, dx.$$

The remainder of the proof follows in the same way as in [6, Proposition 3.1]. ∎

Remark 2.1 As in [6, Remark 3.1], we obtain also

$$(1 + |z|)\, \|(z\,I - B_T)^{-1} g\|_{L^p(I; L^p_{\vec{\mu}}(\Omega))} \lesssim \|g\|_{L^p(I; L^p_{\vec{\mu}}(\Omega))}.$$

As it is clear that, for each t, we have

$$[(A + z\,I)^{-1} h](t) = (-\Delta + z\,I)^{-1}(h(t)),$$

we can use the decomposition in regular and singular parts of the solution of the Helmholtz equation (1.1) obtained in [6] (see [6, (2.4)]) and rewrite (2.2) as

$$u = u_R + \sum_{j=1}^{J} \eta_j \sum_{k \in \mathbb{N}: 0 < \lambda'_j = k\lambda_j < 2 - \frac{2}{p} - \mu_j} u_{\lambda'_j}, \tag{2.3}$$

where

$$u_R = \frac{1}{2\pi i} \int_\gamma R(z)(z\,I - B_T)^{-1} g \, dz, \quad u_{\lambda'_j} = \frac{1}{2\pi i} \int_\gamma \left\langle T_{\lambda'_j}(z), (z\,I - B_T)^{-1} g \right\rangle \tilde{\psi}_{\lambda'_j, z} \, dz, \tag{2.4}$$

with $R(z) : L^p_{\vec{\mu}}(\Omega) \to V^{2,p}_{\vec{\mu}}(\Omega)$ the operator which gives the regular part of the solution of (1.1), $T_{\lambda'_j}(z) : L^p_{\vec{\mu}}(\Omega) \to \mathbb{C} : g \mapsto c_{\lambda'_j}(z) = \langle T_{\lambda'_j}(z), g\rangle$ the one which gives the singular coefficient of the solution of (1.1); η_j is a radial cut-off function such that $\eta_j \equiv 1$ in a small ball centered at S_j and $\eta_j \equiv 0$ outside a larger ball; $P_{j,\lambda'_j}(s) = \sum_{i=0}^{l_{j,\lambda'_j}-1} \frac{s^i}{i!}$ with $l_{j,\lambda'_j} > 2 - \mu_j - \frac{2}{p} - \lambda'_j$ and $\tilde{\psi}_{\lambda'_j,z}(r,\theta) = P_{j,\lambda'_j}(r\sqrt{z})e^{-r\sqrt{z}}\, r^{\lambda'_j} \sin(\lambda'_j\theta)$. Recall that $V^{2,p}_{\vec{\mu}}(\Omega)$ is defined as the closure of $\mathcal{C}^\infty_S(\Omega) = \{v \in \mathcal{C}^\infty(\overline{\Omega}) \mid S_j \notin \operatorname{supp} v\}$ with respect to the norm

$$\|u\|_{V^{2,p}_{\vec{\mu}}(\Omega)} = \Big(\sum_{|\gamma|\le 2} \int_\Omega |D^\gamma u(x)|^p\, w^p(x)\, r^{(|\gamma|-k)p}(x)\, dx\Big)^{1/p}.$$

For more details, see [6, end of Section 2].

Proposition 2.2. *Let the assumptions (H) be satisfied and denote $\sigma_j := 1 - \frac{1}{p} - \frac{\mu_j+\lambda'_j}{2}$. Then for all $s \in]0, \min(1-\sigma_j, 1/p)[$, for all $g \in W^{s,p}(I, L^p_{\vec{\mu}}(\Omega))$, there exist $q_{\lambda'_j} \in W^{s+\sigma_j,p}(I)$ and $E_{\lambda'_j}$ such that $u_{\lambda'_j}$ defined by (2.4) can be written as*

$$u_{\lambda'_j} = (E_{\lambda'_j} *_t q_{\lambda'_j})\, r^{\lambda'_j} \sin(\lambda'_j\theta). \tag{2.5}$$

Moreover we have

$$q_{\lambda'_j} = \frac{1}{2\pi i}\int_\gamma \Big\langle T_{\lambda'_j}(z), (z\,I - B_T)^{-1} g\Big\rangle\, dz,$$

$$E_{\lambda'_j}(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi t} P_{j,\lambda'_j}(r\sqrt{i\xi})\, e^{-r\sqrt{i\xi}}\, d\xi, \tag{2.6}$$

and the operator $U : W^{s,p}(I, L^p_{\vec{\mu}}(\Omega)) \to W^{s+\sigma_j,p}(I) : g \mapsto q_{\lambda'_j}$ is continuous.

