On the Existence of Almost Automorphic Solutions to Some Abstract Hyperbolic Differential Equations

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Abstract

In this paper we give some sufficient conditions, which do ensure the existence and uniqueness of a compact almost automorphic solution to some hyperbolic differential equations. As an illustration, we study the existence and uniqueness of almost automorphic solutions to a one-dimensional heat equation with small delays.

1 Introduction

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space and let $\mathbb{X}_\alpha$, for $\alpha \in (0,1)$, be an abstract intermediate Banach space between $D(A)$, the domain of a linear operator A defined on $\mathbb{X}$, and $\mathbb{X}$. Examples of those $\mathbb{X}_\alpha$ include, among others, the fractional spaces $D((-A)^\alpha)$, the reel interpolation spaces $D_A(\alpha, \infty)$ due to both Lions and Peetre, and the Hölder spaces $D_A(\alpha)$, which coincide with the continuous interpolation spaces that had been introduced in the literature by Da Prato and Grisvard.

In [3] some sufficient conditions for the existence and uniqueness of an almost automorphic solution to the differential equation

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

Received by the editors September 2008.
Communicated by J. Mawhin.
2000 *Mathematics Subject Classification* : 44A35; 42A85; 42A75.
Key words and phrases : Sectorial operator; analytic semigroup; hyperbolic semigroup; almost automorphic function; heat equation with delay.

where A is a sectorial operator whose corresponding analytic semigroup $(T(t))_{t\geq 0}$ is hyperbolic, equivalently, $\sigma(A) \cap i\mathbb{R} = \emptyset$, where $\sigma(A)$ denotes the spectrum of the linear operator A.

In this paper we extend the previous-mentioned result to the more general class of differential equations given by

$$\frac{d}{dt}\left[u(t) + f(t, Bu(t))\right] = Au(t) + g(t, Cu(t)), \quad t \in \mathbb{R} \tag{1.2}$$

where $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is a sectorial operator whose corresponding analytic semigroup $(T(t))_{t\geq 0}$ is hyperbolic, i.e. $\sigma(A) \cap i\mathbb{R} = \{\emptyset\}$, the operator B, C are arbitrary densely defined closed linear operators on $\mathbb{X}$, and f, g are some jointly continuous functions satisfying some additional assumptions (Theorem 3.5). Applications include the study of almost automorphic solutions to as one-dimensional heat equation with small delays.

In this paper, as in [3, 12, 14], we consider a general intermediate space $\mathbb{X}_\alpha$ between $D(A)$ and $\mathbb{X}$. In contrast with the fractional power spaces considered in some recent papers of the author et al. [15, 16], the interpolation and Hölder spaces, for instance, depend only on $D(A)$ and $\mathbb{X}$ and can be explicitly expressed in many concrete cases. The literature related to those intermediate spaces is very extensive, in particular, we refer the reader to the excellent book by A. Lunardi [27], which contains a comprehensive presentation on this topic and related issues.

The existence of almost periodic, asymptotically almost periodic, pseudo almost periodic, and almost automorphic solutions is one of the most attractive topics in qualitative theory of differential equations due to their significance and applications in physics, mathematical biology, control theory, physics and others.

The concept of almost automorphy, which is the central issue in this paper, was first initiated by Bochner in his landmark paper [2]. Since then the theory of almost automorphic functions has found several developments and applications in the theory of abstract differential equations, partial differential equations, functional-differential equations, and integro-differential equations. For more on these and related issues, we refer the reader to [7], [8], [9], [10], [11], [28], [29], [32], [33], and [34] and the references therein. Note that (1.2) includes delay cases and related topics. Thus it is more convenient to consider the so-called compact almost automorphy [7, 25] rather than the classical almost automorphy [2, 28, 32].

Some recent contributions on almost automorphic and asymptotically almost automorphic solutions to differential and partial differential equations have recently been made in [3, 7, 8, 9, 10, 28, 29]. However, the existence and uniqueness of almost automorphic solutions to (1.2) in the case when A is sectorial is an important topic with some interesting applications, which is still an untreated question, is the main motivation of the present paper. Among other things, we will make extensive use of the method of analytic semigroups associated with sectorial operators and the Banach's fixed-point principle to derive sufficient conditions for the existence and uniqueness of an almost automorphic (mild) solution to (1.2).

2 Preliminaries

This section is devoted to some preliminary facts needed in the sequel. Throughout the rest of this paper, $(\mathbb{X}, \|\cdot\|)$ stands for a Banach space, A is a sectorial linear operator (see Definition 2.2), which is not necessarily densely defined, and B, C are (possibly unbounded) linear operators such that $A+B+C$ is not trivial, as each solution to (1.2) belongs to $D(A+B+C) = D(A) \cap D(B) \cap D(C)$. Now if A is a linear operator on $\mathbb{X}$, then $\rho(A)$, $\sigma(A)$, $D(A)$, $N(A)$, $R(A)$ stand for the resolvent, spectrum, domain, kernel, and range of A. The space $B(\mathbb{X}, \mathbb{Y})$ denotes the Banach space of all bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$ equipped with its natural norm with $B(\mathbb{X}, \mathbb{X}) = B(\mathbb{X})$.

To deal with almost automorphic solutions we will need to introduce some classical notions. Throughout the rest of the paper, $(\mathbb{Z}, \|\cdot\|_{\mathbb{Z}})$, $(\mathbb{W}, \|\cdot\|_{\mathbb{W}})$ stand for abstract Banach spaces. In addition to that $C(\mathbb{R}, \mathbb{Z})$ and $BC(\mathbb{R}, \mathbb{Z})$ denote respectively the collection of continuous functions and the collection of bounded continuous functions from $\mathbb{R}$ into $\mathbb{Z}$ equipped with the sup norm defined by

$$\|u\|_\infty := \sup_{t \in \mathbb{R}} \|u(t)\|_{\mathbb{Z}}.$$

Similar definitions apply for both $C(\mathbb{R} \times \mathbb{Z}, \mathbb{W})$ and $BC(\mathbb{R} \times \mathbb{Z}, \mathbb{W})$.

