Numerical quenching for a nonlinear diffusion equation with a singular boundary condition

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Abstract

This paper concerns the study of the numerical approximation for the following boundary value problem

$$\begin{cases} (u^m)_t = u_{xx}, \quad 0 < x < 1, \ t > 0, \\ u_x(0,t) = 0, \quad u_x(1,t) = -u^{-\beta}(1,t), \quad t > 0, \\ u(x,0) = u_0(x) > 0, \quad 0 \leq x \leq 1, \end{cases}$$

where $m \geq 1$, $\beta > 0$. We obtain some conditions under which the solution of a semidiscrete form of the above problem quenches in a finite time and estimate its semidiscrete quenching time. We also establish the convergence of the semidiscrete quenching time. Finally, we give some numerical experiments to illustrate our analysis.

1 Introduction

Consider the following boundary value problem

$$(u^m)_t = u_{xx}, \qquad 0 < x < 1, t > 0, \tag{1}$$

$$u_x(0,t) = 0, u_x(1,t) = -u^{-\beta}(1,t), \qquad t > 0, \tag{2}$$

$$u(x,0) = u_0(x) > 0, \qquad 0 \leq x \leq 1, \tag{3}$$

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where $m \geq 1, \beta > 0, u_0 \in C^2([0,1])$,

$$u_0'(x) < 0, u_0''(x) < 0, x \in (0,1), \tag{4}$$
$$u_0'(0) = 0, u_0'(1) = -u^{-\beta}(1). \tag{5}$$

The equation (1) can be rewritten in the following form

$$u_t = \frac{1}{m} u^{1-m} u_{xx}$$

where $1 - m \leq 0$. Without loss of generality, we may consider the following problem

$$u_t = u^\alpha u_{xx}, 0 < x < 1, t > 0, \tag{6}$$
$$u_x(0,t) = 0, u_x(1,t) = -u^{-\beta}(1,t), t > 0, \tag{7}$$
$$u(x,0) = u_0(x) > 0, 0 \leq x \leq 1, \tag{8}$$

where $\alpha \leq 0$.

Definition 1.1. *We say that the solution u of (6)–(8) quenches in a finite time if there exists a finite time T_q such that $\|u(\cdot,t)\|_{\inf} > 0$ for $t \in [0, T_q)$ but*

$$\lim_{t \to T_q} \|u(\cdot,t)\|_{\inf} = 0,$$

where $\|u(\cdot,t)\|_{\inf} = \min_{0 \leq x \leq 1} u(x,t)$. The time T_q is called the quenching time of the solution u.

The theoretical study of solutions of nonlinear parabolic equations with nonlinear boundary conditions which quench in a finite time has been the subject of investigations of many authors (see [2], [4]-[7] and the references cited therein). In particular in [5], under the conditions given in (4)–(5), Deng and Xu have shown that the solution of (1)–(3) quenches in a finite time at the point $x = 1$ and have estimated its quenching time. Also, in [8]–[12], [16], the theoretical study of the phenomenon of quenching has been handled for other problems. The first condition in (4) allows the solution u to attain its minimum at the point $x = 1$ and the solution u decreases with respect to the second variable because of the last one. The hypotheses in (5) are compatibility conditions which ensure the existence of regular solutions.

In this paper, we are interesting in the numerical study using a semidiscrete form of (6)–(8). We start by the construction of our scheme as follows.

Let I be a positive integer and define the grid $x_i = ih$, $0 \leq i \leq I$, where $h = 1/I$. Approximate the solution u of the problem (6)–(8) by the solution $U_h(t) = (U_0(t), U_1(t), \ldots, U_I(t))^T$ of the following semidiscrete equations

$$\frac{d}{dt} U_i(t) - U_i^\alpha(t) \delta^2 U_i(t) = 0, 0 \leq i \leq I-1, t \in (0, T_q^h), \tag{9}$$
$$\frac{d}{dt} U_I(t) - U_I^\alpha(t) \delta^2 U_I(t) = -\frac{2}{h} U_I^{\alpha-\beta}(t), t \in (0, T_q^h), \tag{10}$$
$$U_i(0) = \varphi_i > 0, 0 \leq i \leq I, \tag{11}$$

where

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, 1 \leq i \leq I-1,$$

$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2}, \quad \delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2}.$$

