A Local extension of proper holomorphic maps between some unbounded domains in $\mathbb{C}^{n*}$

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Abstract

In this paper we study analytic sets extending the graph of proper holomorphic mappings. As applications, we present some results on the local holomorphic extension of proper holomorphic mappings between certain algebraic domains in $\mathbb{C}^n$, not necessarily bounded. Further supplementary results are obtained.

1 Introduction and results

The boundary regularity of proper holomorphic maps between smooth domains in $\mathbb{C}^n$ is still an open problem in full generality. However, positive answers have been obtained in many special cases. For domains in $\mathbb{C}^n$ with real-analytic smooth boundaries, the recent progress is related to the geometric reflection principle in $\mathbb{C}^n$, based on the method of Segre varieties. For related results and without mentioning the entire list, we refer the reader to [24], [15], [17] with references included. Our first purpose in this paper is to prove a local holomorphic extension of proper holomorphic mappings under the assumption that the graph of the mapping extends as an analytic set (see Definition 1). More precisely, we prove the following

Theorem 1. *Let D, D' be arbitrary domains in $\mathbb{C}^n$, $n > 1$, and $f : D \to D'$ be a proper holomorphic mapping. Let $M \subset \partial D$, $M' \subset \partial D'$ be open pieces of the boundaries.*

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Suppose that ∂D is smooth real-analytic and nondegenerate in an open neighborhood of $\bar{M}$ and $\partial D'$ is smooth real-algebraic and nondegenerate in an open neighborhood of $\bar{M}'$. If the cluster set $cl_f(p)$ of a point $p \in M$ contains a point $q \in M'$ and the graph of f extends as an analytic set to a neighborhood of $(p, q) \in \mathbb{C}^n \times \mathbb{C}^n$ (see Definition 1), then f extends holomorphically to a neighborhood of p.

Note that we do not require pseudoconvexity of M or M', and we do not assume that $cl_f(M) \subset M'$. Therefore, a priori $cl_f(p)$ may contain the point at infinity or boundary points which do not lie in M'. Moreover, the following example (appeared in [15]) shows that the extension of the graph of f as an analytic set to a neighborhood of (p, q) does not imply in general that $cl_f(M) \subset M'$ near p.

Example 1. Let $D = \{(z_1, z_2) \in \mathbb{C}^2 : \mathcal{R}e(z_2) + |z_1|^2 < 0\}$ and $D' = \{(z_1', z_2') \in \mathbb{C}^2 : \mathcal{R}e(z_2') + |z_1'|^2 |z_2'|^2 < 0\}$. The map $f : (z_1, z_2) \mapsto (z_1/z_2, z_2)$ is a biholomorphism from D to D'. The graph of f is contained in $\{(z_1, z_2, z_1', z_2') \in \mathbb{C}^2 \times \mathbb{C}^2 : z_1' z_2' - z_1 = 0,\ z_2' = z_2\}$, then f extends as an analytic set to a neighborhood of $(0, 0')$. But $0' \in cl_f(0)$ and $\infty \in cl_f(0)$. Note that in this example the boundary of D' is degenerate (since it contains the complex line $z_2' = 0$).

Theorem 1 is already known if M and M' are smooth, real-analytic hypersurfaces of finite type and additionally $cl_f(M) \subset M'$ (see Theorem 1.1 in [15]), or f is continuous on $D \cup M$ (see [19]), or the graph of the mapping extends as a holomorphic correspondence (see [14]). The proof of all these results uses the method of analytic continuation along Segre varieties.

As an application of Theorem 1, we prove the following

Corollary 1. *Let D and D' be smooth algebraic domains in $\mathbb{C}^n$, $n > 1$, with nondegenerate boundaries and $f : D \to D'$ be a proper holomorphic mapping.*

a) If the cluster set $cl_f(p)$ of a point $p \in \partial D$ contains a point $q \in \partial D'$, then f extends holomorphically to a neighborhood of p and the set of holomorphic extendability of f is an open dense subset of ∂D.

b) If either D or D' has a global holomorphic peak function at infinity, then the set of holomorphic extendability of f is an open dense subset of ∂D.

By a smooth algebraic domain D in $\mathbb{C}^n$ we mean a domain defined globally as $\{z \in \mathbb{C}^n : P(z, \bar{z}) < 0\}$, where P is a real polynomial in $\mathbb{C}^n$ with $dP \neq 0$ on ∂D. We say that its boundary is nondegenerate if $\{z \in \mathbb{C}^n : P(z, \bar{z}) = 0\}$ contains no complex-analytic set with positive dimension. Note that these domains are not necessarily bounded. Recall that a function $\varphi : D \to \mathbb{C}$ is a global holomorphic peak function at infinity on D if φ is holomorphic in D, $|\varphi(z)| < 1$ for all $z \in D$ and $\varphi(z) \to 1$ as $|z| \to \infty$.

Remark 1. The existence of global holomorphic peak functions at infinity is due to Bedford-Fornaess in the case of rigid polynomial domains in $\mathbb{C}^2$ (see [5]). These functions exist also in the case of unbounded convex domains in $\mathbb{C}^n$, which does not contain complex affine lines. Indeed, if D is such a domain, there exist $H_1, \cdots, H_n$ linearly independent hyperplanes such that $\bar{D}$ is on one side of each of these hyperplanes. Up to a linear change of coordinates $\tilde{z} = (\tilde{z}_1, \cdots, \tilde{z}_n)$, we may assume that $H_j = \{\tilde{z} \in \mathbb{C}^n : Re\tilde{z}_j = 0\}$ and D is contained in $\{\tilde{z} \in \mathbb{C}^n : Re\tilde{z}_1 < 0, \cdots, Re\tilde{z}_n < 0\}$ (for details, see Proposition 3.5 in [6]). The image of infinity by the associated Cayley transform is contained in the zero of the function $\prod_{1\le j\le n}(\tilde{z}_j - 1)$. According to [26] (Theorem 6.1.2 page 132), this is the peak set of a holomorphic function. This proves the existence of a global holomorphic peak function at infinity on D. Recall that if Ω is a domain in $\mathbb{C}^n$ and $E \subset \partial\Omega$ is a compact subset, we say that E is a peak set for a holomorphic function f if f is holomorphic in Ω, continuous on $\bar{\Omega}$, $|f(\xi)| < 1$ for every $\xi \in \bar{\Omega}\backslash E$ and $f(\xi) = 1$ for every $\xi \in E$.

In Corollary 1, the function f is not assumed to possess any a priori regularity near p. If D or D' is pseudoconvex, then the holomorphic extendability of f near p will be a consequence of [25] and [17]. If D is a rigid polynomial domain in $\mathbb{C}^n$ ($n > 1$), namely, $D = \{(z_1, 'z) \in \mathbb{C} \times \mathbb{C}^{n-1} : 2Re(z_1) + P('z) < 0\}$, we can follow the proof of Coupet-Pinchuk [8], based on the construction of analytic discs attached to the boundary, to prove the existence of a point $p \in \partial D$ satisfying $cl_f(p) \cap \partial D' \neq \emptyset$ (see Lemma 1.3 in [8]). The proof of Corollary 1 shows the existence of an algebraic set $S^\infty \subset \partial D$ such that f extends holomorphically to a neighborhood of any point from $\partial D\backslash S^\infty$ and for all $t \in S^\infty$, $\lim_{z\to t}|f(z)| = \infty$. Therefore, we get the following decomposition of the boundary $\partial D = S^h \cup S^\infty$, where S^h is the set of holomorphic extendability of the mapping f.

Next, we propose a local version of Corollary 1.

Corollary 2. *Let D, D' be arbitrary domains in $\mathbb{C}^n$, $n > 1$, and $M \subset \partial D$, $M' \subset \partial D'$ be open pieces of the boundaries. Suppose that ∂D (resp. $\partial D'$) is smooth real-algebraic and nondegenerate in an open neighborhood of $\bar{M}$ (resp. $\bar{M}'$). Let $f : D \to D'$ be a proper holomorphic map such that the cluster set $cl_f(M) \subset M'$.*

a) For $n = 2$, f extends across each point of M as a holomorphic map.

b) For $n \geq 2$, suppose that there exist a point $p \in M$ and a neighborhood U of p such that f extends to a uniformly continuous mapping on a dense subset $\mathcal{B}$ of $M \cap U$, possibly with empty interior (see Definition 3). Then f extends across each point of M as a holomorphic map.

Corollary 2 was proved in [24] for proper holomorphic maps between bounded real-analytic domains in $\mathbb{C}^2$. When f is continuous on M, this result is due to Diederich-Pinchuk ([17]). The proof is based on the algebraicity of the mapping with a careful study of $cl_f(M)$, which is the crucial point of the proof. The difficulty is to show that $cl_f(M) \not\subseteq M'_0$, where M'_0 denotes the set of points of M' with degenerate Levi-form. Notice that there is no assumption on the cluster set of M' under f^{-1}.

