Linear Weingarten spacelike hypersurfaces in de Sitter space

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Abstract

In this paper, we give a classification on spacelike linear Weingarten hypersurfaces in de Sitter space $S_1^{n+1}(1)$ according to the sectional curvature or the length of the second fundamental form.

1 Introduction

Let M^n be a complete spacelike hypersurface immersed into de Sitter space S_1^{n+1}. We denote by H, R and S the mean curvature, the normalized scalar curvature and the square of the length of the second fundamental form, respectively.

When M^n has constant H, Goddard [6] conjectured that complete spacelike hypersurfaces with constant H must be totally umbilical. Akutagawa [2] proved that Goddard's conjecture is true when $n = 2$ and $H^2 \leq 1$ or when $n \geq 3$ and $H^2 < 4(n-1)/n^2$ (Ramanathan [13] studied the case $n = 2$ independently). In [10], Montiel proved that Goddard's conjecture is true provided that M^n is compact. Furthermore, he exhibited examples of complete spacelike hypersurfaces with constant H satisfying $H^2 \geq 4(n-1)/n^2$ and being not totally umbilical-the so called hyperbolic cylinders, which are isometric to the Riemannian product $H^1(1-\coth^2 r) \times S^{n-1}(1-\tanh^2 r)$. Montiel [11] proved that complete spacelike hypersurface M^n with $H^2 = 4(n-1)/n^2$ is isometric to a hyperbolic cylinder if M^n has at least two ends. In [7], Ki-Kim-Nakagawa found the upper bound

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of S and they proved that the upper bound can be realized only by hyperbolic cylinders.

When M^n has constant R, Zheng [15] proved that a compact spacelike hypersurface in a de Sitter space S_1^{n+1} is totally umbilical if the sectional curvature of M^n is non-negative and $R < 1$. Later, Cheng and Ishikawa [5] showed that Zheng's result in [15] is also true without additional assumptions on the sectional curvatures of the hypersurface. In [9], Liu obtained a pinching theorem on spacelike hypersurface with constant R, he proved that if $n(1-R) \le \sup S \le D(n,R)$, then either $\sup S = n(1-R)$ and M^n is totally umbilical or $\sup S = D(n,R)$ and M^n is a hyperbolic cylinder, where $D(n,R) = \frac{n}{(n-2)(n-nR-2)}[n(n-1)(1-R)^2 - 4(n-1)(1-R)+n]$.

When M^n is a complete spacelike hypersurface in de Sitter space S_1^{n+1} with $R = kH$, Cheng [4] proved that if the sectional curvature is non-negative and H can obtain its maximum on M^n then M^n is totally umbilical. Shu [14] proved a characteristic theorem concerning such hypersurfaces in terms of the mean curvature H and S. In [8], Li showed that a compact spacelike hypersurface with non-negative sectional curvature is totally umbilical.

In this paper, we will consider spacelike hypersurfaces with $R = aH + b$, which are called linear Weingarten hypersurfaces. This is the generalization of R is constant and $R = kH$. Precisely, we have the following theorems.

Theorem 1.1. *Let M^n be a compact spacelike linear Weingarten hypersurface immersed in the de Sitter space S_1^{n+1} with $R = aH + b$. If $4n(1-b) + (n-1)a^2 \ge 0$ and the sectional curvature of M^n is nonnegative, then M^n is totally umbilical.*

Remark 1.2. When the constant a vanishes identically, a linear Weingarten hypersurface M^n reduces to hypersurface with constant scalar curvature and our Theorem 1.1 reduces to Theorem B of [15]. When the constant b vanishes, we also get the corollary 4.3 of [8].