Here $(0, T_q^h)$ is the maximal time interval on which the solution $U_h(t)$ of (9)–(11) satisfies

$$\|U_h(t)\|_{\inf} > 0,$$

where $\|U_h(t)\|_{\inf} = \min_{0 \leq i \leq I} U_i(t)$. When T_q^h is finite, we say that the solution $U_h(t)$ quenches in a finite time and the time T_q^h is called the semidiscrete quenching time of $U_h(t)$.

In this paper, we give some conditions under which the solution of (9)–(11) quenches in a finite time and estimate its semidiscrete quenching time. We also show that the semidiscrete quenching time converges to the real one when the mesh size goes to zero. Concerning the numerical study, one may also find in [13], [14], [18], [19] some results where the authors have proposed some schemes for the numerical calculation of solutions which present singularities. Our work was motived by the paper in [1] where the authors have used a semidiscrete scheme to study the phenomenon of blow-up for a semilinear heat equation with Dirichlet boundary conditions (we say that a solution blows up in a finite time if it reaches the value infinity in a finite time). Also in [3], the phenomenon of extinction has been treated by a numerical method.

Our paper is written in the following manner. In the next section, we prove some results about the discrete maximum principle. In the third section, under some assumptions, we show that the solution of (9)–(11) quenches in a finite time and estimate its semidiscrete quenching time. In the fourth section, we prove the convergence of the semidiscrete quenching time. Finally in the last section, we give some numerical results.

2 Properties of the semidiscrete scheme

In this section, we give some results about the discrete maximum principle.
The following lemma is a semidiscrete form of the maximum principle.

Lemma 2.1. *Let $a_h(t), b_h(t) \in C^0([0,T), \mathbb{R}^{I+1})$, $b_h(t) \geq 0$ and let $V_h(t) \in C^1([0,T), \mathbb{R}^{I+1})$ such that*

$$\frac{d}{dt}V_i(t) - b_i(t)\delta^2 V_i(t) + a_i(t)V_i(t) \geq 0, 0 \leq i \leq I, t \in (0,T), \tag{12}$$

$$V_i(0) \geq 0, 0 \leq i \leq I. \tag{13}$$

Then we have $V_i(t) \geq 0$, $0 \leq i \leq I$, $t \in (0,T)$.

Proof. Let $T_0 < T$ and define the vector $Z_h(t) = e^{\lambda t} V_h(t)$ where λ is such that $a_i(t) - \lambda > 0$, $0 \leq i \leq I$, $t \in [0, T_0]$. Let $m = \min_{t \in [0,T_0]} \|Z_h(t)\|_{\inf}$. Since for $i \in \{0, ..., I\}$, $Z_i(t)$ is a continuous function, there exists $t_0 \in [0, T_0]$ such that $m = Z_{i_0}(t_0)$ for a certain $i_0 \in \{0, ..., I\}$. It is not hard to see that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0, \tag{14}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \geq 0 \quad \text{if} \quad i_0 = 0, \tag{15}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \geq 0 \quad \text{if} \quad 1 \leq i_0 \leq I-1, \tag{16}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} \geq 0 \quad \text{if} \quad i_0 = I. \tag{17}$$

Using (12), a straightforward computation reveals that

$$\frac{dZ_{i_0}(t_0)}{dt} - b_{i_0}(t_0)\delta^2 Z_{i_0}(t_0) + (a_{i_0}(t_0) - \lambda) Z_{i_0}(t_0) \geq 0. \tag{18}$$

We observe from (14)–(18) that $(a_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0$, which implies that $m = Z_{i_0}(t_0) \geq 0$ because $a_{i_0}(t_0) - \lambda > 0$. Hence $V_h(t) \geq 0$ for $t \in [0, T_0]$ and we have the desired result. ∎

Another version of the maximum principle for semidiscrete equations is the following comparison lemma.