If $D = \{\rho < 0\}$ is a domain in $\mathbb{C}^n$ and c is a real number, we denote by ∂D_c the set defined by $\{\rho = c\}$. Based on the algebraicity result and by analyzing the order of vanishing of the Levi-determinant of the domain, we show the following

Theorem 2. *Let $D = \{P(z,\bar{z}) < 0\}$ and $D' = \{\mathcal{Q}(z',\bar{z}') < 0\}$ be smooth algebraic domains in $\mathbb{C}^n$, $n > 1$, with nondegenerate boundaries. Suppose that either D or D' has a global holomorphic peak function at infinity. Suppose further that one of the following conditions is satisfied :*

(A) *P is plurisubharmonic on D and ∂D_c is nondegenerate for all $c < 0$.*
(B) *$\mathcal{Q}$ is plurisubharmonic on D' and $\partial D'_c$ is nondegenerate for all $c < 0$.*

Then there exists a finite number of irreducible complex-algebraic sets $\hat{A}_1, \cdots, \hat{A}_N$ in $\mathbb{C}^n$ of dimension $n-1$ such that the branch locus V_f of any proper holomorphic mapping $f : D \to D'$ satisfies :

$$V_f \subset \bigcup_{k=1}^{N} \hat{A}_k.$$

For rigid polynomial domains, similar results were proved in [9] and [20]. The integer N is bounded by the degree of the polynomial P. The assumption on the existence of a global holomorphic peak function at infinity on D or D' assures that $cl_f(\partial D) \cap \partial D'$ is non-empty for any proper holomorphic mapping $f : D \to D'$, and this leads to prove the algebraicity of the mapping (see the proof of Corollary 1, second part). The plurisubharmonicity of P and the nondegeneracy of the sets ∂D_c (also the plurisubharmonicity $\mathcal{Q}$ and the nondegeneracy of $\partial D'_c$) are important here to prove that the branch locus extends across the boundary of the domain. We are not able to prove this fact without these assumptions. If D is a bounded pseudoconvex smooth algebraic domain, then conditions (A) and (B) can be dropped; since the branch locus cannot be relatively compact in the domain. Moreover, the proof of Lemma 8 can be adapted in this case (in view of the existence of bounded negative plurisubharmonic exhaustion functions, see for example [13]) to prove that the branch locus extends across the boundary of the domain. Finally, note that for smooth bounded domains, we can give a nice description of the branch locus if the set of weakly pseudoconvex boundary points admits a nice stratification, as it was observed in [3] in the real-analytic case.

As examples of smooth algebraic domains in $\mathbb{C}^n$ (possibly unbounded) defined by $\{P(z,\bar{z}) < 0\}$ and verifying the property : ∂D_c is nondegenerate for all $c \leq 0$, we propose :

1- $D = \{z \in \mathbb{C}^n : P(z,\bar{z}) + 1 < 0\}$, where P is a homogeneous polynomial such that $\{P = -1\}$ is nondegenerate. Set for example, $P(z,\bar{z}) = \mathcal{R}e(z_1^2) + |z_2|^2$, with $z = (z_1, z_2) \in \mathbb{C}^2$.

2- $D = \{(z_1, 'z) \in \mathbb{C} \times \mathbb{C}^{n-1} : \mathcal{R}e(z_1) + \phi('z, \bar{'z}, \mathcal{I}mz_1) < 0\}$ (called semi-rigid domain), where ϕ is a polynomial such that $\{\mathcal{R}e(z_1) + \phi('z, \bar{'z}, \mathcal{I}mz_1) = 0\}$ is nondegenerate .

The following example shows that the nondegeneracy of ∂D does not imply in general the nondegeneracy of ∂D_c for all $c < 0$.

Example 2. Let $D = \{(z_1, z_2) \in \mathbb{C}^2 : P(z_1, z_2) < 0\}$, where $P(z_1, z_2) = 2\mathcal{R}e(z_2) + |z_1|^2|z_2|^2 - 1$. The set $\{P(z_1, z_2) = -1\}$ contains the complex line $z_2 = 0$ and the boundary of D is nondegenerate; since D is strictly pseudoconvex.

Remark 2. Note that any convex domain D in $\mathbb{C}^n$, which does not contain complex affine lines is biholomorphic to a bounded domain in $\mathbb{C}^n$. Indeed, there exist $H_1, \cdots, H_n$ linearly independent hyperplanes such that $\bar{D}$ is on one side of each of these hyperplanes. Up to a linear change of coordinates $\tilde{z} = (\tilde{z}_1, \cdots, \tilde{z}_n)$, we can assume that $H_j = \{\tilde{z} \in \mathbb{C}^n : Re\tilde{z}_j = 0\}$ and D is contained in $\{\tilde{z} \in \mathbb{C}^n : Re\tilde{z}_1 < 0, \cdots, Re\tilde{z}_n < 0\}$. Then the map

$$g : (\tilde{z}_1, \cdots, \tilde{z}_n) \mapsto (\frac{\tilde{z}_1 + 1}{\tilde{z}_1 - 1}, \cdots, \frac{\tilde{z}_n + 1}{\tilde{z}_n - 1})$$

maps D biholomorphically onto a bounded domain Ω in $\mathbb{C}^n$. Moreover, if in addition, D is algebraic, then Ω is also algebraic. In this case, Theorem 2 can be reformulated as follows

Theorem 2-bis. *Let D be a convex smooth algebraic domain in $\mathbb{C}^n$ (possibly unbounded), which does not contain complex affine lines and D' be a smooth algebraic domain in $\mathbb{C}^n$ with nondegenerate boundary. Then there exists a finite number of irreducible complex-algebraic sets $\hat{A}_1, \cdots, \hat{A}_N$ in $\mathbb{C}^n$ of dimension $n-1$ such that the branch locus V_f of any proper holomorphic mapping $f : D \to D'$ satisfies :*

$$V_f \subset \bigcup_{k=1}^{N} \hat{A}_k.$$

As a conclusion from the proof of Theorem 2, one has the following

Corollary 3. *Let $D = \{P(z, \bar{z}) < 0\}$ be a simply connected, smooth algebraic domain in $\mathbb{C}^n$, $n > 1$, with nondegenerate boundary, where P is a plurisubharmonic polynomial on D and $f : D \to D$ be a proper holomorphic self-mapping. Suppose that ∂D_c is nondegenerate for all $c < 0$ and $cl_f(\partial D)$ intersects the boundary of D. Then f is a biholomorphism.*

Corollary 3 generalizes Corollary 1 in [9] for semi-rigid polynomial domains in $\mathbb{C}^n$. It can be also observed as a generalization of the well known result of Alexander [1], stating that any proper holomorphic self-map of the unit ball in $\mathbb{C}^n$ is biholomorphic. A similar result was proved in [23] for bounded smooth algebraic domains in $\mathbb{C}^n$ without the pseudoconvexity assumption.

Remark 3. Since any convex smooth algebraic domain D in $\mathbb{C}^n$, which does not contain complex affine lines is biholomorphic to a bounded smooth algebraic domain, then according to [23], any proper holomorphic self-mapping of D is a biholomorphism.

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2 Notations, definitions and preliminaries

Definition 1. ([15]) *Let $f : D \to D'$ be a holomorphic map between domains in $\mathbb{C}^n$ and $0 \in \partial D$, $0' \in \partial D'$ such that $0' \in cl_f(0)$. If U and U' are small neighborhoods of 0 and $0'$, we say that the graph Γ_f of f extends as an analytic set to $U \times U'$ if there exist an irreducible (closed) complex-analytic set $\mathcal{A} \subset U \times U'$ of pure dimension n and a sequence $a_\nu \to 0$ in $D \cap U$ with neighborhoods $V^\nu \ni a_\nu$ such that $f(a_\nu) \to 0'$ and $\mathcal{A} \supset \Gamma_f \cap (V^\nu \times U')$ for all ν.*

In Definition 1, note that $\mathcal{A}$ does not necessarily contain the whole graph of f over $D \cap U$ (since f is not necessarily continuous at 0).
Now, we recall the definition of a holomorphic correspondence. Let D, D' be domains in $\mathbb{C}^n$ and A be a complex purely n-dimensional subvariety contained in $D \times D'$. We denote by $\pi_1 : A \to D$ and $\pi_2 : A \to D'$ the natural projections. When π_1 is proper, $(\pi_2 \circ \pi_1^{-1})(z)$ is a non-empty finite subset of D' for any $z \in D$ and one may therefore consider the multi-valued mapping $f = \pi_2 \circ \pi_1^{-1}$. Such a map is called a holomorphic correspondence between D and D'; A is said to be the graph of f. Since π_1 is proper, in particular it is a branched analytic covering. Then there exist an $(n-1)$-dimensional complex-analytic subset V_f of the graph of f and an integer m such that π_1 is an m-sheeted covering map from the set $A \backslash \pi_1^{-1}(\pi_1(V_f))$ onto $D \backslash \pi_1(V_f)$. Hence, $f(z) = \{f^1(z), \cdots, f^m(z)\}$ for all $z \in D \backslash \pi_1(V_f)$ and the f^j's are distinct holomorphic functions in a neighborhood of $z \in D \backslash \pi_1(V_f)$. The integer m is called the multiplicity of f and $\pi_1(V_f)$ is its branch locus. One says that f is irreducible if A is irreducible as an analytic set and proper if both π_1 and π_2 are proper.
Definition 2. *Let $f : D \to D'$ be a holomorphic mapping between domains in $\mathbb{C}^n$, Γ_f be the graph of f and z_o be a point in ∂D. We say that f extends as a holomorphic correspondence to a neighborhood U of z_o if there exist an open set $U' \subset \mathbb{C}^n$ and a closed complex-analytic set $A \subset U \times U'$ of pure dimension n, which may possibly be reducible, such that,*
i) $\Gamma_f \cap \{(D \cap U) \times D'\} \subset A$
ii) the natural projection $\pi : A \to U$ is proper.

We will write $z = (z_1, 'z) \in \mathbb{C} \times \mathbb{C}^{n-1}$ for a point $z \in \mathbb{C}^n$. Let M be a smooth real-analytic hypersurface that contains the origin. By $\rho(z, \bar{z})$ we denote a real-analytic defining function of M near 0. In a small neighborhood U of the origin, the complexification $\rho(z, \bar{w})$ of ρ is well-defined by means of a convergent power series in $U \times U$. For $w \in U$, the associated Segre variety is defined as

$$Q_w = \{z \in U : \rho(z, \bar{w}) = 0\}.$$

By the implicit function theorem, it is possible to choose neighborhoods $U_1 \subset\subset U_2$ of the origin such that for any $w \in U_1$, Q_w is a closed, complex hypersurface in U_2 and

$$Q_w = \{(z_1, 'z) \in U_2 : z_1 = h('z, \bar{w})\}, \tag{2.1}$$

where $h('z, \bar{w})$ is holomorphic in $'z$ and antiholomorphic in w. Following the terminology of [16], U_1 and U_2 are usually called a standard pair of neighborhoods of 0. It can be shown that Q_w is independent of the choice of the defining function.