Theorem 1.3. *Let M^n be a complete spacelike linear Weingarten hypersurface immersed in the de Sitter space S_1^{n+1} with $R = aH + b$. Suppose H can attain the maximum on M^n. If $a \neq 0, b < 1$ and the sectional curvature of M^n is non-negative, then M^n is totally umbilical or a hyperbolic cylinder $H^1(1-\coth^2 r) \times S^{n-1}(1-\tanh^2 r)$.*

Remark 1.4. When the constant b vanishes identically and a is positive, Theorem 1.3 reduces to Theorem 1 of [4]. It should be pointed out that Cheng [4] omitted the hyperbolic cylinder $H^1(1-\coth^2 r) \times S^{n-1}(1-\tanh^2 r)$, which is isometric to

$$M^n = \{x \in S_1^{n+1} \mid x_2^2 + \cdots + x_{n+1}^2 = \coth^2 r\},$$

where r is a positive constant and $n > 2$. Such hyperbolic cylinders have constant H and constant R with

$$H = \frac{1}{n}(\coth r + (n-1)\tanh r) > 0, \quad R = 1 - \frac{1}{n}\left(2 + (n-2)\tanh^2 r\right) > 0.$$

It is easy to see that $H^1(1-\coth^2 r) \times S^{n-1}(1-\tanh^2 r)$ satisfies the condition of Theorem 1 in [4] for every positive constant r and it is not totally umbilical.

Theorem 1.5. *Let M^n be a complete spacelike linear Weingarten hypersurface immersed in the de Sitter space S_1^{n+1} with $R = aH + b$. Suppose H can attain the maximum on M^n. If $a \neq 0, b < 1$ and $S \leq 2\sqrt{n-1}$, then either M^n is totally umbilical or $S = 2\sqrt{n-1}$ $(n \geq 3)$ and M^n is isometric to a hyperbolic cylinder $H^1(1-\coth^2 r) \times S^{n-1}(1-\tanh^2 r)$.*

2 Preliminaries

Let M^n be an n-dimensional spacelike hypersurface immersed in the de Sitter space S_1^{n+1}. We choose a local field of pseudo-Riemannian orthonormal frames $\{e_1, \cdots, e_{n+1}\}$ in S_1^{n+1} such that, restricted to M^n, $e_1, \cdots, e_n$ are tangent to M^n, and the vector e_{n+1} is normal to M^n. Let $\{\omega_1, \cdots, \omega_{n+1}\}$ be the dual frame field. In this paper, we make the following convention on the range of indices:

$$1 \leq A,\ B,\ C \leq n+1; \qquad 1 \leq i,\ j,\ k \leq n.$$

Then the structure equations of S_1^{n+1} are given by

$$\begin{aligned}
d\omega_A &= \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \qquad \omega_{AB} + \omega_{BA} = 0,\\
d\omega_{AB} &= \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2}\sum_{C,D} K_{ABCD}\omega_C \wedge \omega_D,\\
K_{ABCD} &= \varepsilon_A \varepsilon_B \left(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}\right),
\end{aligned}$$

where $\varepsilon_i = 1$ and $\varepsilon_{n+1} = -1$. We restrict these forms to M, then we have $\omega_{n+1} = 0$, and the induced metric ds^2 of M is written as $ds^2 = \sum_i \omega_i^2$. We may put

$$\omega_{in+1} = \sum_j h_{ij}\omega_j, \qquad h_{ij} = h_{ji}. \tag{2.1}$$

The quadratic form $B = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j \otimes e_{n+1}$ is the second fundamental form of M^n. We denote $L = (h_{ij})_{n\times n}$ and $S = \sum h_{ij}^2$. The mean curvature vector ξ of M^n is defined by

$$\xi = \frac{1}{n}\sum_i h_{ii} e_{n+1}.$$

The length of the mean curvature vector is called the mean curvature of M^n, denote by H. When $\xi \neq 0$, we choose e_{n+1} to assure

$$H = \frac{1}{n}\sum_i h_{ii}^{n+1} > 0.$$

We can obtain the structure equations of M^n

$$\begin{aligned}
d\omega_i &= \sum_j \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0,\\
d\omega_{ij} &= \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l,
\end{aligned}$$