Proof. Recall that for all $f \in L^p_{\vec{\mu}}(\Omega)$, the mapping $\mathbb{C} \to \mathbb{C} : z \mapsto \Big\langle T_{\lambda'_j}(z), f\Big\rangle$ is holomorphic on $\mathcal{A} := \{z \in \mathbb{C} \mid |\arg(z)| < \theta_A\}$ and continuous on $\overline{\mathcal{A}}$ (see [6]).
Step 1: Extension. Let us consider the extension of g to $\Omega \times \mathbb{R}$, defined by

$$\tilde{g}(x,t) = g(x,t) \text{ if } t \in [0,T], \qquad \tilde{g}(x,t) = 0 \text{ if } t \notin [0,T],$$

and denote by $\tilde{u}_z = (z\,I - B_\infty)^{-1}\tilde{g}$, the solution of

$$z\tilde{u} - \partial_t \tilde{u} = \tilde{g} \quad \text{in } \Omega\times\mathbb{R}, \qquad \tilde{u}(\cdot,0) = 0 \quad \text{in } \Omega.$$

Observe that, by uniqueness of the solution of the Cauchy problem, we have $\tilde{u}_z|_{[0,T]\times\Omega} = (z\,I - B_T)^{-1} g$. Moreover we easily see that

$$\begin{array}{rcll} \tilde{u}_z(x,t) & = & 0, & \text{if } t < 0, \\ & = & -\displaystyle\int_0^t e^{z(t-s)} g(x,s)\, ds, & \text{if } t \in [0,T], \\ & = & -e^{zt}\displaystyle\int_0^T e^{-zs} g(x,s)\, ds, & \text{if } t > T. \end{array}$$

Consider the function

$$\tilde{u}_{\lambda'_j}(x,t) = \frac{1}{2\pi i}\int_\gamma \Big\langle T_{\lambda'_j}(z), (z\,I - B_\infty)^{-1}\tilde{g}\Big\rangle\, \tilde{\psi}_{\lambda'_j,z}(r,\theta)\, dz. \tag{2.7}$$

Observe that $\tilde{u}_{\lambda'_j}|_{\Omega\times[0,T]} = u_{\lambda'_j}$ and that, for $t > T$, $z = \rho e^{\pm i\theta_0}$ with $\rho > 0$, $\theta_0 = \frac{\pi}{2} + \delta$, using [6, (2.6)], we have

$$\begin{aligned}
\left|\left\langle T_{\lambda'_j}(z), (z\,I - B_\infty)^{-1}\tilde{g}\right\rangle \tilde{\psi}_{\lambda'_j,z}(r,\theta)\right| &\lesssim \left|\left\langle T_{\lambda'_j}(z), (z\,I - B_\infty)^{-1}\tilde{g}\right\rangle\right| \\
&\lesssim |e^{z(t-T)}|\ \|T_{\lambda'_j}(z)\|_{(L^p_{\vec{\mu}}(\Omega))'} \left\|\int_0^T e^{z(T-s)}g(x,s)\,ds\right\|_{L^p_{\vec{\mu}}(\Omega)} \\
&\lesssim e^{-\rho|\cos\theta_0|(t-T)}\frac{1}{1+\rho^{\sigma_j}}\left(\int_0^T e^{-q\rho|\cos\theta_0|(T-s)}\,ds\right)^{1/q}\|g\|_{L^p(0,T;L^p_{\vec{\mu}}(\Omega))} \\
&\lesssim e^{-\rho|\cos\theta_0|(t-T)}\frac{1}{1+\rho^{\sigma_j}}\|g\|_{L^p(0,T;L^p_{\vec{\mu}}(\Omega))}.
\end{aligned}$$

On the other hand, for $0 < t < 2T$ and $|z| = \rho$ we have, by Remark 2.1,

$$\begin{aligned}
|\left\langle T_{\lambda'_j}(z), (z\,I - B_\infty)^{-1}\tilde{g}\right\rangle \tilde{\psi}_{\lambda'_j,z}(r,\theta)| &\lesssim |\left\langle T_{\lambda'_j}(z), (z\,I - B_\infty)^{-1}\tilde{g}\right\rangle| \\
&\lesssim |\left\langle T_{\lambda'_j}(z), (z\,I - B_{2T})^{-1}\tilde{g}\right\rangle| \lesssim \frac{1}{1+\rho^{\sigma_j}}\frac{1}{1+\rho}\|g\|_{L^p(0,T;L^p_{\vec{\mu}}(\Omega))}.
\end{aligned}$$

Step 2: For all $x \in \Omega$, the function $\tilde{u}_{\lambda'_j}(x,\cdot) \in L^2(\mathbb{R})$ and hence admits a partial Fourier transform in t. For all $t > 2T$ by the previous considerations, we have

$$\begin{aligned}
|\tilde{u}_{\lambda'_j}(x,t)| &\lesssim \left|\int_\gamma \left\langle T_{\lambda'_j}(z), (z\,I - B_\infty)^{-1}\tilde{g}\right\rangle \tilde{\psi}_{\lambda'_j,z}(r,\theta)\,dz\right| \\
&\lesssim \int_0^\infty e^{-\rho|\cos\theta_0|(t-T)}\,d\rho\ \|g\|_{L^p(0,T;L^p_{\vec{\mu}}(\Omega))} \lesssim \frac{1}{t-T}\,\|g\|_{L^p(0,T;L^p_{\vec{\mu}}(\Omega))}.
\end{aligned}$$

For $t < 2T$ we use a similar argument using here the last estimate of Step 1. This shows that, for all $x \in \Omega$, $\tilde{u}_{\lambda'_j}(x,\cdot) \in L^2(\mathbb{R})$, and we can take its partial Fourier transform in t.