We denote by $S = S(U)$ the set of Segre varieties $\{Q_w,\ w \in U\}$ and λ the so-called Segre map defined by

$$\lambda:\ \ U \to S$$
$$w \mapsto Q_w.$$

Let $I_w := \{z \in U : Q_w = Q_z\}$ be the fiber of λ over Q_w. For any $w \in M$, the set I_w is a complex variety of M. If the hypersurface M is nondegenerate (it contains no complex-analytic set with positive dimension), then for any $w \in M$ there exists a neighborhood U_w of w such that $I_w \cap U_w$ is finite. The set S admits the structure of a complex-analytic variety of finite dimension such that the map λ is a finite antiholomorphic branched covering. The set I_w contains at most as many points as the sheet number of λ. We next list some important properties of Q_w and I_w (see e.g. [10] and [18]).

(a) $z \in Q_w \Longleftrightarrow w \in Q_z$.

(b) $z \in Q_z \Longleftrightarrow z \in M$.

(c) $I_w = \bigcap\{Q_z : z \in Q_w\}$.

(d) The Segre map $\lambda : w \mapsto Q_w$ is locally one-to one near strictly pseudoconvex points of M.

Let $f : D \to D'$ be a proper holomorphic mapping between domains in $\mathbb{C}^n$ with smooth real-analytic boundaries which extends as a holomorphic correspondence F to a neighborhood of a point $p \in \partial D$. Assume that $p = 0$, $f(p) = 0'$ and choose standard neighborhoods $U_2 \supset\supset U_1 \ni 0$ and $U_2' \supset\supset U_1' \ni 0'$. Then we have the following invariance property for the Segre variety under F (see [16]) :

$$\text{For all } (w, w') \in \text{graph}(F) \cap (U_1 \times U_1'),\ \ F(Q_w) \subset Q'_{w'}. \tag{2.2}$$

This means that any branch of F maps any point from Q_w to $Q'_{w'}$ for any point $w' \in F(w)$.

Definition 3. *Under the hypothesis of Corollary* 2, *we say that f extends to a uniformly continuous mapping on $\mathcal{B} \subset M \cap U$ if for all $\epsilon > 0$, there exists $\alpha > 0$ such that for all $z \in D$ and $w \in \mathcal{B}$,*

$$|z - w| < \alpha\ \Rightarrow |f(z) - f(w)| < \epsilon.$$

Finally, recall that if $f : D \to \mathbb{C}^n$ is a holomorphic mapping and $z \in \partial D$, then the cluster set $cl_f(z)$ is defined as :

$$cl_f(z) = \{w \in \mathbb{C}^n \cup \{\infty\} : \lim_{j\to\infty} f(z_j) = w,\ \text{for } z_j \in D \text{ and } z_j \to z\}.$$

3 Proof of theorem 1.

The proof consists of two steps : first, to show that the mapping extends as a holomorphic correspondence to a neighborhood of p by using the process of analytic continuation along paths on the boundary and finally to apply the result of [14], which shows that all extending correspondences are holomorphic mappings.

3.1 Extension as a holomorphic correspondence.

Without loss of generality we assume that $p = 0$, $q = 0'$ and 0 is not in the envelope of holomorphy of D. Let U, U' be neighborhoods of 0 and $0'$ respectively, and $\mathcal{A} \subset U \times U'$ be the irreducible, closed, complex-analytic set of dimension n extending the graph Γ_f of f in the sense of Definition 1. Let $\pi_1 : \mathcal{A} \to U$ be the coordinate projection to the first component and let $E = \{z \in U : \dim \pi_1^{-1}(z) \geq 1\}$. We denote by $F : U \backslash E \to \mathbb{C}^n$ the multiple-valued map corresponding to $\mathcal{A}$; that is,

$$F(w) = \{w' : (w, w') \in \mathcal{A}\}.$$

We denote by S_F its branch locus (i.e., for $z \in U \backslash E$, $z \in S_F$ if the coordinate projection π_1 is not locally biholomorphic near $\pi_1^{-1}(z)$). The crucial point in the proof is to show that $\pi_1^{-1}(0) \cap E = \emptyset$ (i.e., $\pi_1^{-1}(0)$ is discrete). We denote $U^- = D \cap U$ and $U^+ = U \backslash \bar{D}$. We need the following observation.

Lemma 1. *$\mathcal{A} \cap (U^+ \times U') \neq \emptyset$.*

Proof. We follow the ideas of [19]. By contradiction assume that $\mathcal{A} \cap (U^+ \times U') = \emptyset$. Let A be the irreducible component of $\mathcal{A} \cap (U \times U')$ which contains $\Gamma_f \cap (U^- \times U')$. It follows that $A \not\subset (M \cap U) \times U'$. Let L be a complex line in $\mathbb{C}^n$ which contains 0 and is transverse to M such that $\Gamma_f \cap \{(U^- \cap L) \times U'\} \neq \emptyset$. We may choose U such that $U^- \cap L$ is connected. Let $\tilde{A}$ be the irreducible component of $A \cap \{(U \cap L) \times U'\}$ containing $\Gamma_f \cap \{(U^- \cap L) \times U'\}$. The analytic set $\tilde{A}$ has pure complex dimension 1 and it contains $(0, 0')$. Moreover, $\tilde{A} \not\subset (M \cap U) \times U'$. We consider two cases:

- If $\tilde{A} \cap \{(M \cap U) \times U'\}$ is discrete, then by the continuity principle we deduce that $(0, 0')$ is in the envelope of holomorphy of $U^- \times U'$.

- If $\tilde{A} \cap \{(M \cap U) \times U'\}$ is not discrete, then no open subset of $\tilde{A}$ can be contained in $\tilde{A} \cap \{(M \cap U) \times U'\}$. Now, the strong disc theorem shows that $(0, 0')$ is again in the envelope of holomorphy of $U^- \times U'$. Hence, 0 is in the envelope of holomorphy of D. Indeed, if $g \in \mathcal{O}(U^-)$, we may regard it as a function $\tilde{g} \in \mathcal{O}(U^- \times U')$. Then $\tilde{g}$ extends to a neighborhood of $(0, 0')$ and the uniqueness theorem shows that the extension of $\tilde{g}$ is also independent of the variables $z' \in U'$. Hence, g extends holomorphically across 0. This is a contradiction; since 0 is not in the envelope of holomorphy of D. ∎

As a consequence of Lemma 1, we deduce the following result due to Diederich-Pinchuk [15].

Lemma 2. *There exists an open set $\Gamma \subset M \cap U$ such that*

1) *f extends holomorphically to a neighborhood of $U^- \cup \Gamma$, and the graph of f near any point $(z, f(z))$, $z \in \Gamma$, is contained in $\mathcal{A}$.*

2) *$0 \in \bar{\Gamma}$ and $\lim_{\substack{z \to 0 \\ z \in \Gamma}} f(z) = 0'$.*

Proof. Let $V_F = \{(z, z') \in \mathcal{A} : z \in S_F\}$. Since the complex dimension of V_F is at most $n - 1$ (because $\mathcal{A}$ is irreducible and the projection π_1 is locally biholomorphic in an open set of $\mathcal{A}$), then $\mathcal{A} \backslash V_F$ is connected by paths. Without loss of generality we may assume that $M \cap U = \partial D \cap U$. Let $(a, b) \in \Gamma_f \cap (\mathcal{A} \backslash V_F) \cap (U^- \times U')$

and $(a', b') \in (\mathcal{A} \backslash V_F) \cap (U^+ \times U')$, and connect them by a path $\gamma \subset \mathcal{A} \backslash V_F$. It follows that $\pi_1(\gamma) \cap M \neq \emptyset$. Let z_o be the point where $\pi_1(\gamma)$ first intersects M. Then f extends analytically along $\pi_1(\gamma)$ from a to z_o and the graph of f over this part of $\pi_1(\gamma)$ is contained in $\mathcal{A} \backslash V_F$. It follows that z_o is a point of holomorphic extendability for f. The second part follows from the fact that U and U' may be chosen arbitrarily small. ■

The proof of Theorem 1 uses some ideas of Shafikov developed in [22] to study the analytic continuation of holomorphic correspondences and equivalence of domains.

Lemma 3. *There exists a holomorphic change of variables such that in the new coordinates* $Q_0 \not\subset E$.