and the Gauss equation

$$R_{ijkl} = \left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right) - \left(h_{ik}h_{jl} - h_{il}h_{jk}\right), \tag{2.2}$$

where $\{R_{ijkl}\}$ is the component of the curvature tensor of M^n. Let R_{ij} and R denote the components of the Ricci curvature and the normalized scalar curvature of M^n respectively. From (2.2) we have

$$R_{ik} = (n-1)\delta_{ik} - \sum_j (h_{ik}h_{jj} - h_{ij}h_{jk}), \tag{2.3}$$

$$n(n-1)R = n(n-1) - n^2H^2 + S. \tag{2.4}$$

Let h_{ijk} denote the covariant derivative of h_{ij} so that

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}.$$

Then by exterior differentiation of (2.1), we obtain the Codazzi equation

$$h_{ijk} = h_{ikj}. \tag{2.5}$$

Next, we define the second covariant derivative of h_{ij} by

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} + \sum_m h_{mjk}\omega_{mi} + \sum_m h_{imk}\omega_{mj} + \sum_m h_{ijm}\omega_{mk}.$$

By exterior differentiation of (2.5), we can get the following Ricci identity

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}. \tag{2.6}$$

The laplacian of h_{ij} is defined by $\triangle h_{ij} = \sum_k h_{ijkk}$. From (2.5) and (2.6) we obtain

$$\triangle h_{ij} = \sum_k h_{kkij} + \sum_{k,m} h_{mk}R_{mijk} + \sum_{m,k} h_{im}R_{mkjk}. \tag{2.7}$$

Since $\frac{1}{2}\Delta S = \sum_{i,j,k}\left(h_{ijk}\right)^2 + \sum_{i,j} h_{ij}\Delta h_{ij}$, then it follows from (2.7) that

$$\frac{1}{2}\triangle S = \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j,k} h_{ij}h_{kkij} + \sum_{i,j,k,m} h_{ij}h_{mk}R_{mijk} + \sum_{i,j,k,m} h_{ij}h_{im}R_{mkjk}. \tag{2.8}$$

We choose $e_1, \cdots, e_n$ such that $h_{ij} = \lambda_i\delta_{ij}$, then (2.8) becomes

$$\frac{1}{2}\triangle S = \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2. \tag{2.9}$$

Let $T = \sum_{i,j} T_{ij}\omega_i\omega_j$ be a symmetric tensor on M^n defined by

$$T_{ij} = nH\delta_{ij} - h_{ij}.$$

We introduce an operator $\Box$ associated to T acting on $f \in C^2(M^n)$ by

$$\Box f = \sum_{i,j} T_{ij} f_{ij} = \sum_{i,j} (nH\delta_{ij} - h_{ij}) f_{ij}. \tag{2.10}$$

Setting $f = nH$ in (2.10) and from (2.4) we obtain

$$\begin{aligned} \Box(nH) &= \sum_{i,j} (nH\delta_{ij} - h_{ij})(nH)_{ij} \\ &= \sum_i (nH)(nH)_{ii} - \sum_i \lambda_i (nH)_{ii} \\ &= \frac{1}{2}\Delta(nH)^2 - \sum_i (nH_i)^2 - \sum_i \lambda_i (nH)_{ii} \\ &= \frac{1}{2}\Delta S - \frac{n(n-1)}{2}\Delta R - n^2|\nabla H|^2 - \sum_i \lambda_i (nH)_{ii}. \end{aligned} \tag{2.11}$$

From (2.9) and (2.11), we have

$$\Box(nH) = \sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 - \frac{n(n-1)}{2}\Delta R + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2. \tag{2.12}$$

We introduce an operator

$$L = \Box + \frac{n-1}{2} a\Delta.$$

Then it follows from $R = aH + b$ that

$$L(nH) = \Box(nH) + \frac{n-1}{2} a\Delta(nH) = \Box(nH) + \frac{1}{2} n(n-1)\triangle R. \tag{2.13}$$

Substituting (2.12) into (2.13), we have

$$L(nH) = \sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2. \tag{2.14}$$

Proposition 2.1. Let M^n be an n-dimensional spacelike linear Weingarten hypersurface immersed in the de Sitter space S_1^{n+1} with $R = aH + b$. If $a \neq 0, b < 1$, then L is elliptic.