2.1 Almost Automorphy

Definition 2.1. A strongly continuous function $\varphi : \mathbb{R} \mapsto \mathbb{X}$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n \in \mathbb{N}}$, there exists a subsequence $(s_n)_{n \in \mathbb{N}}$ of $(s'_n)_{n \in \mathbb{N}}$ such that

$$\psi(t) := \lim_{n \mapsto \infty} \varphi(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\varphi(t) = \lim_{n \mapsto \infty} \psi(t - s_n)$$

for each $t \in \mathbb{R}$.

The range of an almost automorphic function is relatively compact on $\mathbb{X}$ and hence is bounded. We denote the space of almost automorphic functions $\varphi : \mathbb{R} \mapsto \mathbb{X}$ by $AA(\mathbb{X})$. It is well-known that $(AA(\mathbb{X}), \|\cdot\|_\infty)$ is a Banach space, see, e.g., [28].

Among other things, almost automorphic functions satisfy the following properties.

Theorem 2.2. ([28, Theorem 2.1.3]) *If $\varphi, \varphi_1, \varphi_2 \in AA(\mathbb{X})$, then*

(i) $\varphi_1 + \varphi_2 \in AA(\mathbb{X})$,

(ii) $\lambda\varphi \in AA(\mathbb{X})$ *for any scalar* λ,

(iii) $\varphi_\alpha \in AA(\mathbb{X})$ *where* $\varphi_\alpha : \mathbb{R} \to \mathbb{X}$ *is defined by* $\varphi_\alpha(\cdot) = \varphi(\cdot + \alpha)$,

(iv) *the range* $\mathcal{R}_\varphi := \{\varphi(t) : t \in \mathbb{R}\}$ *is relatively compact in* $\mathbb{X}$*, thus* φ *is bounded in norm,*

(v) *if* $\varphi_n \to \varphi$ *uniformly on* $\mathbb{R}$ *where each* $\varphi_n \in AA(\mathbb{X})$*, then* $\varphi \in AA(\mathbb{X})$ *too.*

(vi) *if* $\psi \in L^1(\mathbb{R})$*, then* $\varphi * \psi \in AA(\mathbb{R})$*, where* $\varphi * \psi$ *is the convolution of* φ *with* ψ *on* $\mathbb{R}$*.*

Remark 2.3. The function ψ in the Definition 2.1 above is measurable, but not necessarily continuous. Moreover, if ψ is continuous, then φ is uniformly continuous, see details in [28, Theorem 2.6].

Example 2.4. A classical example of an almost automorphic function, which is not almost periodic is the function defined by

$$\varphi(t) = \cos\left(\frac{1}{2 + \sin\sqrt{2}t + \sin t}\right), \quad t \in \mathbb{R}.$$

It can be shown that φ is not uniformly continuous, and hence is not almost periodic.

Let $l^\infty(\mathbb{X})$ denote the space of all bounded (two-sided) sequence in $\mathbb{X}$. It is equipped with its corresponding sup norm defined for each sequence $x = (x_n)_{n\in\mathbb{Z}} \in l^\infty(\mathbb{X})$ by: $\|x\|_\infty := \sup_{n\in\mathbb{Z}} \|x_n\|$.

Definition 2.5. A sequence $x = (x_n)_{n\in\mathbb{Z}} \in l^\infty(\mathbb{X})$ is said to be almost automorphic if for every sequence of integers (k'_n), there exists a subsequence (k_n) such that

$$y_p := \lim_{n\to\infty} x_{p+k_n}$$

is well defined for each $p \in \mathbb{Z}$, and

$$\lim_{n\to\infty} y_{p-k_n} = x_p$$

for each $p \in \mathbb{Z}$.

The collection of all these almost automorphic sequences is denoted by $aa(\mathbb{X})$.

Definition 2.6. A continuous function $\Phi : \mathbb{R} \times \mathbb{Z} \mapsto \mathbb{W}$ is said to be almost automorphic in $t \in \mathbb{R}$ for each $z \in \mathbb{Z}$ if for every sequence of real numbers $(\sigma_n)_\mathbb{N}$ there exists a subsequence $(s_n)_\mathbb{N}$ of $(\sigma_n)_\mathbb{N}$ such that

$$\Psi(t,z) := \lim_{n\mapsto\infty} \Phi(t+s_n, z) \text{ in } \mathbb{W}$$

is well defined for each $t \in \mathbb{R}$ and each $z \in \mathbb{Z}$ and

$$\Phi(t,u) = \lim_{n\mapsto\infty} \Psi(t-s_n, z) \text{ in } \mathbb{W}$$

for each $t \in \mathbb{R}$ and for every $z \in \mathbb{Z}$.

The collection of such functions will be denoted by $AA(\mathbb{Z}, \mathbb{W})$.

Using [28, Theorem 2.2.6, p. 22] one easily obtains the following.

Theorem 2.7. *Let $F : \mathbb{R} \times \mathbb{Z} \mapsto \mathbb{W}$ be an almost automorphic function in $t \in \mathbb{R}$ for each $z \in \mathbb{Z}$ and assume that F satisfies a Lipschitz condition in z uniformly in $t \in \mathbb{R}$. Let $\phi : \mathbb{R} \mapsto \mathbb{Z}$ be almost automorphic. Then the function $\Phi : \mathbb{R} \mapsto \mathbb{W}$ defined by $\Phi(t) := F(t, \phi(t))$ is almost automorphic.*

In addition to the above-mentioned notions, the present setting requires the introduction of the concept of compact almost automorphy, see, e.g., [25].