Proof. The ideas of the proof were given in [22]. Assume that $Q_0 \subset E$. From Proposition 4.1 of [21] there exists a point $t \in \Gamma \backslash E$ such that $Q_0 \cap Q_t \neq \emptyset$ (Γ is the set defined in Lemma 2). Let $h : \tilde{U} \to \mathbb{C}^n$ be the germ of the mapping f defined in a neighborhood $\tilde{U}$ of t. We shrink $\tilde{U}$ and choose V in such a way that for any $w \in V$, the set $Q_w \cap \tilde{U}$ is connected. Observe that if V is small enough then $Q_w \cap \tilde{U} \neq \emptyset$ for any $w \in V$, as $w \in Q_t$ implies $t \in Q_w$. Note that V is a neighborhood of $Q_t \cap \tilde{U}$; since if $w \in Q_t$, then $t \in Q_w$ and $Q_w \cap \tilde{U} \neq \emptyset$. Following the ideas in [10] and [16], we define

$$X = \{(w, w') \in V \times \mathbb{C}^n : h(Q_w \cap \tilde{U}) \subset Q'_{w'}\}.$$

We would like to have $Q_w \cap \tilde{U}$ connected for any $w \in V$ to avoid ambiguity in the condition $h(Q_w \cap \tilde{U}) \subset Q'_{w'}$; since different components of $Q_w \cap \tilde{U}$ could be mapped a priori to different Segre varieties. Let $\mathcal{Q}(w', \bar{w}')$ be a defining polynomial function of M'. Let $z \in \tilde{U}$ and $z' = h(z)$. The inclusion $h(Q_w \cap \tilde{U}) \subset Q'_{w'}$ can be expressed as

$$\mathcal{Q}(h(z), \bar{w}') = 0 \quad \text{for any } z \in Q_w \cap \tilde{U}. \tag{3.1}$$

Therefore by property (2.1) of Segre varieties we can choose $\tilde{U}$ in the form $\tilde{U} = \tilde{U}_1 \times {}'\tilde{U} \subset \mathbb{C} \times \mathbb{C}^{n-1}$ such that $Q_w = \{(k('z, \bar{w}), 'z),\ 'z \in {}'\tilde{U}\}$, and (3.1) is equivalent to

$$\mathcal{Q}(h(k('z, \bar{w}), 'z)\,, \bar{w}') = 0, \text{ for any } 'z \in {}'\tilde{U}. \tag{3.2}$$

Thus, X is defined by an infinite system of holomorphic equations in $(\bar{w}, \bar{w}')$. By the Noetherian property of the ring of holomorphic functions, we can choose finitely many points $'z^1, \cdots, 'z^m$ so that (3.2) can be written as a finite system

$$\sum_{|J| \leq d'} \alpha_J^k(w) w'^J = 0,$$

where $k = 1, \cdots, m$, d' is the degree of $\mathcal{Q}$ in w' and α_J^k are holomorphic functions in w. We define the closure of X in $V \times \mathbb{P}^n$ in the following way. Let $\tilde{t} = (t_0, t_1, \cdots, t_n)$ be homogeneous coordinates in $\mathbb{P}^n$, and let $w'_j = t_j/t_0$ and $t = (t_1, \cdots, t_n)$. Then

$$\sum_{|J| \leq d'} \alpha_J^k(w)(t/t_0)^J = 0, \quad k = 1, \cdots, m$$

is a system of equations homogeneous in $\tilde{t}$ that defines an analytic function in $V \times \mathbb{P}^n$. We denote this variety by $\bar{X}$. Clearly, the restriction of $\bar{X}$ to $V \times (\mathbb{P}^n \backslash H_0) = V \times \mathbb{C}^n$ coincides with the set defined by (3.2). Here $H_0 = \{t_0 = 0\}$ is the hyperplane at infinity. Let $\pi : \bar{X} \to V$ and $\pi' : \bar{X} \to \mathbb{P}^n$ be the natural projections. According to [22] (Lemma 3), h extends as a holomorphic correspondence to $V \backslash (\Lambda_1 \cup \Lambda_2)$, where $\Lambda_1 = \pi(\pi'^{-1}(H_0))$ and $\Lambda_2 = \pi\{(w, w') \in \bar{X} : \dim \pi^{-1}(w) \geq 1\}$. It is easy to see that Λ_1 is a complex manifold of dimension at most $n-1$, and according to [21] (Proposition 3.3), Λ_2 is a complex-analytic set of dimension at most $n-2$. By considering dimension, we may assume that $Q_0 \cap V \not\subset \Lambda_2$. Also, we may assume that $Q_0 \cap V \not\subset \Lambda_1$; since otherwise we can perform a linear fractional transformation such that H_0 is mapped onto another complex hyperplane $H \subset \mathbb{P}^n$ with $H \cap M' = \emptyset$. Thus, by the holomorphic extension along Q_t we can find points in Q_0 where h extends as a holomorphic correspondence. This implies that in the new coordinates $Q_0 \not\subset E$. ∎

As a consequence, we deduce the following

Lemma 4. *$\pi_1^{-1}(0)$ is discrete.*

Proof. It suffices to show that $0 \notin E$. In view of Lemma 3, we may assume that $Q_0 \not\subset E$. By contradiction, suppose that $0 \in E$. It follows that there exist a point $b \in Q_0$ and a small neighborhood $U_b \ni b$ such that $U_b \cap E = \emptyset$. As in the proof of Lemma 3, we may choose U and U_b so small such that for any $z \in U$, the set $Q_z \cap U_b$ is non-empty and connected. Let $\Sigma = \{z \in U : Q_z \cap U_b \subset S_F\}$. We define

$$X = \{(w, w') \in (U \backslash \Sigma) \times \mathbb{C}^n : F(Q_w \cap U_b) \subset Q'_{w'}\}.$$

We follow the convention of using the right prime to denote the objects in the target domain. For instance, $Q'_{w'}$ will mean the Segre variety of w' with the respect to the hypersurface M'.

We prove the following properties of X.

Claim 1.

i) *X is not empty;*

ii) *X is a complex-analytic set in $(U \backslash \Sigma) \times \mathbb{C}^n$;*

iii) *X is closed in $(U \backslash \Sigma) \times \mathbb{C}^n$;*

iv) *$\Sigma \times \mathbb{C}^n$ is a removable singularity for X.*

Proof. i) In view of Lemma 2, there exists a sequence $\{a_j\} \subset \Gamma \backslash (E \cup \Sigma)$ such that $a_j \to 0$ as $j \to \infty$, f extends holomorphically across a_j and the graph of f near $(a_j, f(a_j))$ is contained in $\mathcal{A}$. It follows by the invariance property of Segre varieties (see (2.2)) that $(a_j, f(a_j)) \in X$ and so $X \neq \emptyset$.

ii) Let $(w, w') \in X$. Consider an open simply connected set $\Omega \subset U_b \backslash S_F$ such that $Q_w \cap \Omega \neq \emptyset$. The branches of F are globally defined in Ω. Since $Q_w \cap U_b$ is connected, the inclusion $F(Q_w \cap U_b) \subset Q'_{w'}$ is equivalent to

$$F^j(Q_w \cap \Omega) \subset Q'_{w'}, \quad j = 1, \cdots, m,$$

where the F^j denote the branches of F in Ω. Recall that $\mathcal{Q}(w',\bar{w}')$ denotes a defining polynomial function of M'. The inclusion $F^j(Q_w \cap \Omega) \subset Q'_{w'}$, $j = 1, \cdots, m$ can be expressed as

$$\mathcal{Q}(F^j(z), \bar{w}') = 0 \text{ for any } z \in Q_w \cap \Omega, \quad j = 1, \cdots, m.$$

As in the proof of Lemma 3, we can choose Ω in the form $\Omega = \Omega_1 \times {}'\Omega \subset \mathbb{C} \times \mathbb{C}^{n-1}$ such that $Q_w = \{(k('z, \bar{w}), 'z), \ 'z \in {}'\Omega\}$, and

$$\mathcal{Q}(F^j(k('z, \bar{w}), 'z)\ , \bar{w}') = 0, \text{ for any } 'z \in {}'\Omega. \tag{3.3}$$

Thus, X is defined by an infinite system of holomorphic equations in $(\bar{w}, \bar{w}')$. By the Noetherian property of the ring of holomorphic functions, we can choose finitely many points $'z^1, \cdots, 'z^m$ so that (3.3) can be written as a finite system

$$\sum_{|J| \le d'} \alpha_J^k(w) w'^J = 0,$$

where $k = 1, \cdots, m$, d' is the degree of $\mathcal{Q}$ in w' and α_J^k are holomorphic functions in w. Thus, X is a complex-analytic set in $(U \backslash \Sigma) \times \mathbb{C}^n$.

iii) The set X is closed in $(U \backslash \Sigma) \times \mathbb{C}^n$. Indeed; let (w^j, w'^j) be a sequence in X that converges to $(w_o, w'_o) \in (U \backslash \Sigma) \times \mathbb{C}^n$, as $j \to \infty$. Since $Q_{w^j} \to Q_{w_o}$ and $Q'_{w'^j} \to Q'_{w'_o}$, from the inclusion $F(Q_{w^j} \cap U_b) \subset Q'_{w'^j}$ we obtain

$$F(Q_{w_o} \cap U_b) \subset Q'_{w'_o},$$

which implies that $(w_o, w'_o) \in X$ and thus, X is a closed set.

iv) Now, let us show that $\Sigma \times \mathbb{C}^n$ is a removable singularity for X. Let $t \in \Sigma$. It follows that $\bar{X} \cap (\{t\} \times \mathbb{C}^n) \subset \{t\} \times \{z' : \ F(Q_t \cap U_b) \subset Q'_{z'}\}$. If $w' \in F(Q_t \cap U_b) \subset Q'_{z'}$, then $z' \in Q'_{w'}$. Since $\dim_{\mathbb{C}} Q'_{w'} = n - 1$, then $\{z' : \ F(Q_t \cap U_b) \subset Q'_{z'}\}$ has dimension at most $2n - 2$ and $\bar{X} \cap (\Sigma \times \mathbb{C}^n)$ has 2n-dimensional measure zero. Now, Bishop's theorem can be applied to conclude that $\Sigma \times \mathbb{C}^n$ is a removable singularity for X. ∎

Now, we continue with the proof of Lemma 4. Let $\{a_j\}$ be a sequence in $\Gamma \backslash (E \cup \Sigma)$ such that $a_j \to 0$ as $j \to \infty$. Then, f extends holomorphically across a_j and the graph of f near $(a_j, f(a_j))$ is contained in $\mathcal{A}$. Moreover, for small neighborhoods $U_j \ni a_j$ we have :

$$X|_{U_j \times \mathbb{C}^n} = \mathcal{A}|_{U_j \times \mathbb{C}^n}. \tag{3.4}$$

We denote by $\bar{X}$ the closure of X in $U \times \mathbb{C}^n$. Without loss of generality we may assume that $\bar{X}$ is irreducible. Then in view of (3.4) and by the uniqueness theorem (see for instance [7]) we deduce that $\bar{X} = \mathcal{A}$.