Step 3: The partial Fourier transform in t of $\tilde{u}_{\lambda'_j}(x,\cdot)$ satisfies, for all $\xi \neq 0$,

$$\mathcal{F}_t(\tilde{u}_{\lambda'_j})(x,\xi) = -\left\langle T_{\lambda'_j}(i\xi), \mathcal{F}_t(\tilde{g})(\cdot,\xi)\right\rangle\ \tilde{\psi}_{\lambda'_j,i\xi}(x).$$

As $\tilde{u}_{\lambda'_j}(x,\cdot) \in L^2(\mathbb{R})$, using [17, Cor 1, p.154], we know that

$$\mathcal{F}_t(\tilde{u}_{\lambda'_j})(x,\xi) = \lim_{k\to\infty}\int_{-k}^k e^{-it\xi}\tilde{u}_{\lambda'_j}(x,t)\,dt.$$

Hence by the above computations we have, for $k > 2T$,

$$\begin{aligned}
&\int_{-k}^k\int_{\mathbb{R}}\left|\left\langle T_{\lambda'_j}(\rho e^{i\,\mathrm{sgn}(\rho)\theta_0}), (\rho e^{i\,\mathrm{sgn}(\rho)\theta_0}I - B_\infty)^{-1}\tilde{g}\right\rangle \tilde{\psi}_{\lambda'_j,\rho e^{i\,\mathrm{sgn}(\rho)\theta_0}}(x)e^{-i\xi t}e^{i\,\mathrm{sgn}(\rho)\theta_0}\right|\,d\rho\,dt \\
&\quad\lesssim \Bigg(\int_0^{2T}\int_0^{+\infty}\frac{1}{1+\rho^{\sigma_j}}\frac{1}{1+\rho}\,d\rho\,dt + \\
&\qquad\qquad \int_{2T}^k\int_0^{+\infty}\frac{1}{1+\rho^{\sigma_j}}e^{-\rho\,|\cos\theta_0|(t-T)}\,d\rho\,dt\Bigg)\|g\|_{L^p(0,T;L^p_{\vec{\mu}}(\Omega))} \\
&\quad\lesssim \Bigg(\int_0^{2T}\int_0^{+\infty}\frac{1}{1+\rho^{\sigma_j}}\frac{1}{1+\rho}\,d\rho\,dt + \\
&\qquad\qquad \int_{2T}^k\frac{1}{|\cos\theta_0|(t-T)}\,dt\Bigg)\|g\|_{L^p(0,T;L^p_{\vec{\mu}}(\Omega))} < +\infty.
\end{aligned}$$

Hence, by Fubini's theorem, we obtain

$$\begin{aligned}\mathcal{F}_t(\tilde{u}_{\lambda'_j})(x,\xi) &= \frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z), \mathcal{F}_t((zI-B_\infty)^{-1}\tilde{g})(\cdot,\xi)\right\rangle \tilde{\psi}_{\lambda'_j,z}(x)\,dz \\ &= \frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z), \frac{\mathcal{F}_t(\tilde{g})(\cdot,\xi)}{z-i\xi}\right\rangle \tilde{\psi}_{\lambda'_j,z}(x)\,dz.\end{aligned}$$

The rest of the proof follows [6, Step 2 of the Proof of Proposition 3.2] observing that, by Hölder inequality, we have

$$\begin{aligned}\|\mathcal{F}_t(\tilde{g})(\cdot,\xi)\|^p_{L^p_{\vec{\mu}}(\Omega)} &= \int_\Omega w^p(x)\left|\int_{\mathbb{R}} e^{-i\xi t}\tilde{g}(x,t)\,dt\right|^p dx \\ &\lesssim \int_\Omega w^p(x)\left(\int_{\mathbb{R}}|\tilde{g}(x,t)|\,dt\right)^p dx \\ &\lesssim \int_\Omega w^p(x)\left(\int_0^T |g(x,t)|\,dt\right)^p dx \lesssim \|g\|^p_{L^p(I;L^p_{\vec{\mu}}(\Omega))}.\end{aligned}$$

Step 4: The operator $U : W^{s,p}(I;L^p_{\vec{\mu}}(\Omega)) \to W^{s+\sigma_j,p}(I) : g \mapsto q_{\lambda'_j}$ with $q_{\lambda'_j}$ given by (2.6) is continuous. By the results of [8], as $0<s<1/p$, we know that

$$W^{s,p}(I;L^p_{\vec{\mu}}(\Omega)) = \left\{g\in E \mid \int_0^\infty \rho^{sp}\|B_T(B_T-\rho e^{\pm i(\frac{\pi}{2}+\delta)}I)^{-1}g\|^p_E\,\rho^{-1}d\rho<\infty\right\}.$$

We have a similar characterization of $W^{s+\sigma_j,p}(I)$ by considering the operator

$$N : D(N)\subset L^p(I)\to L^p(I) : u\mapsto \partial_t u \quad\text{with}\quad D(N)=\{u\in W^{1,p}(I)\mid u(0)=0\}.$$