Definition 2.8. A continuous function $F : \mathbb{R} \mapsto \mathbb{Z}$ is said to be compact almost automorphic if for every sequence of real numbers $(\sigma_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (\sigma_n)_{n\in\mathbb{N}}$ such that $G(t) := \lim_{n\to\infty} F(t + s_n)$, and $F(t) = \lim_{n\to\infty} G(t - s_n)$ uniformly on compact subsets of $\mathbb{R}$. The collection of those functions will be denoted by $AA_c(\mathbb{Z})$.

Let $AP(\mathbb{Z})$ denote the space of almost periodic functions $f : \mathbb{R} \mapsto \mathbb{Z}$. It is well-known that $AP(\mathbb{Z}), AA_c(\mathbb{Z})$, and $AA(\mathbb{Z})$ are closed subsets of $(BC(\mathbb{R}, \mathbb{Z}), \|\cdot\|_\infty)$ with

$$AP(\mathbb{Z}) \subset AA_c(\mathbb{Z}) \subset AA(\mathbb{Z}) \subset BC(\mathbb{R}, \mathbb{Z}).$$

In view of the above, the proof of the next lemma is straightforward.

Lemma 2.9. *The space $AA_c(\mathbb{Z})$ endowed with the sup norm is a Banach space.*

Definition 2.10. A continuous function $F : \mathbb{R} \times \mathbb{Z} \mapsto \mathbb{W}$, $(t, u) \mapsto F(t, u)$ is said to be compact almost automorphic in $t \in \mathbb{R}$, if for every sequence of real numbers $(\sigma_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{\mathbb{N}} \subset (\sigma_n)_{\mathbb{N}}$ such that

$$G(t, z) := \lim_{n\mapsto\infty} F(t + s_n, z), \text{ and } F(t, z) = \lim_{n\mapsto\infty} G(t - s_n, z) \text{ in } \mathbb{W},$$

where the limits are uniform on compact subset of $\mathbb{R}$, for each $z \in \mathbb{Z}$. The space of such functions will be denoted by $AA_c(\mathbb{Z}, \mathbb{W})$.

We have the following composition result.

Theorem 2.11. *[11] Let $F \in AA_c(\mathbb{Z}, \mathbb{W})$ and let $\varphi \in AA_c(\mathbb{Z})$. Assume that F is Lipschitz, that is, there exists $L > 0$ such that*

$$\|F(t, x) - F(t, y)\|_{\mathbb{W}} \leq L\, \|x - y\|_{\mathbb{Z}}, \ \forall t \in \mathbb{R}, \forall x, y \in \mathbb{Z}. \tag{2.1}$$

Then the $\mathbb{W}$-valued function G defined by $G(t) := F(t, \varphi(t))$ is in $AA_c(\mathbb{W})$.

Proof. Let $(s'_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Now, fix a subsequence $(s_n)_{n\in\mathbb{N}}$ of $(s'_n)_{n\in\mathbb{N}}$, $G \in BC(\mathbb{R} \times \mathbb{Z}; \mathbb{W})$ and $\psi \in BC(\mathbb{R}, \mathbb{Z})$ so that the pair $G, (s_n)_{n\in\mathbb{N}}$ is associated with F as in Definition 2.10 and the pair ψ, $(s_n)_{n\in\mathbb{N}}$ is associated with φ as in Definition 2.8. Let $K \subset \mathbb{R}$ be an arbitrary compact and let $\varepsilon > 0$. Since $\mathcal{R}(\psi) = \{\psi(t) : t \in \mathbb{R}\}$ is relatively compact, there exist points $x_i \in \mathbb{Z}, i = 1, ...n$, such that for each $t \in \mathbb{R}$ one can find $i(t) \in \{1, ...n\}$ with

$$\|\psi(t) - x_{i(t)}\|_{\mathbb{Z}} \leq \varepsilon.$$

Let N_ε be a natural number such that $\|F(s+s_n, x_i) - G(s, x_i)\|_{\mathbb{W}} \leq \varepsilon, \forall s \in K$ and for all $i = 1, ...n$, whenever $n \geq N_\varepsilon$. In view of the above, for each $s \in K$, and $n \geq N_\varepsilon$,

$$\begin{aligned}
&\|F(t+s_n, \varphi(t+s_n)) - G(t, \psi(t))\|_{\mathbb{W}} \\
&\leq \ \|F(t+s_n, \varphi(t+s_n)) - F(t+s_n, \psi(t))\|_{\mathbb{W}} \\
&+ \ \|F(t+s_n, \psi(t)) - F(t+s_n, x_{i(t)})\|_{\mathbb{W}} \\
&\quad +\|F(t+s_n, x_{i(t)}) - G(t, x_{i(t)})\|_{\mathbb{W}} + \|G(t, x_{i(t)}) - G(t, \psi(t))\|_{\mathbb{W}} \\
&\leq \ L\|\varphi(t+s_n) - \varphi(t)\|_{\mathbb{Z}} + L\|\psi(t) - x_{i(t)}\|_{\mathbb{Z}} + \varepsilon \\
&+ \ L\|x_{i(t)} - \psi(t)\|_{\mathbb{Z}}
\end{aligned}$$

which proves that the convergence is uniform on K.