Proof. If $H = 0$, we have $R = b < 1$. It follows from (2.4) that $S = n(n-1)$ $(R-1) < 0$. This is impossible. Therefore we have $H > 0$. It follows from (2.4) and $R = aH + b$ that

$$S = n^2H^2 + n(n-1)(aH + b - 1). \tag{2.15}$$

It follows that

$$a = \frac{1}{n(n-1)H}\left(S - n^2H^2 + n(n-1)(1-b)\right). \tag{2.16}$$

For any i, from (2.16) we have

$$\begin{aligned}
& nH-\lambda_i+\frac{n-1}{2}a \\
= \; & nH-\lambda_i+\frac{1}{2nH}\Big(S-n^2H^2+n(n-1)(1-b)\Big) \\
= \; & \left\{\frac{1}{2}(\sum_j \lambda_j)^2-\lambda_i\sum_j\lambda_j+\frac{1}{2}\sum_j\lambda_j^2+\frac{1}{2}n(n-1)(1-b)\right\}(nH)^{-1} \\
= \; & \left\{\sum_j \lambda_j^2+\frac{1}{2}\sum_{l\neq j}\lambda_l\lambda_j-\lambda_i\sum_j\lambda_j+\frac{1}{2}n(n-1)(1-b)\right\}(nH)^{-1} \\
= \; & \left\{\sum_{i\neq j}\lambda_j^2+\frac{1}{2}\sum_{\substack{l\neq j\\ l,j\neq i}}\lambda_l\lambda_j+\frac{1}{2}n(n-1)(1-b)\right\}(nH)^{-1} \\
= \; & \frac{1}{2}\left\{\sum_{i\neq j}\lambda_j^2+(\sum_{j\neq i}\lambda_j)^2+n(n-1)(1-b)\right\}(nH)^{-1}.
\end{aligned}$$

It follows from $b<1$ that

$$nH-\lambda_i+\frac{n-1}{2}a>0. \tag{2.17}$$

Thus L is an elliptic operator. ∎

Lemma 2.2. *Let M^n be an n-dimensional spacelike linear Weingarten hypersurface immersed in the de Sitter space S_1^{n+1} with $R=aH+b$. If $(n-1)a^2+4n(1-b)\geq 0$, then we have*

$$\sum_{i,j,k}h_{ijk}^2\geq n^2|\nabla H|^2. \tag{2.18}$$

Moreover, suppose that the equality holds on M^n in (2.18). Then either H is constant on M^n or $r(L)=1$, where $r(L)$ denotes the rank of L.

Proof. From (2.4) and $R=aH+b$, we have

$$S=n^2H^2+n(n-1)(aH+b-1). \tag{2.19}$$

Taking the covariant derivative of (2.19), we have

$$2\sum_{i,j}h_{ij}h_{ijk}=S_k=\Big(2n^2H+n(n-1)a\Big)H_k \tag{2.20}$$

for every k. Hence, by Cauchy-Schwartz's inequality, we have

$$\sum_{i,j}h_{ij}^2\sum_{i,j,k}h_{ijk}^2\geq(n^2H+\frac{1}{2}n(n-1)a)^2|\nabla H|^2, \tag{2.21}$$

that is

$$S\sum_{i,j,k} h_{ijk}^2 \geq (n^2H + \frac{1}{2}n(n-1)a)^2|\nabla H|^2. \tag{2.22}$$