Hence if $s+\sigma_j<1/p$, we have

$$W^{s+\sigma_j,p}(I) = \left\{g\in L^p(I)\mid \int_0^\infty \tau^{(s+\sigma)p}\|N(N+\tau I)^{-1}g\|^p_{L^p(I)}\,\tau^{-1}d\tau<\infty\right\},$$

while if $s+\sigma_j>1/p$, defining $W^{s+\sigma_j,p}_{\text{left}}(I)=\{g\in W^{s+\sigma_j,p}(I)\mid g(0)=0\}$, we have

$$W^{s+\sigma_j,p}_{\text{left}}(I) = \left\{g\in L^p(I)\mid \int_0^\infty \tau^{(s+\sigma)p}\|N(N+\tau I)^{-1}g\|^p_{L^p(I)}\,\tau^{-1}d\tau<\infty\right\}.$$

Claim 1: For $\tau\geq 0$, we have

$$N(N+\tau I)^{-1}q_{\lambda'_j} = \frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z), B_T(zI-B_T)^{-1}g\right\rangle \frac{dz}{z+\tau}. \tag{2.8}$$

First observe that

$$\begin{aligned}N(N+\tau I)^{-1}q_{\lambda'_j} &= \frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z), B_T(B_T+\tau I)^{-1}(zI-B_T)^{-1}g\right\rangle dz \\ &= \left(\frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z), B_\infty(B_\infty+\tau I)^{-1}(zI-B_\infty)^{-1}\tilde{g}\right\rangle dz\right)\Bigg|_{\Omega\times[0,T]}.\end{aligned}$$

Let us show that we can take the Fourier transform in t of

$$\frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z), B_\infty(B_\infty+\tau I)^{-1}(zI-B_\infty)^{-1}\tilde{g}\right\rangle dz.$$

We have

$$\begin{aligned} B_\infty(B_\infty+\tau I)^{-1}(zI-B_\infty)^{-1}\tilde g &= (zI-B_\infty)^{-1}\tilde g-\tau\,(B_\infty+\tau I)^{-1}(zI-B_\infty)^{-1}\tilde g \\ &=: \tilde v_z(x,t)-\tau\,\tilde v_{z\tau}(x,t). \end{aligned}$$

Observe that, for $t>T$, we have $\tilde v_z(x,t)=-e^{z(t-T)}\int_0^T e^{z(T-s)}g(x,s)\,ds$ and

$$\begin{aligned} \tilde v_{z\tau}(x,t) &= -\int_0^T e^{-\tau(t-s)}\int_0^s e^{z(s-\sigma)}g(x,\sigma)\,d\sigma ds \\ &\qquad -\int_T^t e^{-\tau(t-s)}e^{z(s-T)}\int_0^T e^{z(T-\sigma)}g(x,\sigma)\,d\sigma ds \\ &= -\int_0^T \frac{e^{z(t-\sigma)}-e^{-\tau(t-\sigma)}}{z+\tau}g(x,\sigma)\,d\sigma. \end{aligned}$$

Hence, for $\tau\geq 0$ and if $t>2T$ we have as above, using [6, (2.6)],

$$\begin{aligned} &\left|\frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z),B_\infty(B_\infty+\tau I)^{-1}(zI-B_\infty)^{-1}\tilde g\right\rangle dz\right| \\ &\lesssim \frac{1}{t-T}\|g\|_{L^p(I;L^p_{\vec\mu}(\Omega))}+\left|\int_\gamma \frac{\tau|e^{z(t-T)}|}{|z+\tau|}\left|\left\langle T_{\lambda'_j}(z),\int_0^T e^{z(T-\sigma)}g(x,\sigma)\,d\sigma\right\rangle\right| dz \right.\\ &\qquad\left. +\int_\gamma \frac{\tau|e^{-\tau(t-T)}|}{|z+\tau|}\left|\left\langle T_{\lambda'_j}(z),\int_0^T e^{-\tau(T-\sigma)}g(x,\sigma)\,d\sigma\right\rangle\right| dz\right| \\ &\leq \left(\frac{1}{t-T}+\left|\int_\gamma \frac{\tau}{|z+\tau|}|e^{z(t-T)}|\,dz\right|\right.\\ &\qquad\left.+\left|\tau e^{-\tau(t-T)}\int_\gamma \frac{1}{(|z+\tau|)(1+|z|^{\sigma_j})}\,dz\right|\right)\|g\|_{L^p(I;L^p_{\vec\mu}(\Omega))} \\ &\lesssim \left(\frac{1}{t-T}+\frac{1}{\sin\theta_0}\frac{1}{|\cos\theta_0|}\frac{1}{t-T}\right.\\ &\qquad\left.+\tau e^{-\tau(t-T)}\int_1^\infty \frac{1}{1+\rho^{\sigma_j}}\frac{1}{\rho\sin\theta_0}\,d\rho+\frac{e^{-\tau(t-T)}}{\sin\theta_0}\right)\|g\|_{L^p(I;L^p_{\vec\mu}(\Omega))}. \end{aligned}$$

We conclude that this function belongs to $L^2(\mathbb{R},L^p_{\vec\mu}(\Omega))$ and we can take its Fourier transform in t. By Cauchy theorem, we obtain, as in [6], that its Fourier transform in t is given by

$$-\left\langle T_{\lambda'_j}(i\xi),\mathcal{F}_t(\tilde g)(\cdot,\xi)\right\rangle \frac{i\xi}{i\xi+\tau}. \tag{2.9}$$