Arguing as previously it follows that $G(t-s_n, \varphi(t-s_n)) - F(t, \varphi(t))$ converges uniformly to 0 on compact sets of $\mathbb{R}$. This completes the proof. ∎

2.2 Sectorial Linear Operators and their Associated Semigroups

Definition 2.12. A linear operator $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ (not necessarily densely defined) is said to be sectorial if the following hold: there exist constants $\omega \in \mathbb{R}$, $\theta \in (\frac{\pi}{2}, \pi)$, and $M > 0$ such that

$$\rho(A) \supset S_{\theta,\omega} := \{\lambda \in \mathbb{C} : \lambda \neq \omega, \ |\arg(\lambda - \omega)| < \theta\}, \text{ and} \qquad (2.2)$$

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda - \omega|}, \quad \lambda \in S_{\theta,\omega}. \qquad (2.3)$$

The class of sectorial operators is very rich and contains most of classical operators encountered in the literature. Two examples of sectorial operators are given as follows:

Example 2.13. Let $p \geq 1$ and let $\mathbb{X} = L^p(\mathbb{R})$ be the Lebesgue space equipped with its norm $\|\cdot\|_p$ defined by

$$\|\varphi\|_p = \left(\int_{\mathbb{R}} |\varphi(x)|^p dx\right)^{1/p}.$$

Define the linear operator A on $L^p(\mathbb{R})$ by

$$D(A) = W^{2,p}(\mathbb{R}), \ \ A(\varphi) = \varphi'', \ \ \forall \varphi \in D(A).$$

It can be checked that the operator A is sectorial on $L^p(\mathbb{R})$.

Example 2.14. Let $p \geq 1$ and let $\Omega \subset \mathbb{R}^d$ be open bounded subset with C^2 boundary $\partial\Omega$. Let $\mathbb{X} := L^p(\Omega)$ be the Lebesgue space equipped with the norm, $\|\cdot\|_p$ defined by,

$$\|\varphi\|_p = \left(\int_{\Omega} |\varphi(x)|^p dx\right)^{1/p}.$$

Define the operator A as follows:

$$D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \quad A(\varphi) = \Delta\varphi, \quad \forall \varphi \in D(A),$$

where $\Delta = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$ is the Laplace operator.

It can be checked that the operator A is sectorial on $L^p(\Omega)$.

It is well-known that [27] if A is sectorial, then it generates an analytic semigroup $(T(t))_{t\geq 0}$, which maps $(0,\infty)$ into $B(\mathbb{X})$ and such that there exist $M_0, M_1 > 0$ with

$$\|T(t)\| \leq M_0 e^{\omega t}, \quad t > 0, \tag{2.4}$$

$$\|t(A-\omega)T(t)\| \leq M_1 e^{\omega t}, \quad t > 0. \tag{2.5}$$

Throughout the rest of the paper, we suppose that the semigroup $(T(t))_{t\geq 0}$ is hyperbolic, that is, there exist a projection P and constants $M, \delta > 0$ such that $T(t)$ commutes with P, $N(P)$ is invariant with respect to $T(t)$, $T(t) : R(Q) \mapsto R(Q)$ is invertible, and the following hold

$$\|T(t)Px\| \leq Me^{-\delta t}\|x\| \qquad \text{for } t \geq 0, \tag{2.6}$$

$$\|T(t)Qx\| \leq Me^{\delta t}\|x\| \qquad \text{for } t \leq 0, \tag{2.7}$$

where $Q := I - P$ and, for $t \leq 0$, $T(t) := (T(-t))^{-1}$.

Recall that the analytic semigroup $(T(t))_{t\geq 0}$ associated with A is hyperbolic if and only if

$$\sigma(A) \cap i\mathbb{R} = \emptyset,$$

see, e.g., [19, Prop. 1.15, pp.305].

Definition 2.15. Let $\alpha \in (0,1)$. A Banach space $(\mathbb{X}_\alpha, \|\cdot\|_\alpha)$ is said to be an intermediate space between $D(A)$ and $\mathbb{X}$, or a space of class $\mathcal{J}_\alpha$, if $D(A) \subset \mathbb{X}_\alpha \subset \mathbb{X}$ and there is a constant $c > 0$ such that

$$\|x\|_\alpha \leq c\|x\|^{1-\alpha}\|x\|_A^\alpha, \qquad x \in D(A), \tag{2.8}$$

where $\|\cdot\|_A$ is the graph norm of A.

Concrete examples of $\mathbb{X}_\alpha$ include $D((-A^\alpha))$ for $\alpha \in (0,1)$, the domains of the fractional powers of A, the real interpolation spaces $D_A(\alpha,\infty)$, $\alpha \in (0,1)$, defined as follows

$$\begin{cases} D_A(\alpha,\infty) := \{x \in \mathbb{X} : [x]_\alpha = \sup_{0<t\leq 1} \|t^{1-\alpha}AT(t)x\| < \infty\} \\ \|x\|_\alpha = \|x\| + [x]_\alpha, \end{cases}$$

the abstract Hölder spaces $D_A(\alpha) := \overline{D(A)}^{\|.\|_\alpha}$ as well as the complex interpolation spaces $[\mathbb{X}, D(A)]_\alpha$, see A. Lunardi [27] for details.