On the other hand, it follows from (2.19) that

$$\begin{aligned} & \left(n^2H + \frac{1}{2}n(n-1)a\right)^2 - n^2S \\ = \ & n^2\left(n^2H^2 + n(n-1)Ha - S\right) + \frac{1}{4}n^2(n-1)^2a^2 \\ = \ & n^3(n-1)(1-b) + \frac{1}{4}n^2(n-1)^2a^2 \\ = \ & \frac{1}{4}n^2(n-1)\left((n-1)a^2 + 4n(1-b)\right). \end{aligned} \tag{2.23}$$

Since $(n-1)a^2 + 4n(1-b) \geq 0$, we have

$$\left(n^2H + \frac{1}{2}n(n-1)a\right)^2 \geq n^2S. \tag{2.24}$$

It follows from (2.22) and (2.24) that

$$S\sum_{i,j,k} h_{ijk}^2 \geq (n^2H + \frac{1}{2}n(n-1)a)^2|\nabla H|^2 \geq n^2S|\nabla H|^2. \tag{2.25}$$

Hence either $S = 0$ and $\sum_{i,j,k} h_{ijk}^2 = n^2|\nabla H|^2$ or $\sum_{i,j,k} h_{ijk}^2 \geq n^2|\nabla H|^2$.

We suppose $\sum_{i,j,k} h_{ijk}^2 = n^2|\nabla H|^2$ on M^n. Then inequalities in (2.21), (2.22), (2.24) and (2.25) become equalities.

If $(n-1)a^2 + 4n(1-b) > 0$, then $(n^2H + \frac{1}{2}n(n-1)a)^2 > n^2S$ from (2.23). Since the second equality in (2.25) holds, we have $|\nabla H| = 0$ and hence H is constant on M^n.

If $(n-1)a^2 + 4n(1-b) = 0$, then from (2.23) we have $(n^2H + \frac{1}{2}n(n-1)a)^2 = n^2S$. This together with (2.20) forces that

$$S_k^2 = 4n^2SH_k^2, \qquad k = 1, \cdots, n. \tag{2.26}$$

Since the equality holds in (2.21), there exists a real function c_k on M^n such that

$$h_{ijk} = c_kh_{ij}, \qquad i,\ j = 1, \cdots, n, \tag{2.27}$$

for every k. Taking the sum on both sides of equation (2.27) with respect to $i = j$, we get

$$H_k = c_kH, \qquad k = 1, \cdots, n. \tag{2.28}$$

From (2.27), we have

$$S_k = 2\sum_{ij} h_{ij}h_{ijk} = 2c_kS, \qquad k = 1, \cdots, n. \tag{2.29}$$

Multiplying both sides of equations in (2.29) by H and by using (2.28), we have

$$HS_k = 2H_kS, \qquad k = 1, \cdots, n. \tag{2.30}$$

It follows from (2.26) and (2.30) that

$$H_k^2 S = H_k^2 n^2 H^2, \qquad k = 1, \cdots, n. \tag{2.31}$$

We assume that H is not constant. Then there exists a k_0 such that H_{k_0} is not zero. Hence from (2.31) we have

$$S = n^2 H^2. \tag{2.32}$$

On the other hand, multiplying both sides of equations in (2.27) by H and by using (2.28), we have

$$Hh_{ijk} = H_k h_{ij}, \qquad i,\ j,\ k = 1, \cdots, n. \tag{2.33}$$

Taking the sum on both sides of (2.33) with respect to $j = k$ and from (2.5), we have

$$(nH)H_i = \sum_j H_j h_{ij}, \quad i = 1, \cdots, n. \tag{2.34}$$

We choose $e_1, \cdots, e_n$ such that $h_{ij} = \lambda_i \delta_{ij}$, then (2.34) becomes

$$(nH - \lambda_i)H_i = 0, \qquad i = 1, \cdots, n. \tag{2.35}$$

Since H_{k_0} is not zero, we have

$$\lambda_{k_0} = nH. \tag{2.36}$$

It follows from (2.32) and (2.36) that $\lambda_k = 0$ for all $k \neq k_0$ on M^n. Hence $r(L) = 1$ on M^n. ■

Remark 2.3. When $b < 1$, then $(n-1)a^2 + 4n(1-b) > 0$. It follows from the proof of Lemma 2.2 that $\sum_{i,j,k} h_{ijk}^2 \geq n^2|\nabla H|^2$. Moreover, if the equality holds, then H is constant.