In the same way, we can take the Fourier transform in t of the right-hand side of (2.8) since, for $t>2T$ we have

$$\begin{aligned} &\left|\frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z),B_\infty(zI-B_\infty)^{-1}\tilde g(x,t)\right\rangle \frac{dz}{z+\tau}\right| \\ &\qquad=\left|-\frac{1}{2\pi i}\int_\gamma z e^{z(t-T)}\left\langle T_{\lambda'_j}(z),\int_0^T e^{z(T-s)}g(x,s)\,ds\right\rangle\frac{dz}{z+\tau}\right|\lesssim \frac{1}{t-T}. \end{aligned}$$

Hence by Cauchy theorem, as in [6], its Fourier transform in t is given by (2.9).

As the Fourier transform of the two functions coincide, the two functions are equal.

Claim 2: For $0<s<\min(1-\sigma_j,1/p)$, *the operator* $U: W^{s,p}(I;L^p_{\vec\mu}(\Omega))\to W^{s+\sigma_j,p}(I):$ $g\mapsto q_{\lambda'_j}$ *is continuous.* The proof is the same as the corresponding one in [6].

Conclusion. By Step 3, we have, for all $\xi \neq 0$,

$$\mathcal{F}_t(\tilde{u}_{\lambda'_j})(x,\xi) = -\left\langle T_{\lambda'_j}(i\xi), \mathcal{F}_t(\tilde{g})(\cdot,\xi)\right\rangle \tilde{\psi}_{\lambda'_j,i\xi}(x). \tag{2.10}$$

Let

$$\tilde{q}_{\lambda'_j}(t) = \frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z), (z\,I - B_\infty)^{-1}\tilde{g}(\cdot,t)\right\rangle dz. \tag{2.11}$$

As previously we can take its Fourier transform and we see, applying again the Cauchy theorem as above, that its Fourier transform is given by

$$\mathcal{F}(\tilde{q}_{\lambda'_j})(\xi) = \frac{1}{2\pi i}\int_\gamma \left\langle T_{\lambda'_j}(z), \frac{\mathcal{F}_t(\tilde{g})(\cdot,\xi)}{z - i\xi}\right\rangle dz = -\left\langle T_{\lambda'_j}(i\xi), \mathcal{F}_t(\tilde{g})(\cdot,\xi)\right\rangle.$$

Consider the function $E_{\lambda'_j}(x,t)$ which has as Fourier transform in t

$$\mathcal{F}_t(E_{\lambda'_j})(x,\xi) = P_{j,\lambda'_j}(r\sqrt{i\xi})\, e^{-r\sqrt{i\xi}}.$$

As $P_{j,\lambda'_j}(r\sqrt{i\xi})\, e^{-r\sqrt{i\xi}} \in L^\infty(\mathbb{R})$ and by [18, p.113], $L^\infty(\mathbb{R}) \subset \mathcal{S}'$, we have also by [18, Thm-Def 3.3, p.114] that $E_{\lambda'_j}(x,\cdot) \in \mathcal{S}'$. Now observe that by [18, p.112], $\mathcal{S}' \subset \mathcal{D}'$. As $\tilde{q}_{\lambda_j} \in L^2(\mathbb{R})$, there exists a sequence $(q_n)_n \subset \mathcal{D}(\mathbb{R})$ such that $q_n \to \tilde{q}_{\lambda_j}$ in $L^2(\mathbb{R})$. By [18, Thm 6.3, p.120] or [17, Thm 6, p.160], as $\mathcal{F}_t(E_{\lambda'_j})$ is bounded, we have that

$$\mathcal{F}_t(E_{\lambda'_j} * q_n) = \mathcal{F}_t(E_{\lambda'_j})\,\mathcal{F}_t(q_n) \to \mathcal{F}_t(E_{\lambda'_j})\,\mathcal{F}_t(\tilde{q}_{\lambda_j}), \qquad \text{in } L^2(\mathbb{R}).$$

Hence, we have $E_{\lambda'_j} * q_n \to E_{\lambda'_j} * \tilde{q}_{\lambda_j}$, in $L^2(\mathbb{R})$, which proves that

$$\tilde{u}_{\lambda'_j} = (E_{\lambda'_j} *_t \tilde{q}_{\lambda'_j})\, r^{\lambda'_j} \sin(\lambda'_j\theta)$$

and the result follows. ■

As in [6] we can extend the previous Proposition to $g \in L^p(I, L^p_{\vec{\mu}}(\Omega))$.

Theorem 2.3. *Let the assumptions (H) be satisfied and denote $\sigma_j := 1 - \frac{1}{p} - \frac{\mu_j + \lambda'_j}{2}$. Then for all $g \in L^p(I, L^p_{\vec{\mu}}(\Omega))$, the problem (0.1) has a unique strong solution u which can be written in the form*

$$u = u_R + \sum_{j=1}^{J} \eta_j \sum_{k\in\mathbb{N}: 0<\lambda'_j = k\lambda_j < 2 - \frac{2}{p} - \mu_j} u_{\lambda'_j},$$

where u_R (resp. $u_{\lambda'_j}$) is given by (2.4) (resp. (2.5)) with $q_{\lambda'_j} \in W^{\sigma_j,p}(I)$ and $E_{\lambda'_j}$ given by (2.6). Moreover the mapping $L^p(I, L^p_{\vec{\mu}}(\Omega)) \to W^{\sigma_j,p}(I) : g \mapsto q_{\lambda'_j}$ is continuous.