For a hyperbolic analytic semigroup $(T(t))_{t\geq 0}$, one can easily check that similar estimations as both (2.6) and (2.7) still hold with norms $\|\cdot\|_\alpha$. In fact, as the part of A in $R(Q)$ is bounded, it follows from (2.7) that

$$\|AT(t)Qx\| \leq C' e^{\delta t}\|x\| \qquad \text{for } t \leq 0.$$

Hence, from (2.8) there exists a constant $c(\alpha) > 0$ such that

$$\|T(t)Qx\|_\alpha \leq c(\alpha)e^{\delta t}\|x\| \qquad \text{for } t \leq 0. \tag{2.9}$$

In addition to the above, the following holds

$$\|T(t)Px\|_\alpha \leq \|T(1)\|_{B(\mathbb{X},\mathbb{X}_\alpha)}\|T(t-1)Px\| \qquad \text{for } t \geq 1,$$

and hence from (2.6), one obtains

$$\|T(t)Px\|_\alpha \leq M' e^{-\delta t}\|x\|, \qquad t \geq 1,$$

where M' depends on α. For $t \in (0,1]$, by (2.5) and (2.8)

$$\|T(t)Px\|_\alpha \leq M'' t^{-\alpha}\|x\|.$$

Hence, there exist constants $M(\alpha) > 0$ and $\gamma > 0$ such that

$$\|T(t)Px\|_\alpha \leq M(\alpha)t^{-\alpha}e^{-\gamma t}\|x\| \qquad \text{for } t > 0. \tag{2.10}$$

3 Main results

To study the existence and uniqueness of pseudo almost periodic solutions to (1.2) we need to introduce the notion of mild solution to it.

Definition 3.1. Let $\alpha \in (0,1)$. A bounded continuous function $u : \mathbb{R} \mapsto \mathbb{X}_\alpha$ is said to be a mild solution to (1.2) provided that the function $s \to AT(t-s)Pf(s, Bu(s))$ is integrable on $(-\infty, t)$, $s \to AT(t-s)Qf(s, Bu(s))$ is integrable on (t, ∞) for each $t \in \mathbb{R}$, and

$$\begin{aligned} u(t) &= -f(t, Bu(t)) - \int_{-\infty}^{t} AT(t-s)Pf(s, Bu(s))ds \\ &+ \int_{t}^{\infty} AT(t-s)Qf(s, Bu(s))ds + \int_{-\infty}^{t} T(t-s)Pg(s, Cu(s))ds \\ &- \int_{t}^{\infty} T(t-s)Qg(s, Cu(s))ds \end{aligned}$$

for each $\forall t \in \mathbb{R}$.

Throughout the rest of the paper we denote by $\Gamma_1, \Gamma_2, \Gamma_3$, and Γ_4, the nonlinear integral operators defined by

$$(\Gamma_1 u)(t) := \int_{-\infty}^{t} AT(t-s)Pf(s, Bu(s))ds, \;\; (\Gamma_2 u)(t) := \int_{t}^{\infty} AT(t-s)Qf(s, Bu(s))ds,$$

$$(\Gamma_3 u)(t) := \int_{-\infty}^{t} T(t-s)Pg(s, Cu(s))ds, \;\; \text{and}$$

$$(\Gamma_4 u)(t) := \int_{t}^{\infty} T(t-s)Qg(s, Cu(s))ds.$$

To study (1.2) we require the following assumptions:

(H1) The operator A is sectorial and generates a hyperbolic (analytic) semigroup $(T(t))_{t\geq 0}$.

(H2) Let $0 < \alpha < 1$. Then $\mathbb{X}_\alpha = D((-A^\alpha))$, or $\mathbb{X}_\alpha = D_A(\alpha, p), 1 \leq p \leq +\infty$, or $\mathbb{X}_\alpha = D_A(\alpha)$, or $\mathbb{X}_\alpha = [\mathbb{X}, D(A)]_\alpha$. We also assume that $B, C : \mathbb{X}_\alpha \longrightarrow \mathbb{X}$ are bounded linear operators.

(H3) Let $0 < \alpha < \beta < 1$, and $f : \mathbb{R} \times \mathbb{X} \longrightarrow \mathbb{X}_\beta$ be a compact almost automorphic function in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{X}$, $g : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ be compact almost automorphic in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{X}$.

(H4) The functions f, g are uniformly Lipschitz with respect to the second argument in the following sense: there exists $K > 0$ such that

$$\|f(t,u) - f(t,v)\|_\beta \leq K\|u - v\|,$$

and

$$\|g(t,u) - g(t,v)\| \leq K\|u - v\|$$

for all $u, v \in \mathbb{X}$ and $t \in \mathbb{R}$.

In order to show that Γ_1 and Γ_2 are well defined, we need the following estimates.

Lemma 3.2. *Let* $0 < \alpha, \beta < 1$. *Then*

$$\|AT(t)Qx\|_\alpha \leq ce^{\delta t}\|x\|_\beta \qquad \text{for } t \leq 0, \tag{3.1}$$

$$\|AT(t)Px\|_\alpha \leq ct^{\beta-\alpha-1}e^{-\gamma t}\|x\|_\beta, \qquad \text{for } t > 0. \tag{3.2}$$

Proof. As for (2.9), the fact that the part of A in $R(Q)$ is bounded yields

$$\|AT(t)Qx\| \leq ce^{\delta t}\|x\|_\beta, \quad \|A^2T(t)Qx\| \leq ce^{\delta t}\|x\|_\beta, \qquad \text{for } t \leq 0,$$

since $\mathbb{X}_\beta \hookrightarrow \mathbb{X}$. Hence, from (2.8) there is a constant $c(\alpha) > 0$ such that

$$\|AT(t)Qx\|_\alpha \leq c(\alpha)e^{\delta t}\|x\|_\beta \qquad \text{for } t \leq 0.$$

Furthermore,

$$\begin{aligned}\|AT(t)Px\|_\alpha &\leq \|AT(1)\|_{B(\mathbb{X},\mathbb{X}_\alpha)}\|T(t-1)Px\| \\ &\leq ce^{-\delta t}\|x\|_\beta, \qquad \text{for } t \geq 1.\end{aligned}$$