Lemma 2.4. *[12] Let μ_i $(1 \leq i \leq n)$ be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2$, where $\beta =$ constant ≥ 0. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3 \tag{2.37}$$

and the equality holds if and only if at least $(n-1)$ of the μ_i are equal.

3 Proof of Theorems

Proof of Theorem 1.1. Since M^n is compact, we take integration over M^n on both sides of (2.12) and have

$$0 = \int_M (\sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2) + \frac{1}{2}\int_M \sum_{i,j} R_{ijij}\,(\lambda_i - \lambda_j)^2. \tag{3.1}$$

Since the sectional curvature of M^n is non-negative and from Lemma 2.2, we conclude that

$$\sum_{i,j,k} h_{ijk}^2 = n^2|\nabla H|^2, \tag{3.2}$$

and

$$R_{ijij}(\lambda_i - \lambda_j)^2 = (1 - \lambda_i\lambda_j)(\lambda_i - \lambda_j)^2 = 0. \tag{3.3}$$

It follows from (3.3) that $\lambda_i = \lambda_j$ or $R_{ijij} = 1 - \lambda_i\lambda_j = 0$ when $\lambda_i \neq \lambda_j$. We conclude that M^n has at most two distinct principal curvature. In fact, without loss of generality, we assume that M^n has three distinct principle curvature $\lambda_{i_1}, \lambda_{i_2}, \lambda_{i_3}$. Then $\lambda_{i_1}\lambda_{i_2} = \lambda_{i_2}\lambda_{i_3} = 1$ and hence $\lambda_{i_1} = \lambda_{i_3}$. This is a contradiction. Hence we have that M^n has at most two distinct principal curvature.

It follows from (3.2) and Lemma 2.2 that either H is constant or $r(L) = 1$. If $r(L) = 1$, then there exists a k_0 such that $\lambda_{k_0} = nH$ and $\lambda_k = 0$ for $k \neq k_0$, which together with (3.3) shows that $H = 0$. This is a contradiction. Hence we have that H is constant, which together with (3.2) shows that λ_i is constant for every i. From the congruence theorem in [1] and the compactness of M^n, we conclude that M is totally umbilical. This completes the proof of Theorem 1.1. ∎

Proof of Theorem 1.3. It follows from (2.14) and Remark 2.3 that

$$L(nH) = \sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2 \geq 0, \tag{3.4}$$

here we used the assumption that the sectional curvature of M^n is non-negative. Since L is elliptic and H can obtain its maximum on M, we deduce that H is constant. Thus

$$\sum_{i,j,k} h_{ijk}^2 = n^2|\nabla H|^2 = 0, \tag{3.5}$$

and

$$\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2 = \sum_{i,j}(1 - \lambda_i\lambda_j)(\lambda_i - \lambda_j)^2 = 0. \tag{3.6}$$

It follows from (3.5) that λ_i is constant for every i. From (3.6) we have $\lambda_i = \lambda_j$ or $R_{ijij} = 1 - \lambda_i\lambda_j = 0$ when $\lambda_i \neq \lambda_j$. Similar to the proof of Theorem 1.1 , we get M^n has at most two distinct constant principal curvature. If all the principle curvatures are equal, we have that M^n is totally umbilical. Otherwise, without loss of generality, we may suppose that

$$\lambda_1 = \cdots = \lambda_k = \lambda, \quad \lambda_{k+1} = \cdots = \lambda_n = \mu,$$

for some $k = 1, \cdots, n-1$, and $\lambda\mu = 1$.