3 Regularity of $q_{\lambda'_j} \to (\frac{\partial}{\partial t} - \Delta)(\eta_j u_{\lambda'_j})$

In order to consider the regularity of u_R we observe that u_R satisfies

$$\partial_t u_R - \Delta u_R = g - \sum_{j=1}^{J} \sum_{k\in\mathbb{N}:0<\lambda'_j=k\lambda_j<2-\frac{2}{p}-\mu_j} (\partial_t(\eta_j u_{\lambda'_j}) - \Delta(\eta_j u_{\lambda'_j})). \tag{3.1}$$

Hence we need informations on the regularity of $\partial_t(\eta_j u_{\lambda'_j}) - \Delta(\eta_j u_{\lambda'_j})$. This is the aim of this section.

Lemma 3.1. *The kernel H defined on $\mathbb{R}^+ \times \mathbb{R}$ by*

$$H(r,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \sqrt{i\xi}\, e^{-r\sqrt{i\xi}}\, e^{i\xi t}\, d\xi \tag{3.2}$$

satisfies, for all $\ell \in \mathbb{N}$,

$$\left|\frac{\partial^\ell}{\partial r^\ell} H(r,t)\right| \lesssim (|t| + r^2)^{-\frac{3+\ell}{2}}. \tag{3.3}$$

Proof. Let E be the elementary solution of the heat equation in $\mathbb{R}^2$, i.e.,

$$E(r,t) = \frac{M(t)}{\sqrt{4\pi t}} e^{-\frac{r^2}{4t}}, \tag{3.4}$$

where $M(t) = 1$ if $t > 0$ and $M(t) = 0$ if $t < 0$. Recall that E is a tempered distribution. We easily check that the partial Fourier transform $\mathcal{F}_t E$ in t of E is given by

$$\mathcal{F}_t E(r,\xi) = \frac{e^{-|r|\sqrt{i\xi}}}{2\sqrt{i\xi}}.$$

As

$$\mathcal{F}_t(\frac{\partial^2}{\partial r^2}E) = \frac{\partial^2}{\partial r^2}(\mathcal{F}_t E) = \frac{\sqrt{i\xi}}{2} e^{-|r|\sqrt{i\xi}} - \delta_0(r) = \mathcal{F}_t(\frac{H(|r|,t)}{2} - \delta_0(r)\delta_0(t)),$$

and since $\mathcal{F}_t$ is an isomorphism from $\mathcal{S}'(\mathbb{R}^2)$ into itself, we deduce that

$$H(|r|,t) = 2\frac{\partial^2}{\partial r^2}E(r,t) + 2\delta_0(r)\delta_0(t).$$

Hence, for $r > 0$, $H(r,t) = 2\frac{\partial^2 E}{\partial r^2}(r,t)$ and we conclude as in [6]. ∎

Theorem 3.2. *Under assumptions (H) and recalling that $\sigma_j = 1 - \frac{1}{p} - \frac{\mu_j + \lambda'_j}{2}$, the mapping $q_{\lambda'_j} \to (\frac{\partial}{\partial t} - \Delta)(\eta_j u_{\lambda'_j})$ is continuous from $W^{\sigma_j,p}(I)$ into $L^p(I; L^p_{\vec{\mu}}(\Omega))$.*

Proof. Recall that, by [6, Remark 3.2], $0 < \sigma_j < 1$.

Case 1: $P_{j,\lambda'_j} \equiv 1$ i.e. $\lambda'_j + \mu_j - 1 + \frac{2}{p} > 0$. As in the proof of Proposition 2.2, consider the functions $\tilde{q}_{\lambda'_j}$ given by (2.11) and $\tilde{u}_{\lambda'_j}$ given by (2.7).

Let us take the Fourier transform in t of $f(x,t) = \eta_j(r)(\frac{\partial}{\partial t} - \Delta)\tilde{u}_{\lambda'_j}(x,t)$. We obtain

$$\mathcal{F}_t f(x,\xi) = \eta_j(r)\,\mathcal{F}_t((\frac{\partial}{\partial t} - \Delta)\tilde{u}_{\lambda'_j}) = \eta_j(r)\,(i\xi\, I - \Delta)\,\mathcal{F}_t(\tilde{u}_{\lambda'_j}).$$