Let $\hat{F}$ be the multiple-valued mapping corresponding to $\bar{X}$. By construction, for any $a' \in \hat{F}(0)$, $\hat{F}(0) = I'_{a'}$. Since $0' \in \hat{F}(0) \cap M'$, it follows that $\hat{F}(0) \subset M'$ and so $\hat{F}(0)$ is a finite set. Thus, if V is a bounded open neighborhood of $\hat{F}(0)$, we may choose U such that $\bar{X} \cap (U \times \partial V) = \emptyset$. Otherwise; there exists a sequence $(z_j, z'_j)_j$ in X such that $(z_j)_j$ converges to 0 and $(z'_j)_j$ converges to $z'_o \in \partial V$ as $j \to \infty$. This implies that $(0, z'_o) \in \bar{X}$ and $z'_o \notin \hat{F}(0)$: a contradiction. Then $\hat{F} : U \to V$ defines a holomorphic correspondence extending f. This contradicts the fact that $0 \in E$ and completes the proof of Lemma 4. ∎

3.2 Conclusion of the proof of Theorem 1.

In view of Lemma 4, f extends as a holomorphic correspondence to a neighborhood U of 0. Then f extends to U as an algebroid m-valued mapping $\hat{f} = (\hat{f}^1, \cdots, \hat{f}^n)$ whose components $w_\nu = \hat{f}^\nu(z)$ satisfy polynomial equations

$$w_\nu^m + a_{1\nu}(z){w_\nu}^{m-1} + \cdots + a_{m\nu}(z) = 0,\ \nu = 1, \cdots, n$$

with holomorphic coefficients $a_{\mu\nu} \in \mathcal{O}(U)$. In particular, the map f extends continuously to $\bar{D} \cap U$. Then after an appropriate shrinking of U and U' the map $f : D \cap U \to D' \cap U'$ defines a proper holomorphic mapping that extends as a holomorphic correspondence in a neighborhood of 0. According to [14], this extension is in fact a holomorphic mapping. ∎

4 Proof of corollary 1 and 2

4.1 Proof of Corollary 1.

a) First, we prove that the mapping f is algebraic (i.e., the graph of the mapping is contained in an irreducible complex n-dimensional algebraic set in $\mathbb{C}^n \times \mathbb{C}^n$). If D is not pseudoconvex, there exist $\tilde{p} \in \partial D$ and a neighborhood U of $\tilde{p}$ such that f extends holomorphically to U. By moving slightly $\tilde{p}$, we may assume that f extends to a biholomorphic mapping in a neighborhood of $\tilde{p}$. The classical Webster's theorem ([27]) implies that f extends to an algebraic mapping. Assume now that D is pseudoconvex, which implies that D' is also pseudoconvex. In view of [25] and [17], f extends holomorphically to a neighborhood of p. Then, as above we can conclude that f is algebraic by using Webster's theorem. Now, it follows from Theorem 1 that f extends holomorphically to a neighborhood of p. To finish the proof, we have to show that the set of holomorphic extendability of f is an open dense subset of ∂D. Let $S^h = \{z \in \partial D : cl_f(z) \cap \partial D' \neq \emptyset\}$ and $S^\infty = \partial D \backslash S^h$.
Claim 2. *S^h is a dense subset of ∂D.*
Proof. Since f is algebraic, its components f^j, $j = 1, \cdots, n$ are also algebraic. Then there exist polynomials

$$\mathcal{P}_j(z, w_j) = a_j^{m_j}(z){w_j}^{m_j} + \cdots + a_j^1(z)w_j + a_j^0(z), \qquad j = 1, \cdots, n$$

where $a_j^{k_j}(.)$ are holomorphic polynomials for all $k_j \in \{0, \cdots, m_j\}$ such that

$$\mathcal{P}_j(z, f^j(z)) = 0, \text{ for all } z \in D \text{ and } j = 1, \cdots, n.$$

Without loss of generality, we may assume that $a_j^{m_j} \not\equiv 0$ on $\mathbb{C}^n$ for all $j = 1, \cdots, n$. If $\tilde{p} \in S^\infty$, there exists $j \in \{1, \cdots, n\}$ such that $a_j^{m_j}(\tilde{p}) = 0$. It follows that the polynomial function $\tilde{a} = \prod_{1 \le j \le n} a_j^{m_j}$ vanishes identically on S^∞. If S^∞ has an interior point, then by the boundary uniqueness theorem (see for instance [7]) the polynomial $\tilde{a}$ vanishes identically on $\mathbb{C}^n$, which implies that $a_j^{m_j} \equiv 0$ on $\mathbb{C}^n$ for some j. This contradiction completes the proof of Claim 2. ∎

Now, the assertion follows from Theorem 1, Claim 2 and the algebraicity of the mapping.

b) First, assume that D' has a global holomorphic peak function at infinity. It suffices to prove that S^∞ has no interior point. There exists a holomorphic function φ on D' satisfying $|\varphi(w)| < 1$ on D' and $\varphi(w) \to 1$ as $|w| \to \infty$. Set $G(z) = \varphi \circ f(z) - 1$. If S^∞ has an interior point $\tilde{p} \in \partial D$, then the function $G(z) \to 0$ as z tends to a boundary point close to $\tilde{p}$. By the boundary uniqueness theorem we get that $f \equiv \infty$ on D : a contradiction.

Assume now, that D has a global holomorphic peak function at infinity. Consider the proper holomorphic correspondence $f^{-1} : D' \to D$.

Claim 3. $S'^\infty = \{q \in \partial D' : \limsup_{z' \to q} |f^{-1}(z')| = \infty\}$ is nowhere dense.

Proof. Suppose that S'^∞ has an interior point $q_0 \in \partial D'$. There exists a holomorphic function ϕ on D satisfying $|\phi(w)| < 1$ on D and $\phi(w) \to 1$ as $|w| \to \infty$. The function $G(z') = \prod_{1 \le j \le m} [\phi \circ g^j(z') - 1]$ is holomorphic in $D' \backslash \sigma'$, $\sigma' \subset D'$ is a complex-analytic set of dimension $\le n-1$ and $g^1, \cdots, g^m$ are the branches of f^{-1}. Since $G(z')$ is bounded ($|G(z')| \le 2^m$), then it extends as a holomorphic function on D'. The function $G(z') \to 0$ as z' tends to a boundary point close to q_0. By the boundary uniqueness theorem, we get that one of the branch $g^j \equiv \infty$ on D'. This contradiction completes the proof of the claim. ∎

Claim 3 shows in particular, that $cl_f(\partial D) \cap \partial D'$ is not empty. Then the assertion follows from the result of Corollary 1-a). ∎

4.2 Proof of Corollary 2.

The idea of the proof is as in Corollary 1, but the method of proof is different; since here the domains are only algebraic in a neighborhood of an open piece of the boundary. In view of Theorem 1, it suffices to prove that f is algebraic. Note that here $cl_f(M) \subset M'$.

a) First, assume that $n = 2$. Without loss of generality we may assume that M is connected. Let $U \subset \mathbb{C}^2$ be an open neighborhood such that $M = \partial D \cap U = \{z \in U : P(z, \bar{z}) = 0\}$ with P a real polynomial and $dP \neq 0$ on $\bar{M}$. We denote by M_s^+ the set of all strictly pseudoconvex points of M and M_s^- the set of all strictly pseudoconcave points of M. Let M^+ (resp. M^-) denote the interior of $\overline{M_s^+}$ (resp. $\overline{M_s^-}$). The set M^+ is the pseudoconvex part of M and M^- is the pseudoconcave part of M. It is known that $M^- \subset \hat{D}$, where $\hat{D}$ denotes always the envelope of holomorphy of D. Let $T = \{z \in M : \mathcal{L}_P(z) = 0\}$, where $\mathcal{L}_P$ is the Levi-form of the boundary restricted to $T_z^c M$, the complex tangent space at z to M. The set T is a real-algebraic set in M of dimension at most 2. It admits a locally finite semi-algebraic stratification as, $T = T_0 \cup T_1 \cup T_2$, where T_i for $i = 0, 1, 2$ are disjoint union of connected real-algebraic submanifolds of M of real dimension i. The set $M_b := M \backslash \{M^+ \cup M^-\}$ is called the border set in M. It is a closed semi-algebraic

subset of M and $M_b \subset T$. Let $M_e := M_b \cap (T_0 \cup T_1)$, be called the exceptional set. The set M_e is a pluripolar set. It was shown in [12] that $M_b \backslash M_e \subset \hat{D}$.
Let $U' \subset \mathbb{C}^2$ be an open neighborhood such that $M' = \partial D' \cap U' = \{z \in U' : \mathcal{Q}(z', \bar{z}') = 0\}$ with $\mathcal{Q}$ a real polynomial and $d\mathcal{Q} \neq 0$ on $\bar{M}'$. We follow the same notations as above by using the right prime to denote the objects in the target domain. Let $p \in M$, then there exists a point $q \in M'$ such $q \in cl_f(p)$. We consider several cases :

(1) - If $p \in M^-$, then there exists a neighborhood V of p such that f extends holomorphically to V. Hence, by moving slightly p (if necessary), we may assume that f extends biholomorphically near p. Now, the classical Webster's theorem implies that f extends to an algebraic mapping.

(2) - If $p \in M^+$ and $q \in M'^+$, then in view of [25] and [17], f extends holomorphically to a neighborhood of p. Hence, we can conclude as above that f is algebraic.