Now for $t \in (0,1]$, by (2.5) and (2.8), one has

$$\|AT(t)Px\|_\alpha \leq ct^{-\alpha-1}\|x\|,$$

and

$$\|AT(t)Px\|_\alpha \leq ct^{-\alpha}\|Ax\|,$$

for each $x \in D(A)$. Thus, by reiteration Theorem (see [27]), it follows that

$$\|AT(t)Px\|_\alpha \leq ct^{\beta-\alpha-1}\|x\|_\beta$$

for every $x \in \mathbb{X}_\beta$ and $0 < \beta < 1$, and hence, there exist constants $M(\alpha) > 0$ and $\gamma > 0$ such that

$$\|T(t)Px\|_\alpha \leq M(\alpha)t^{\beta-\alpha-1}e^{-\gamma t}\|x\|_\beta \qquad \text{for } t > 0. \qquad \blacksquare$$

Lemma 3.3. *Under assumptions* (**H1**)-(**H2**)-(**H3**)-(**H4**), *the integral operators* Γ_3 *and* Γ_4 *defined above map* $AA_c(\mathbb{X}_\alpha)$ *into itself.*

Proof. Let $u \in AA_c(\mathbb{X}_\alpha)$. Since $C \in B(\mathbb{X}_\alpha, \mathbb{X})$ it follows that $Cu \in AA_c(\mathbb{X})$. Setting $h(t) = g(t, Cu(t))$ and using Theorem 2.11 it follows that $h \in AA_c(\mathbb{X})$.

For a given sequence $(\sigma_n)_{n\in\mathbb{N}}$ of real numbers, fix a subsequence $(s_n)_{n\in\mathbb{N}}$, and a continuous functions $p \in BC(\mathbb{R}; \mathbb{X})$ such that $h(t+s_n)$ converges to $p(t)$ in $\mathbb{X}$, and $p(t-s_n)$ converges to h in $\mathbb{X}$, uniformly on compact sets of $\mathbb{R}$. Using (2.10) it follows that

$$\|T(t-s)Ph(s)\|_\alpha \leq M(\alpha)(t-s)^{-\alpha}e^{-\gamma(t-s)}\|h(s)\| \tag{3.3}$$

and hence the function $s \mapsto T(t-s)Ph(s)$ is integrable over $(-\infty, t)$ for each $t \in \mathbb{R}$. Furthermore, since

$$w(t+s_n) = \int_{-\infty}^{t} T(t-s)Ph(s+s_n)ds, \quad t \in \mathbb{R},\ n \in \mathbb{N}, \tag{3.4}$$

using the estimate (3.3) and the Lebesgue Dominated Convergence Theorem, it follows that $w(t+s_n)$ converges to $z(t) = \int_{-\infty}^{t} T(t-s)p(s)ds$ for each $t \in \mathbb{R}$ in $\mathbb{X}_\alpha$.

The remaining task is to prove that the convergence is uniform on all compact sets in $\mathbb{R}$. Let $K \subset \mathbb{R}$ be an arbitrary compact and let $\varepsilon > 0$. Fix $L > 0$ and $N_\varepsilon \in \mathbb{N}$ such that $K \subset [\frac{-L}{2}, \frac{L}{2}]$ with $\int_{L/2}^{\infty} s^{-\alpha}e^{-\gamma s}ds < \varepsilon$ and

$$\|h(s+s_n) - p(s)\| \leq \varepsilon,\ \ n \geq N_\varepsilon,\ s \in [-L, L].$$

Clearly,

$$\begin{aligned}
\|w(t+s_n) - z(t)\|_\alpha &\leq \int_{-\infty}^{t} \|T(t-s)P(h(s+s_n) - p(s))\|_\alpha ds \\
&\leq M(\alpha)\int_{-\infty}^{-L}(t-s)^{-\alpha}e^{-\gamma(t-s)}\|h(s+s_n) - p(s)\|ds \\
&\quad + M(\alpha)\int_{-L}^{t}(t-s)^{-\alpha}e^{-\gamma(t-s)}\|h(s+s_n) - p(s)\|ds \\
&\leq 2M(\alpha)\|h\|_\infty \int_{t+L}^{\infty} s^{-\alpha}e^{-\gamma s}ds + M(\alpha)\varepsilon \int_0^\infty s^{-\alpha}e^{-\gamma s}ds \\
&\leq 2M(\alpha)\|h\|_\infty \int_{\frac{L}{2}}^{\infty} s^{-\alpha}e^{-\gamma s}ds + M(\alpha)\varepsilon \int_0^\infty s^{-\alpha}e^{-\gamma s}ds \\
&\leq M(\alpha)\varepsilon \left(2\|h\|_\infty + \int_0^\infty s^{-\alpha}e^{-\gamma s}ds\right),
\end{aligned}$$

which proves that the convergence is uniform on K, by the fact that the last estimate is independent of $t \in K$. Proceeding as previously, one can similarly prove that $z(t-s_n)$ converges to w uniformly on compact sets in $\mathbb{R}$. The proof for Γ_4 is similar to that of Γ_3. ∎

Lemma 3.4. *Under assumptions* (**H1**)-(**H2**)-(**H3**)-(**H4**), *the integral operators* Γ_1 *and* Γ_2 *defined above map* $AA_c(\mathbb{X}_\alpha)$ *into itself.*

Proof. One follows along the same lines as in the proof of Lemma 3.3. Let $u \in AA_c(\mathbb{X}_\alpha)$. Since $B \in B(\mathbb{X}_\alpha, \mathbb{X})$ it follows that $Bu \in AA_c(\mathbb{X})$. Setting $\tilde{h}(t) = f(t, Cu(t))$ and using Theorem 2.11 it follows that $h \in AA_c(\mathbb{X}_\beta)$.