As in Step 3 of the proof of Proposition 2.2, we have

$$\mathcal{F}_t(\tilde{u}_{\lambda'_j})(x,\xi) = -\left\langle T_{\lambda'_j}(i\xi), \mathcal{F}_t(\tilde{g})(\cdot,\xi)\right\rangle \tilde{\psi}_{\lambda'_j,i\xi}(x).$$

and hence

$$\begin{aligned}(i\xi\, I - \Delta)\, \mathcal{F}_t(\tilde{u}_{\lambda'_j}) &= -c_{\lambda'_j}(i\xi)\, (i\xi\, I - \Delta)\, (e^{-r\sqrt{i\xi}}\, r^{\lambda'_j} \sin(\lambda'_j\theta)) \\ &= -c_{\lambda'_j}(i\xi)\, \sqrt{i\xi}\, e^{-r\sqrt{i\xi}}\, r^{\lambda'_j-1} \sin(\lambda'_j\theta)\, (2\lambda'_j+1),\end{aligned}$$

with $c_{\lambda'_j}(i\xi) = \langle T_{\lambda'_j}(i\xi), \mathcal{F}_t(\tilde{g})(\cdot,\xi)\rangle = -\mathcal{F}_t(\tilde{q}_{\lambda'_j})(\xi)$. Using the kernel H given by (3.2), as previously, we obtain that

$$f(x,t) = (H *_t \tilde{q}_{\lambda'_j})(r)\, (2\lambda'_j+1)\, r^{\lambda'_j-1} \sin(\lambda'_j\theta)\, \eta_j(r).$$

As

$$\int_{\mathbb{R}} H(r,s)\, ds = \int_{\mathbb{R}} e^{-it\xi} H(r,t)\, dt\Big|_{\xi=0} = \mathcal{F}_t H(r,0) = \sqrt{i\xi} e^{-r\sqrt{i\xi}}\Big|_{\xi=0} = 0,$$

we have

$$f(x,t) = (2\lambda'_j+1)\, r^{\lambda'_j-1} \sin(\lambda'_j\theta)\, \eta_j(r) \int_{\mathbb{R}} H(r,s)\, [\tilde{q}_{\lambda'_j}(t-s) - \tilde{q}_{\lambda'_j}(t)]\, ds.$$

From this point on, the proof proceeds as in [6].

Case 2: $\deg(P_{j,\lambda'_j}) = l_{j,\lambda'_j} - 1 \geq 1$. This case is treated as in [6] using Lemma 3.1. ∎

4 Application of Dore-Venni's approach [7]

Now we are able to consider the regularity of u_R and to prove our main result.

Theorem 4.1. *Let $p \geq 2$, Ω be a bounded polygonal domain of $\mathbb{R}^2$. Denote by $S_j, j = 1,\ldots,J$, the vertices of $\partial\Omega$ enumerated clockwise and, for $j \in \{1,2,\ldots,J\}$, let ψ_j be the interior angle of Ω at the vertex S_j and $\lambda_j = \frac{\pi}{\psi_j}$. For all $j = 1,\ldots,J$, let μ_j satisfies*

$$-\lambda_j < \mu_j < \frac{2p-2}{p}, \quad |\mu_j| < \frac{2\sqrt{p-1}}{p}\lambda_j,$$

and, for all $k \in \mathbb{Z}^*$, $2 - \frac{2}{p} - \mu_j \neq k\lambda_j$ and $\mu_j + k\lambda_j \neq 1$. For every $g \in L^p(0,T; L^p_{\vec{\mu}}(\Omega))$, there exists a unique solution $u \in L^p(0,T; L^p_{\vec{\mu}}(\Omega))$ of

$$\begin{aligned} \partial_t u - \Delta u = g, &\quad \text{in } \Omega \times]0,T[, \\ u = 0, &\quad \text{on } \partial\Omega \times [0,T], \\ u(\cdot,0) = 0, &\quad \text{in } \Omega. \end{aligned}$$

Moreover u admits the decomposition

$$u = u_R + \sum_{j=1}^{J} \eta_j \sum_{k\in\mathbb{N}: 0<\lambda'_j=k\lambda_j<2-\frac{2}{p}-\mu_j} u_{\lambda'_j},$$

with

$$u_R \in L^p(I; V^{2,p}_{\vec{\mu}}(\Omega)) \cap W^{1,p}(I; L^p_{\vec{\mu}}(\Omega)) \text{ and } u_{\lambda'_j} = (E_{\lambda'_j} *_t q_{\lambda'_j}) r^{\lambda'_j} \sin(\lambda'_j\theta),$$

where $q_{\lambda'_j} \in W^{\sigma_{j,\lambda'_j},p}(I)$ and $E_{\lambda'_j}(x,t) = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}} e^{i\xi t} P_{j,\lambda'_j}(r\sqrt{i\xi})\, e^{-r\sqrt{i\xi}}\, d\xi$, with $\sigma_{j,\lambda'_j} = -\dfrac{\mu_j+\lambda'_j}{2} + 1 - \dfrac{1}{p}$.

Proof. As in [6], we prove that u_R defined by (2.4) satisfies, for all $\theta \in]0,1[$,

$$u_R \in L^p(I; (L^p_{\vec{\mu}}(\Omega), V^{2,p}_{\vec{\mu}}(\Omega))_\theta).$$

We now observe that u_R is a strong solution of (3.1) with a right-hand side in $L^p(I; L^p_{\vec{\mu}}(\Omega))$ according to the previous results.