(3) - If $p \in M^+$ and $q \in M'^- \cup M'_b$. Let $\Upsilon := \{z \in M : cl_f(z) \subset M'_e\}$. Since M'_e is pluripolar, there exists $\phi \in PSH(\mathbb{C}^2)$, $\phi \not\equiv -\infty$ such that $\phi_{/M'_e} \equiv -\infty$. Then $\psi = \phi \circ f \in PSH(D)$ and $\psi(z) \to -\infty$ as $z \to z^o \in \Upsilon$. The set Υ has no interior point, as otherwise $\psi \equiv -\infty$ and this is a contradiction. Hence, by moving slightly p, we may assume that $q \in M'^- \cup (M'_b \backslash M'_e)$. We need the following observation due to Diederich-Pinchuk [16].

Lemma 5. *Let $f : D \to D'$ be a proper holomorphic mapping. Assume that there exist $a \in \partial D$ and $a' \in \partial D'$ such that $a' \in cl_f(a)$. Then $a \in \hat{D}$, if $a' \in \hat{D}'$.*

Since $M'^- \cup (M'_b \backslash M'_e) \subset \hat{D}'$, then in view of Lemma 5 we deduce that $p \in \hat{D}$. Hence, again Webster's theorem implies that f extends to an algebraic mapping.

b) Next, assume that $n \geq 2$. We denote by M_s^+ the set of strictly pseudoconvex points of M and M_s^- the set of strictly pseudoconcave points of M. The set of points where the Levi-form $\mathcal{L}_P$ has eigenvalues of both signs on $T^c(M)$ and no zero will be denoted by $M^{\pm}$ and by M^0 we mean the set of points of M where $\mathcal{L}_P$ has at least one eigenvalue 0 on $T^c(M)$. Note that M^0 is a closed real-algebraic set of dimension at most $2n - 2$. We have

$$M = M_s^+ \cup M_s^- \cup M^{\pm} \cup M^0.$$

It is well known that $M_s^- \cup M^{\pm} \subset \hat{D}$. Then if M is not pseudoconvex, the same argument used in a)-(1) shows that the mapping f is algebraic. For the rest of the proof we may suppose that M is pseudoconvex. Let $p \in \mathcal{B} \subset M \cap U$ be a strictly pseudoconvex boundary point (such a point exists; since M is pseudoconvex and $\mathcal{B}$ is dense). It suffices to prove that f extends holomorphically to a neighborhood of p. We consider several cases:

- Assume, that $q = f(p) \in M_s'^+$. Then in view of [25], f extends continuously to a neighborhood of p and in view of [17], f extends holomorphically to a neighborhood of p.

- Assume next, that $q = f(p) \in M'^{\pm} \cup M_s'^-$. Then by Lemma 5, we deduce that $p \in \hat{D}$. Hence, f extends holomorphically to a neighborhood of p.

- Finally, assume that $q = f(p) \in M'_0$. We need the following lemma (appeared in [17] in the case of continuous CR-mapping between real-analytic hypersurfaces in $\mathbb{C}^n$).

Lemma 6. *Let $N' \subset M'$ be a real $\mathcal{C}^2$-smooth generic manifold of real dimension at most $2n-2$ that contains q and let V be a neighborhood of $p \in M_s^+$. Then $cl_f(M \cap V) \not\subseteq N'$.*

Proof. We follow the ideas of [17] with some minor modifications. If $\dim_{\mathbb{R}} N' < 2n-2$, we can always find a generic manifold in M' of dimension $2n-2$ which contains N'. Then without loss of generality, we may assume that $\dim_{\mathbb{R}} N' = 2n-2$ and $q = 0'$. There exists a complex plane $L' \ni 0'$ such that $L' \cap N'$ is a totally real-manifold of real dimension 2 near $0'$. For $a' \in \mathbb{C}^n$, let $L_{a'}$ be the complex plane parallel to L' and passing through a'. For a small neighborhood V' of $0'$ in $\mathbb{C}^n$ the intersection $L_{a'} \cap N' \cap V'$ is a totally real-manifold of real dimension 2. There exists a strictly plurisubharmonic function $\varphi_{a'}$ on V' such that:

- $\varphi_{a'} \geq 0$
- $\varphi_{a'} = 0$ on $L_{a'} \cap N' \cap V'$.

Since p is a strictly pseudoconvex point, then in a new coordinates we may assume that $p = 0$ and the defining function r of D can be written near 0 as

$$r(z, \bar{z}) = 2Re(z_1) + |'z|^2 + o(|z|^2).$$

Let $a \in D \cap V$ be a point closed to 0 such that $f(a) \in V'$. Set $H_a := \{z \in D \cap V : z_1 = a_1\}$. Notice that H_a is a complex-manifold of dimension $n-1$ and $H_a \subset\subset V$. Set $A_a := H_a \cap f^{-1}(L_{f(a)} \cap V')$, which is a complex-analytic set in $D \cap V$. Since f is proper and $\dim_{\mathbb{C}} L_{f(a)} = 2$, the complex dimension of A_a is at least 1. If $cl_f(M \cap V) \subset N'$, we would have $cl_f(\partial H_a) \subset N'$. The function $g_a = \varphi_{f(a)} \circ f$ is plurisubharmonic and positive on A_a and $cl_{g_a}(\partial A_a) = 0$. It follows by the maximum principle that $g_a \equiv 0$ on A_a. But $\varphi_{f(a)}$ is strictly plurisubharmonic, then f is constant on A_a with image in N'. This is a contradiction; since $f(a) \in D'$. ∎

Now, we continue with the proof of Corollary 2. We want to show that there exists an open set $U^1 \subset U$ such that $cl_f(M \cap U^1) \not\subseteq M'_0$.
The set M'_0 can be stratified as $M'_0 = \cup_k N'_k$ by smooth generic manifolds N'_k of dimension k less or equal to $2n-2$. Let j_o be the largest index such that $cl_f(M \cap U) \cap N'_j \neq \emptyset$. Then $cl_f(M \cap U) \cap N'_{j_o} \neq \emptyset$ and $cl_f(M \cap U) \cap N'_j = \emptyset$ for all $j > j_o$. Let $b' \in cl_f(p) \cap N'_{j_o}$. Therefore, there exists a sequence $\{p_k\}$ in D such that $p_k \to p$ and $f(p_k) \to b'$. Let ϵ be a positive real number so that $D(b', 2\epsilon) \cap N'_j = \emptyset$ for all $j < j_o$ and let $k_1 = k_1(\epsilon)$ be an integer such that for all $k > k_1$, $|f(p_k) - b'| < \epsilon$

Claim 4. *There exists a point $a \in M_s^+ \cap U$ with the following properties : f is continuous at a and $f(a) \in D(b', 2\epsilon)$.*

Proof. Let $\{a_k\}$ be a sequence in $\mathcal{B}$ with $a_k \to p$. Since f extends to a uniformly continuous function on $\mathcal{B}$, there exist a real number $\alpha > 0$ and an integer k_2 such for all $z \in D$ and $k > k_2$, $|z - a_k| < \alpha \Rightarrow |f(z) - f(a_k)| < \epsilon$. Starting with some integer k_3, we have $|a_k - p_k| < \alpha$. Hence, for all $k \geq \max(k_1, k_2, k_3)$, $|f(a_k) - b'| < 2\epsilon$. ∎

By Claim 4, there exists $U^1 \subset U$, a neighborhood of a, such that $f(D \cap U^1) \subset D(b', 2\epsilon)$. It follows that $cl_f(M \cap U^1) \subset D(b', 2\epsilon)$. But $cl_f(M \cap U^1) \cap N'_j = \emptyset$ for all $j > j_o$ and $D(b', 2\epsilon) \cap N'_j = \emptyset$ for all $j < j_o$. Hence, if $cl_f(M \cap U^1) \subseteq M'_0$, then $cl_f(M \cap U^1) \subseteq N'_{j_o}$: a contradiction with Lemma 6. Now, as in the previous cases we can show that f is algebraic. This finishes the proof of Corollary 2. ■

5 Proof of theorem 2 and 2-bis, and corollary 3

5.1 Proof of Theorem 2

Let $f : D \to D'$ be a proper holomorphic mapping as in Theorem 2. Following the proof of Corollary 1-b), it is clear that if D or D' has a global holomorphic peak function at infinity, then $cl_f(\partial D)$ intersects $\partial D'$. Now, by repeating the argument used in the proof of Corollary 1-a), we may show that f is algebraic.

(A) First, assume that P is plurisubharmonic on D and ∂D_c is nondegenerate for all $c < 0$. We denote by J_f the Jacobi determinant of f and by $V_f = \{z \in D : J_f(z) = 0\}$ its branch locus. Following [3] and [4], we consider the Levi-determinant of D defined by

$$\Lambda_{\partial D} = -\det \begin{pmatrix} 0 & \frac{\partial P}{\partial \bar{z}_1} & \cdots & \frac{\partial P}{\partial \bar{z}_n} \\ \frac{\partial P}{\partial z_1} & \frac{\partial^2 P}{\partial z_1 \partial \bar{z}_1} & \cdots & \frac{\partial^2 P}{\partial z_1 \partial \bar{z}_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial P}{\partial z_n} & \frac{\partial^2 P}{\partial z_n \partial \bar{z}_1} & \cdots & \frac{\partial^2 P}{\partial z_n \partial \bar{z}_n} \end{pmatrix}.$$

Note that $\Lambda_{\partial D}(z) \geq 0$ for all $z \in \partial D$ and the set

$$\omega(\partial D) = \{z \in \partial D : \Lambda_{\partial D}(z) = 0\}$$

is precisely the set of weakly pseudoconvex boundary points. For any point $p \in \partial D$ we consider also the order of vanishing of the Levi-determinant at p denoted by $\tau(p)$, which is defined as follows: we choose smooth coordinates $x = (x_1, \cdots, x_{2n-1})$ on ∂D such that p corresponds to $x = 0$, and the formal power series

$$\Lambda_{\partial D}(x) = \sum_{j=0}^{\infty} \sum_{|\alpha|=j} a_\alpha x^\alpha,$$

where $\alpha = (\alpha_1, \cdots, \alpha_{2n-1})$ is a multi-index,

$$x^\alpha = x_1^{\alpha_1} \cdots x_{2n-1}^{\alpha_{2n-1}}$$

and $|\alpha| = \alpha_1 + \cdots + \alpha_{2n-1}$. We set

$$\tau(p) = \min\{|\alpha| : a_\alpha \neq 0\}.$$

The definition does not depend on the choice of the coordinates. The number $\tau(p)$ can be also defined as the smallest nonnegative integer m such that there is a tangential differential operator T of order m on ∂D (i.e., T is a differential operator of order m satisfying $TP = 0$) such that $T\Lambda_{\partial D}(p) \neq 0$. The function τ is upper-semicontinuous. Indeed, for each $\lambda > 0$ the set $\{z \in \partial D : \tau(z) < \lambda\}$ is open; since its complement $\{z \in \partial D : T\Lambda_{\partial D}(z) = 0$ for all T, with order $< \lambda\}$ is closed, being the intersection over all T of closed zero sets of the smooth functions $T\Lambda_{\partial D}$. Note that $\{p \in \partial D : \tau(p) = 0\}$ is the set of strictly pseudoconvex boundary points and $\{p \in \partial D : \tau(p) \geq 1\}$ is the set of weakly pseudoconvex boundary points.