For a given sequence $(\sigma_n)_{n\in\mathbb{N}}$ of real numbers, fix a subsequence $(s_n)_{n\in\mathbb{N}}$, and a continuous functions $q \in BC(\mathbb{R}; \mathbb{X}_\beta)$ such that $\tilde{h}(t+s_n)$ converges to $q(t)$ in $\mathbb{X}_\beta$, and $q(t-s_n)$ converges to h in $\mathbb{X}_\beta$, uniformly on compact sets of $\mathbb{R}$. Using (3.2) it follows that

$$\|AT(t-s)P\tilde{h}(s)\|_\alpha \leq c(t-s)^{\beta-\alpha-1}e^{-\gamma(t-s)}\|\tilde{h}(s)\|_\beta \tag{3.5}$$

and hence the function $s \mapsto AT(t-s)P\tilde{h}(s)$ is integrable over $(-\infty, t)$ for each $t \in \mathbb{R}$. Furthermore, since

$$\tilde{w}(t+s_n) = \int_{-\infty}^{t} AT(t-s)P\tilde{h}(s+s_n)ds, \quad t \in \mathbb{R},\ n \in \mathbb{N}, \tag{3.6}$$

using the estimate (3.5) and the Lebesgue Dominated Convergence Theorem, it follows that $\tilde{w}(t+s_n)$ converges to $\tilde{z}(t) = \int_{-\infty}^{t} AT(t-s)q(s)ds$ for each $t \in \mathbb{R}$ in the fractional space $\mathbb{X}_\alpha$.

The remaining task is to prove that the convergence is uniform on all compact sets in $\mathbb{R}$. Let $K \subset \mathbb{R}$ be an arbitrary compact and let $\varepsilon > 0$. Fix $L > 0$ and $N_\varepsilon \in \mathbb{N}$ such that $K \subset [\frac{-L}{2}, \frac{L}{2}]$ with $\int_{L/2}^{\infty} s^{\beta-\alpha-1}e^{-\gamma s} < \varepsilon$ and

$$\|\tilde{h}(s+s_n) - p(s)\|_\beta \leq \varepsilon, \ \ n \geq N_\varepsilon,\ s \in [-L, L].$$

Using the notation $\|L\|_{\infty,\beta} = \sup_{s\in\mathbb{R}} \|L(s)\|_\beta$, for each $t \in K$, one has:

$$\begin{aligned}
\|\tilde{w}(t+s_n) - z(t)\|_\alpha &\leq \int_{-\infty}^{t} \|AT(t-s)P(\tilde{h}(s+s_n) - q(s))\|_\alpha ds \\
&\leq c\int_{-\infty}^{-L} (t-s)^{\beta-\alpha-1}e^{-\gamma(t-s)}\|\tilde{h}(s+s_n) - q(s)\|_\beta ds \\
&\quad + c\int_{-L}^{t} (t-s)^{\beta-\alpha-1}e^{-\gamma(t-s)}\|\tilde{h}(s+s_n) - q(s)\|_\beta ds \\
&\leq 2c\|\tilde{h}\|_{\infty,\beta}\int_{t+L}^{\infty} s^{\beta-\alpha-1}e^{-\gamma s}ds + c\varepsilon\int_{0}^{\infty} s^{\beta-\alpha-1}e^{-\gamma s}ds \qquad (3.7) \\
&\leq 2c\|\tilde{h}\|_{\infty,\beta}\int_{\frac{L}{2}}^{\infty} s^{\beta-\alpha-1}e^{-\gamma s}ds + c\varepsilon\int_{0}^{\infty} s^{\beta-\alpha-1}e^{-\gamma s}ds \qquad (3.8) \\
&\leq c\varepsilon\left(2\|\tilde{h}\|_{\infty,\beta} + \int_{0}^{\infty} s^{\beta-\alpha-1}e^{-\gamma s}ds\right), \qquad (3.9)
\end{aligned}$$

which proves that the convergence is uniform on K, by the fact that the last estimate is independent of $t \in K$. Proceeding as previously, one can similarly prove that $\tilde{z}(t-s_n)$ converges to $\tilde{w}$ uniformly on compact sets in $\mathbb{R}$. The proof for Γ_2 is similar to that of Γ_1. ■

Throughout the rest of the paper, the constant $k(\alpha)$ denotes the bound of the embedding $\mathbb{X}_\beta \hookrightarrow \mathbb{X}_\alpha$, that is,

$$\|u\|_\alpha \leq k(\alpha)\|u\|_\beta \ \text{ for each } u \in \mathbb{X}_\beta.$$

Theorem 3.5. *Under the assumptions* **(H1)-(H2)-(H3)-(H4)**, *the evolution equation* (1.2) *has a unique almost automorphic mild solution whenever* $\Theta < 1$, *where*

$$\Theta = K\varpi\left[k(\alpha) + \frac{c}{\delta} + c\frac{\Gamma(\beta-\alpha)}{\gamma^{\beta-\alpha}} + \frac{M(\alpha)\Gamma(1-\alpha)}{\gamma^{1-\alpha}} + \frac{c(\alpha)}{\delta}\right],$$

and $\varpi = \max(\|B\|_{B(\mathbb{X}_\alpha,\mathbb{X})}, \|C\|_{B(\mathbb{X}_\alpha,\mathbb{X})})$.

Proof. Consider the nonlinear operator $\mathbb{Q}$ on $AA_c(\mathbb{X}_\alpha)$ given by

$$\begin{aligned} \mathbb{Q}u(t) &= -f(t,Bu(t)) - \int_{-\infty}^t AT(t-s)Pf(s,Bu(s))ds \\ &+ \int_t^\infty AT(t-s)Qf(s,Bu(s))ds + \int_{-\infty}^t T(t-s)Pg(s,Cu(s))ds \\ &- \int_t^\infty T(t-s)Qg(s,Cu(s))ds \end{aligned}$$

for each $t \in \mathbb{R}$.