Then we apply Dore-Venni's approach [7] (see also Theorem 2.2 of [6]) with $E = L^p(I; L^p_{\vec{\mu}}(\Omega))$, and

$$A : D(A) \subset E \to E : u \mapsto -\Delta u, \quad \text{with} \quad D(A) = L^p(I; D(\Delta_{p,\vec{\mu}})),$$
$$B : D(B) \subset E \to E : u \mapsto \partial_t u, \quad \text{with} \quad D(B) = W^{1,p}_{\text{left}}(I; L^p_{\vec{\mu}}(\Omega)).$$

The assumptions (H_3), (H_4), (H_5) of [6] can be verified as in [6]. To verify (H_6) we apply the following result of Coifman - Weiss (see [3] or for example [1]).

If $-C$ is the infinitesimal generator of a strongly continuous contraction semi-group in E which preserves the positivity then there exists $K > 0$ such that, for all $s \in \mathbb{R}$,

$$\|C^{is}\| \leq K(1 + |s|)\, e^{\frac{\pi}{2}|s|}.$$

For what concerns the operator A, the argument is the same as in [6]. For what concerns B, we already know that $-B$ is the generator of a C_0 semigroup of contractions S. It remains to verify that S preserves the positivity. As usual it suffices to check that its resolvent preserves positivity: Namely for $\lambda > 0$ consider the solution $u \in D(B)$ of

$$\partial_t u + \lambda u = f \geq 0, \quad u(0) = 0.$$

Then $u(x,t) = (B + \lambda\, I)^{-1} f = \int_0^t e^{-\lambda(t-s)} f(x,s)\, ds$ which is clearly non negative.

We conclude as in [6] that $u_R \in L^p(I; V^{2,p}_{\vec{\mu}}(\Omega)) \cap W^{1,p}(I; L^p_{\vec{\mu}}(\Omega))$. ■

References

[1] **H. Amann**, *Linear and quasilinear parabolic problems*, Monographs in Mathematics, Birkhäuser Verlag, Basel, Boston, Berlin, 1995.

[2] **N.T. Anh and N.M. Hung**, *Asymptotic formulas for solutions of parameter-depending elliptic boundary-value problems in domains with conical points*, Electron. J. Differential Equations, 2009 (2009), No. 125, 1-21.

[3] **R.R. Coifman and G. Weiss**, *Transference methods in analysis*, C.B.M.S.-A.M.S. 31, 1976.

[4] **G. Da Prato and P. Grisvard**, *Sommes d'opérateurs linéaires et équations différentielles opérationnelles*, J. Math. Pure Appl. 54 (1975), 305-387.

[5] **C. De Coster and S. Nicaise**, *Singular behavior of the solution of the Helmholtz equation in weighted L^p-Sobolev spaces*, Adv. Differential Equations, 16 (2011), 165-198.

[6] **C. De Coster and S. Nicaise**, *Singular behavior of the solution of the periodic-Dirichlet heat equation in weighted L^p-Sobolev spaces*, Adv. Differential Equations, 16 (2011), 221-256.

[7] **G. Dore and A. Venni**, *On the closedness of the sum of two closed operators*, Math. Z. 196 (1987), 189-201.

[8] **P. Grisvard**, *Equations différentielles abstraites*, Ann. Scient. Ec. Norm. Sup. 2 (1969), 311-395.

[9] **P. Grisvard**, *Edge behavior of the solution of an elliptic problem*, Math. Nachr. 132 (1987), 281-299.

[10] **P. Grisvard**, *Singular behavior of elliptic problems in non hilbertian Sobolev spaces*, J. Math. Pures Appl. 74 (1995), 3-33.

[11] **V.A. Kozlov**, *Coefficients in the asymptotic solutions of the Cauchy boundary-value parabolic problems in domains with a conical point*, Sibirskii Mat. Zhurnal 29 (1988), 75-89.

[12] **V.A. Kozlov and V.G. Maz'ya**, *Singularities of solutions of the first boundary value problem for the heat equation in domains with conical points. II*, Soviet Math. (Iz. VUZ) 31 (1987), 49-57.

[13] **A.I. Nazarov**, *L_p-estimates for the solution to the Dirichlet problem and to the Neumann problem for the heat equation in a wedge with edge of arbitrary codimension*, J. Math. Sciences 106 (2001), 2989-3014.

[14] **A.I. Nazarov**, *Dirichlet problem for quasilinear parabolic equations in domains with smooth closed edges*, Amer. Math. Soc. Transl. 209 (2003), 115-141.

[15] **J. Prüss and G. Simonett**, *H^∞-Calculus for the sum of non-commuting operators*, Trans. A.M.S. 359 (2007), 3549-3565.

[16] **V.A. Solonnikov**, *L_p-estimates for solutions of the heat equation in a dihedral angle*, Rend. Matematica, Serie VII, vol. 21, Roma (2001), 1-15.

[17] **K. Yosida**, *Functional Analysis, Fourth Edition*, Springer Verlag, Berlin, Heidelberg, New York, 1974.

[18] **C. Zuily**, *Eléments de distributions et d'équations aux dérivées partielles*, Dunod, Paris, 2002.

Université de Valenciennes et du Hainaut Cambrésis
LAMAV, FR CNRS 2956,
Institut des Sciences et Techniques de Valenciennes
F-59313 Valenciennes Cedex 9, France
Colette.DeCoster,Serge.Nicaise@univ-valenciennes.fr