Lemma 7. *Let $f : D \to D'$ be a proper holomorphic mapping as in Theorem 2. Then for all $p \in S^h$, the set of holomorphic extendability of f, $\tau(p) \geq \tau(f(p))$ and the inequality holds if and only if f is branched at p.*

Proof. Let $p \in S^h \cap \{J_f \neq 0\}$. Since $\mathcal{Q}$ is a defining function for D', $\nabla(\mathcal{Q} \circ f)(p) \neq 0$. Then $\mathcal{Q} \circ f$ is a local defining function of D in a neighborhood of p, and by the chain rule we have:

$$\Lambda_{\mathcal{Q}\circ f}(p) = |J_f(p)|^2 \Lambda_{\mathcal{Q}}(f(p)).$$

Hence, we are able to deduce the lemma. ■

Following [3] (or [9]), there is a semi-algebraic stratification for $\omega(\partial D)$ given by

$$\omega(\partial D) = A_1 \cup A_2 \cup A_3 \cup A_4$$

where A_4 is a closed, real-algebraic set of dimension at most $2n - 4$ and $A_2 \cup A_3 \cup A_4$ is also a closed, real algebraic set of dimension at most $2n - 3$. Further, A_1, A_2 and A_3 are either empty or smooth, real-algebraic manifolds; A_2 and A_3 have dimension $2n - 3$, and A_1 has dimension $2n - 2$. When A_2 and A_3 are non-empty, A_2 and A_3 are CR manifolds with

$$\dim_{\mathbb{C}} T^c A_2 = n - 2 \text{ and } \dim_{\mathbb{C}} T^c A_3 = n - 3.$$

Recall that $\dim_{\mathbb{C}} T^c A_j$ denotes the complex dimension of the complex tangent space to A_j ($j \in \{2, 3\}$). Finally, the function τ is constant on every connected component of A_1.

Lemma 8. *Let W be an irreducible component of V_f and $\mathcal{E}_W := \overline{W} \cap \partial D$.*

1) *There exists an open dense subset O_W of $\mathcal{E}_W$ such that for all $p \in O_W$:*

i) *W extends across the boundary of D as a pure $(n-1)$-dimensional polynomial variety in $\mathbb{C}^n$ and $\mathcal{E}_W$ is a polynomial submanifold of dimension $2n - 3$ in a neighborhood of p.*

ii) *f extends holomorphically in a neighborhood of p.*

2) *$\overline{W}$ does not intersect the set $\partial D \backslash w(\partial D)$ of strictly pseudoconvex boundary points.*

Proof. 1) -i) Since W is an irreducible algebraic set in D of dimension $n - 1$, there exists an irreducible polynomial h in $\mathbb{C}^n$ such that $W = \{z \in D : h(z) = 0\}$. If W does not extend across ∂D, the polynomial P will be negative on $\hat{W} = \{z \in \mathbb{C}^n :$

$h(z) = 0\}$. According to [7] (Proposition 2, page 76), there exists an analytic cover $\pi : \hat{W} \to \mathbb{C}^{n-1}$. Let $g^1, \cdots, g^k$ be the branches of π^{-1} which are locally defined and holomorphic in $\mathbb{C}^{n-1} \setminus \sigma$, with $\sigma \subset \mathbb{C}^{n-1}$ a complex-analytic set of dimension at most $n-2$. Consider the function $\hat{P}(w) = \sup\{P \circ g^1(w), \cdots, P \circ g^k(w)\}$. Since π is an analytic cover, $\hat{P}$ extends as a plurisubharmonic function to $\mathbb{C}^{n-1}$. Then there exists a negative constant c such that $\hat{P} \equiv c$; since $\hat{P}$ is negative. It follows that for all $w_0 \in \mathbb{C}^{n-1} \setminus \sigma$, there exist a neighborhood $U_{w_0} \ni w_0$ and an integer $s \in \{1, \cdots, k\}$ such that $P \circ g^s \equiv c$ on U_{w_0}. This contradicts the nondegeneracy of ∂D_c.

Let us verify now that there exists an open dense subset O_W of $\mathcal{E}_W$ such that for all $p \in \mathcal{O}_W$, $\mathcal{E}_W$ is a polynomial submanifold of dimension $2n-3$ in a neighborhood of p. We may choose a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ such that $v := \dfrac{\partial^\alpha h}{\partial z^\alpha}$ vanishes on W; but ∇v is not identically zero on W. We need the following observation (appeared in [2] in the strictly pseudoconvex case).

Claim 5. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ with real-analytic and nondegenerate boundary. For any point $p \in \partial\Omega$, there exists $\epsilon_0 > 0$ so that for any $\epsilon \in (0, \epsilon_0)$, there is some $\delta \in (0, \epsilon)$ with the following property : for any point $q \in \Omega \cap B(p, \delta)$, there exists a plurisubharmonic function φ_q on $\Omega \cap B(p, \epsilon_0)$, continuous on $\bar{\Omega} \cap B(p, \epsilon_0)$ such that $\varphi_q(q) = 1$ and $0 < \varphi_q(z) < \dfrac{1}{2}$ for all $z \in D \cap \partial B(p, \epsilon)$.*

Proof. According to [11], there exists a local plurisubharmonic peak function at p (i.e., there exist a small $\epsilon_0 > 0$ and a function $\psi_p \in PSH(\Omega \cap B(p, \epsilon_0)) \cap \mathcal{C}^0(\bar{\Omega} \cap B(p, \epsilon_0))$ such that $\psi_p(p) = 1$ and $\psi_p(z) < 1$ on $(\bar{\Omega} \cap B(p, \epsilon_0)) \setminus \{p\}$). Set $\phi_p = e^{\psi_p - 1}$ and for $0 < \epsilon < \epsilon_0$, let $M = \max\{\phi_p(z) : z \in \bar{\Omega} \cap \partial B(p, \epsilon)\}$. Note that $0 < M < 1$. The set $\{z \in \bar{\Omega} \cap B(p, \epsilon) : \phi_p(z) > M\}$ is an open neighborhood of p in $\bar{\Omega} \cap B(p, \epsilon)$, so there is some $\delta > 0$ so that $0 < M < \phi_p(z)$ for all $z \in \Omega \cap B(p, \delta)$. For any $q \in \Omega \cap B(p, \delta)$, let $h_q(z) = \phi_p(z)/\phi_p(q)$. Then $h_q(q) = 1$ and for all $z \in \bar{\Omega} \cap \partial B(p, \epsilon)$, $0 < h_q(z) < M/\phi_p(q) < 1$. It follows that for $N > 0$ large enough, the function $\varphi_q(z) = (h_q(z))^N$ has the properties given by the claim. ∎

Let $W_1 = \{z \in W : (\partial v/\partial z_1)(z) = 0\}$. Using the irreducibility assumption, W_1 is a nowhere dense subvariety of W. First, we show that for $p \in \mathcal{E}_W$, $(\partial v/\partial z_1)$ can not vanish everywhere on $\{|z-p| \leq \epsilon_0\} \cap \mathcal{E}_W$ for any $\epsilon_0 > 0$. We may assume that ϵ_0 is the real number given by Claim 5 corresponding to p. Let $\epsilon \in (0, \epsilon_0)$ and a corresponding $\delta > 0$. Let $z^1 \in (W \setminus W_1) \cap B(p, \delta)$. By Claim 5, there exists a function $\varphi \in PSH(D \cap B(p, \epsilon_0)) \cap \mathcal{C}^0(\bar{D} \cap B(p, \epsilon_0))$ such that $\varphi(z^1) = 1$ and $0 < \varphi(z) < \dfrac{1}{2}$ for all $z \in D \cap \partial B(p, \epsilon)$. Note that for any natural number N, the function φ^N is plurisubharmonic; since by construction, φ is the exponential of a plurisubharmonic function. Assume that $(\partial v/\partial z_1)$ vanishes everywhere on $\{|z-p| \leq \epsilon_0\} \cap \mathcal{E}_W$. There exists a constant $C > 0$ such that $|(\partial v/\partial z_1)(z)| < C$ for all $z \in B(p, \epsilon_0)$. By the maximum principle, $0 < |(\partial v/\partial z_1)(z)|(\varphi^N(z)) < C/2^N$ for all integer N and for all $z \in W \cap B(p, \epsilon)$. Letting $N \to \infty$, we conclude that $(\partial v/\partial z_1)(z^1) = 0$, which is a contradiction. Then there exists an open dense subset $\mathcal{O}_W$ of $\mathcal{E}_W$ such that for any $q \in \mathcal{O}_W$, $(\partial v/\partial z_1)(q) \neq 0$. For a fixed $q \in \mathcal{O}_W$, there exists a neighborhood U of q in $\mathbb{C}^n$ such that $(\partial v/\partial z_1)$ vanishes nowhere on U. Then $\tilde{W} = \{z \in U : v(z) = 0\}$ is a polynomial submanifold of U. Since W

extends across the boundary of D as a variety, a useful consequence of this fact is that $\tilde{W}$ has dimension $2n-3$. Otherwise, the Hausdorff dimension of $\tilde{W}$ will be less than or equal to $2n-4$. Then $\tilde{W}\backslash(\tilde{W}\cap\partial D)$ will be connected (see [7]). This implies that $\tilde{W}$ cannot be separated by ∂D and contradicts the first statement of this lemma.