As we have previously seen, for every $u \in AA_c(\mathbb{X}_\alpha)$, $f(\cdot, Bu(\cdot)) \in AA_c(\mathbb{X}_\beta) \subset AA_c(\mathbb{X}_\alpha)$. In view of Lemma 3.3 and Lemma 3.4, it follows that $\mathbb{Q}$ maps $AA_c(\mathbb{X}_\alpha)$ into itself. To complete the proof one has to show that $\mathbb{Q}$ has a unique fixed-point.

Let $v, w \in AA_c(\mathbb{X}_\alpha)$

$$\begin{aligned} \|\Gamma_1(v)(t) - \Gamma_1(w)(t)\|_\alpha &\le \int_{-\infty}^t \|AT(t-s)P\left[f(s,Bv(s)) - f(s,Bw(s))\right]\|_\alpha ds \\ &\le cK\|B\|_{B(\mathbb{X}_\alpha,\mathbb{X})}\|v-w\|_{\infty,\alpha}\int_{-\infty}^t (t-s)^{\beta-\alpha-1}e^{-\gamma(t-s)}ds \\ &= c\frac{\Gamma(\beta-\alpha)}{\gamma^{\beta-\alpha}}K\|B\|_{B(\mathbb{X}_\alpha,\mathbb{X})}\|v-w\|_{\infty,\alpha}. \end{aligned}$$

Similarly,

$$\begin{aligned} \|\Gamma_2(v)(t) - \Gamma_2(w)(t)\|_\alpha &\le \int_t^\infty \|AT(t-s)Q\left[f(s,Bv(s)) - f(s,Bw(s))\right]\|_\alpha ds \\ &\le cK\|B\|_{B(\mathbb{X}_\alpha,\mathbb{X})}\|v-w\|_{\infty,\alpha}\int_t^{+\infty} e^{\delta(t-s)}ds \\ &= \frac{cK\|B\|_{B(\mathbb{X}_\alpha,\mathbb{X})}}{\delta}\|v-w\|_{\infty,\alpha}. \end{aligned}$$

Now for Γ_3 and Γ_4, we have the following approximations

$$\begin{aligned} \|\Gamma_3(v)(t) - \Gamma_3(w)(t)\|_\alpha &\le \int_{-\infty}^t \|T(t-s)P\left[g(s,Cv(s)) - g(s,Cw(s))\right]\|_\alpha ds \\ &\le \frac{K\|C\|_{B(\mathbb{X}_\alpha,\mathbb{X})}M(\alpha)\Gamma(1-\alpha)}{\gamma^{1-\alpha}}\|v-w\|_{\infty,\alpha}, \end{aligned}$$

and

$$\begin{aligned} \|\Gamma_4(v)(t) - \Gamma_4(w)(t)\|_\alpha &\le \int_t^{+\infty} \|T(t-s)Q\left[g(s,Cv(s)) - g(s,Cw(s))\right]\|_\alpha ds \\ &\le Kc(\alpha)\|C\|_{B(\mathbb{X}_\alpha,\mathbb{X})}\|v-w\|_{\infty,\alpha}\int_t^{+\infty} e^{\delta(t-s)}ds \\ &= \frac{K\|C\|_{B(\mathbb{X}_\alpha,\mathbb{X})}c(\alpha)}{\delta}\|v-w\|_{\infty,\alpha}. \end{aligned}$$

Combining it follows that

$$\|\mathbb{Q}v - \mathbb{Q}w\|_{\infty,\alpha} \leq \Theta \,.\, \|v - w\|_{\infty,\alpha}.$$

Clearly, if $\Theta < 1$, then (1.2) has a unique fixed-point by the Banach fixed point theorem, which obviously is the only (compact) almost automorphic solution to (1.2). ∎

Example 3.6. (The 1-D Heat Equation with Small Delay). For $\sigma \in \mathbb{R}$, consider the one-dimensional heat equation with small delay equipped with Dirichlet conditions:

$$\frac{\partial}{\partial t}[\varphi(t,x) + f(t,\varphi(t-p,x))] = \frac{\partial^2}{\partial x^2}\varphi(t,x) + \sigma\varphi(t,x) + g(t,\varphi(t-p,x)) \quad (3.10)$$

$$\varphi(t,0) = \varphi(t,1) = 0 \quad (3.11)$$

for $t \in \mathbb{R}$ and $x \in [0,1]$, where $p > 0$, and $f, g : \mathbb{R} \times C[0,1] \mapsto C[0,1]$ are some jointly continuous functions.

Take $\mathbb{X} := C[0,1]$, equipped with the sup norm. Define the operator A by

$$A(\varphi) := \varphi'' + \sigma\varphi, \ \ \forall \varphi \in D(A),$$

where $D(A) := \{\varphi \in C^2[0,1], \varphi(0) = \varphi(1) = 0\} \subset C[0,1]$.

Clearly A is sectorial, and hence is the generator of an analytic semigroup. In addition to the above, the resolvent and spectrum of A are respectively given by

$$\rho(A) = \mathbb{C} - \{-n^2\pi^2 + \sigma : n \in \mathbb{N}\} \ \text{ and } \ \sigma(A) = \{-n^2\pi^2 + \sigma : n \in \mathbb{N}\}$$

so that $\sigma(A) \cap i\mathbb{R} = \{\emptyset\}$ whenever $\sigma \neq n^2\pi^2$.

Theorem 3.7. *Under assumptions* **(H3)-(H4)**, *if $\sigma \neq n^2\pi^2$ for each $n \in \mathbb{N}$, then the heat equation with small delay* (3.10)-(3.11) *has a unique $\mathbb{X}_\alpha$-valued compact almost automorphic mild solution whenever K is small enough.*

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