We can prove $k = 1$ or $n-1$. In fact, if $1 < k < n-1$, we have $\lambda^2 \leq 1$, $\mu^2 \leq 1$ from $R_{ijij} = 1 - \lambda_i\lambda_j \geq 0$. This together with $\lambda\mu = 1$ shows that $\lambda = \mu = 1$ or $\lambda = \mu = -1$, which contradicts $\lambda \neq \mu$. Hence we have $k = 1$ or $n-1$.

We assume $\lambda = \tanh r$, $\mu = \coth r$. Since the sectional curvature of M^n is non-negative and by means of the congruence Theorem of Abe-Koike-Yamaguchi [1], we have that M^n is isometric to a hyperbolic cylinder $S^{n-1}(1-\tanh^2 r) \times H^1(1-\coth^2 r)$. This completes the proof of Theorem 1.3. ∎

Proof of Theorem 1.5. Let $\mu_i = \lambda_i - H$ and $|\Phi|^2 = \sum_i \mu_i^2$, we get

$$\sum_i \mu_i = 0, \quad |\Phi|^2 = S - nH^2, \quad \sum_i \lambda_i^3 = \sum_i \mu_i^3 + 3H|\Phi|^2 + nH^3. \tag{3.7}$$

It follows from (2.2) and (3.7) that (2.14) becomes

$$\begin{aligned} L(nH) &= \sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j}(1-\lambda_i\lambda_j)(\lambda_i - \lambda_j)^2 \\ &= \sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 + |\Phi|^2(n + S - 2nH^2) - nH\sum_i \mu_i^3. \end{aligned} \tag{3.8}$$

From (3.8), Remark 2.3 and Lemma 2.4, we have

$$\begin{aligned} & L(nH) \qquad\qquad (3.9) \\ \geq\ & |\Phi|^2\left(n + S - 2nH^2 - (n-2)H\sqrt{\frac{n}{n-1}}|\Phi|\right) \\ =\ & |\Phi|^2\left(n - \frac{n}{2\sqrt{n-1}}S + \frac{1}{2\sqrt{n-1}}\left((\sqrt{n-1}+1)|\Phi| - (\sqrt{n-1}-1)\sqrt{n}H\right)^2\right) \\ \geq\ & |\Phi|^2\left(n - \frac{n}{2\sqrt{n-1}}S\right), \end{aligned}$$

which together with the assumption of the theorem $S \leq 2\sqrt{n-1}$ shows that

$$L(nH) \geq |\Phi|^2\left(n - \frac{n}{2\sqrt{n-1}}S\right) \geq 0.$$

Since L is elliptic and H can obtain its maximum on M, we deduce that H is constant. Hence

$$|\Phi|^2\left(n - \frac{n}{2\sqrt{n-1}}S\right) = 0.$$

If $S < 2\sqrt{n-1}$, then $|\Phi|^2 = 0$ and M^n is totally umbilical.

If $S = 2\sqrt{n-1}$, all the inequalities in (3.9) become equalities. We have

$$(\sqrt{n-1}+1)|\Phi| - (\sqrt{n-1}-1)\sqrt{n}H = 0.$$

Hence

$$n^2H^2 = n\sqrt{n-1} + 2(n-2).$$

When $n = 2$, we have $|\Phi| = 0$ and M^n is totally umbilical. When $n \geq 3$, since the equality holds in (2.37) of Lemma 2.4, after renumberation if necessary, we can assume

$$\lambda_1 = \cdots = \lambda_{n-1} = \tanh r, \quad \lambda_n = \coth r.$$

Therefore, M^n is isometric to a hyperbolic cylinder $H^1(1-\coth^2 r) \times S^{n-1}(1-\tanh^2 r)$ from the congruence theorem in [1]. This completes the proof of Theorem 1.5. ∎

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