1) -ii) Since f is algebraic, its components f^j, $j=1,\cdots,n$ are also algebraic. Then there exist polynomials

$$\mathcal{P}_j(z,w_j)=a_j^{m_j}(z){w_j}^{m_j}+\cdots+a_j^1(z)w_j+a_j^0(z),\quad j=1,\cdots,n$$

where $a_j^{k_j}(.)$ are holomorphic polynomials for all $k_j\in\{0,\cdots,m_j\}$ such that

$$\mathcal{P}_j(z,f^j(z))=0,\ \forall z\in D \text{ and } j=1,\cdots,n. \tag{5.1}$$

Consider (5.1) only for $z\in W$. First, note that we may assume that for all $j=1,\cdots,n$, there exists $k_j\in\{1,\cdots,m_j\}$ such that $a_j^{k_j}\not\equiv 0$ on W. Without loss of generality, assume that $k_j=m_j$ for all j. If $p\in S^\infty\cap\mathcal{E}_W$ (recall that $S^\infty=\partial D\setminus S^h$), then there exists $j\in\{1,\cdots,n\}$ such that $a_j^{m_j}(p)=0$. So the polynomial function $\tilde{a}=\prod_{1\le j\le n}a_j^{m_j}$ vanishes identically on $S^\infty\cap\mathcal{E}_W$. To Show that $S^h\cap\mathcal{O}_{\mathcal{W}}$ is dense in $\mathcal{O}_{\mathcal{W}}$, suppose by contradiction that $S^\infty\cap\mathcal{O}_{\mathcal{W}}$ has an interior point. Then by the boundary uniqueness theorem the polynomial $\tilde{a}$ vanishes identically on W. As h is irreducible, h divides $a_j^{m_j}$ for some j, contradicting the fact that $a_j^{m_j}\not\equiv 0$ on W for all j.
This contradiction shows that $O_W=S^h\cap\mathcal{O}_{\mathcal{W}}$ has the properties claimed by the lemma.

2) Now, let prove that $\overline{W}$ does not intersect the set of strictly pseudoconvex boundary points. Let $p\in O_W$. As in the proof of Lemma 7, we have

$$\Lambda_{\mathcal{Q}\circ f}(p)=|J_f(p)|^2\Lambda_{\mathcal{Q}}(f(p)).$$

So $O_W\subset w(\partial D)$, which implies that $\mathcal{E}_W\subset w(\partial D)$. ∎

Conclusion of the proof of Theorem 2-(A). The conclusion is similar to [3] and [9]. For completeness we add it here. The algebraic set A_2 contains finitely many components, which we will denote as $\sigma_1,\cdots,\sigma_N$. Since $\dim_{\mathbb{R}}\sigma_j=\dim_{\mathbb{R}}T^c\sigma_j$, then σ_j is a complex-manifold of dimension $n-1$. Let $\hat{A}_j$ be the complex-algebraic set in $\mathbb{C}^n$ such that $\mathrm{Reg}(\hat{A}_j)=\sigma_j$. By considering dimension and CR dimension, we see that $A_3\cap O_W$ and $A_4\cap O_W$ are nowhere dense in O_W. Next we claim that $A_1\cap O_W$ cannot contain an open subset of O_W. By contradiction, let suppose $p\in O_W\subset A_1$. We may choose a sequence $\{q_k\}_k\subset A_1\cap\{J_f\neq 0\}$ such that $q_k\to p$. The mapping f is a local diffeomorphism in a neighborhood of all points q_k and the function τ is constant on every connected component of A_1, then for all k,

$$\tau(p)=\tau(q_k)=\tau(f(q_k)).$$

On the other hand, by Lemma 7,

$$\tau(p)>\tau(f(p)).$$

This is a contradiction; since τ is upper-semicontinuous. We conclude that $A_2 \cap O_W$ contains an open subset of A_2. Thus, it contains an open subset of σ_j for some j. Applying the maximum principle, we conclude that $W \subset \hat{A}_j$. This completes the proof of Theorem 2-(A).

(B) In this case, suppose that $\mathcal{Q}$ is plurisubharmonic on D' and for all $c < 0$, $\partial D'_c$ is nondegenerate. The only crucial point here is to show that the branch locus of f extends across the boundary of D and the rest of the proof is as in the case (A). The set W denotes always an irreducible component of V_f. The set $f(W)$ is an irreducible algebraic set of dimension $n-1$ in D', then there exists an irreducible polynomial $\hat{h}$ in $\mathbb{C}^n$ such that $f(W) = \{z' \in D' : \hat{h}(z') = 0\}$. If W does not extend across ∂D, then $\mathcal{Q}(z') \leq 0$ for all $z' \in W' = \{z' \in \mathbb{C}^n : \hat{h}(z') = 0\}$. By repeating the argument used in the proof of Lemma 8 (first part), we may show that there exists a negative constant c' such that $\{\mathcal{Q} = c'\}$ contains a complex-analytic set with positive dimension. This contradicts the nondegeneracy of $\partial D'_{c'}$. ∎

Proof of Theorem 2-bis. Let $f : D \to D'$ be a proper holomorphic mapping as in Theorem 2-bis. Then f is algebraic (the proof of this fact can be deduced easily from the proof of Corollary 1 if D is bounded and from Remark 1 and the proof of Corollary 1 if D is unbounded). It suffices to show that the branch locus extends across the boundary of the domain and the rest of the proof is as in the proof of Theorem 2-(A). According to Remark 2, D is biholomorphic to a bounded pseudoconvex smooth algebraic domain Ω in $\mathbb{C}^n$. (Recall that the boundary of any bounded real-analytic domain in $\mathbb{C}^n$ is nondegenerate). Let $g : D \to \Omega$ be such a biholomorphism. Note that g is an algebraic mapping. Let $G = f \circ g^{-1} : \Omega \to D'$, W be an irreducible component of V_f and $\mathcal{W} = g(W)$. According to [13], Ω has a bounded negative plurisubharmonic exhaustion function ρ (i.e., a continuous real negative plurisubharmonic function on Ω such that $\{z \in \Omega : \rho(z) < c\}$ is a compact subset of Ω for each constant $c < 0$ and $\partial\Omega = \{\rho = 0\}$). It suffices to prove that $\mathcal{W}$ extends across the boundary of Ω. Let us assume that $\rho(z) \leq 0$ for all $z \in \mathcal{W}$ and argue by contradiction. Since $\mathcal{W}$ is an algebraic set of dimension $n-1$, then again by repeating the argument used in the proof of Lemma 8 (first part) we may show that there exists a negative constant c such that $\{z \in \Omega : \rho(z) = c\}$ contains a complex-analytic set with positive dimension : a contradiction; since $\{z \in \Omega : \rho(z) = c\}$ is a compact. ∎

5.2 Proof of Corollary 3

In view of the simple connectedness of D, it suffices to prove that V_f is empty. Since $cl_f(\partial D) \cap \partial D \neq \emptyset$, by repeating the arguments used in the proof of Corollary 1, we may show that f is algebraic and S^∞ has no interior point in ∂D. Let $z_0 \in S^h \cap \{J_f \neq 0\}$. There exists a neighborhood U of z_0 such that f is a diffeomorphism from $U \cap \partial D$ onto $f(U \cap \partial D)$. Since S^∞ has no interior point and $\partial D = S^h \cup S^\infty$, there exists $\tilde{z}_0 \in U \cap \partial D$ such that $f(\tilde{z}_0) \in S^h$. This proves that $cl_{f^2}(\partial D) \cap \partial D \neq \emptyset$. The same argument shows that $cl_{f^N}(\partial D) \cap \partial D \neq \emptyset$ for all N, where f^N denotes the N-th iteration of f. This leads to prove that f^N is algebraic for all N (see the proof of Corollary 1). Now, the proof of Theorem 2-(A) shows that for all N the variety V_{f^N} has a finite number of irreducible components independent of f^N. Then there exists an integer s such that $V_{f^s} = V_{f^{s+1}}$. We may assume $s = 1$, that is $V_f = V_{f^2}$. Since $V_{f^2} = V_f \cup f^{-1}(V_f)$, it follows that $V_f \subseteq f(V_f)$, where $f(V_f)$ is a complex-analytic variety of D by a theorem of Remmert (see [7]). Hence, we have $V_f = f(V_f)$ because V_f has finitely many components. Assume that V_f is not empty. According to Lemma 8, there exists a boundary point $p \in \overline{V}_f \cap \partial D$ such that f extends holomorphically in a neighborhood of p. Note that for all N, $f^N(p) \in \overline{V}_f$; since $V_f = f(V_f)$. The sequence of numbers $\tau(f^N(p))$ is strictly decreasing and $\tau(p)$ is a finite integer, so there exists an integer N_0 such that $\tau(f^{N_0}(p)) = 0$, which implies that $f^{N_0}(p)$ is a strictly pseudoconvex boundary point, contradicting the fact that $f^{N_0}(p) \in \overline{V}_f \cap \partial D$. This proves that $V_f = \emptyset$ and completes the proof. ■

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