Lemma 2.2. *Let $V_h(t)$, $U_h(t) \in C^1([0, T), \mathbb{R}^{I+1})$, $f \in C^0(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $b_h(t) \geq 0$ such that*

$$\frac{dV_i}{dt} - b_i(t)\delta^2 V_i + f(V_i(t), t) < \frac{dU_i}{dt} - b_i(t)\delta^2 U_i + f(U_i(t), t),\ 0 \leq i \leq I,\ \ t \in (0, T),$$
$$V_i(0) < U_i(0), 0 \leq i \leq I.$$

Then we have $V_i(t) < U_i(t)$, $0 \leq i \leq I$, $t \in (0, T)$.

Proof. Define the vector $Z_h(t) = U_h(t) - V_h(t)$. Let t_0 be the first $t \in (0, T)$ such that $Z_h(t) > 0$ for $t \in [0, t_0)$, but

$$Z_{i_0}(t_0) = 0 \quad \text{for a certain } i_0 \in \{0, ..., I\}.$$

We observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \geq 0 \quad \text{if} \quad i_0 = 0,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \geq 0 \quad \text{if} \quad 1 \leq i_0 \leq I-1,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} \geq 0 \quad \text{if} \quad i_0 = I,$$

which implies that

$$\frac{dZ_{i_0}(t_0)}{dt} - b_{i_0}(t_0)\delta^2 Z_{i_0}(t_0) + f(U_{i_0}(t_0), t_0) - f(V_{i_0}(t_0), t_0) \leq 0.$$

But this estimate contradicts the first strict differential inequality of the lemma and the proof is complete. ■

A direct consequence of Lemmas 2.1 and 2.2 is that the semidiscrete solution is bounded from above by $\|\varphi_h\|_\infty$.

3 Quenching in the semidiscrete problem

In this section, under some assumptions, we show that the solution U_h of (9)–(11) quenches in a finite time and estimate its semidiscrete quenching time.
The following result shows a property of the operator δ^2.

Lemma 3.1. *Let $U_h \in \mathbb{R}^{I+1}$ such that $U_h > 0$. Then we have*

$$\delta^2 U_i^{-\gamma} \geq -\gamma U_i^{-\gamma-1}\delta^2 U_i, \quad 0 \leq i \leq I,$$

where $\gamma > 0$.

Proof . Apply Taylor's expansion to obtain

$$\delta^2 U_0^{-\gamma} = -\gamma U_0^{-\gamma-1}\delta^2 U_0 + (U_1 - U_0)^2 \frac{\gamma(\gamma+1)}{h^2}\theta_0^{-\gamma-2},$$

$$\begin{aligned} \delta^2 U_i^{-\gamma} &= -\gamma U_i^{-\gamma-1}\delta^2 U_i + (U_{i+1} - U_i)^2 \frac{\gamma(\gamma+1)}{2h^2}\theta_i^{-\gamma-2} \\ &+ (U_{i-1} - U_i)^2 \frac{\gamma(\gamma+1)}{2h^2}\eta_i^{-\gamma-2} \quad \text{if} \quad 1 \leq i \leq I-1, \end{aligned}$$

$$\delta^2 U_I^{-\gamma} = -\gamma U_I^{-\gamma-1}\delta^2 U_I + (U_{I-1} - U_I)^2 \frac{\gamma(\gamma+1)}{h^2}\eta_I^{-\gamma-2},$$

where θ_i is an intermediate value between U_i and U_{i+1} and η_i the one between U_{i-1} and U_i. Use the fact that $U_h > 0$ to complete the rest of the proof. ■

The lemma below gives a property of the semidiscrete solution.

Lemma 3.2. *Let U_h be the solution of (9)–(11) and assume that the initial data at (11) obeys*

$$\varphi_{i+1} < \varphi_i, \quad 0 \leq i \leq I-1.$$

Then we have

$$U_{i+1}(t) < U_i(t) \quad \text{for} \quad t \in (0, T_q^h), \quad 0 \leq i \leq I-1.$$

Proof . Let us notice that (9)–(10) can be rewritten as follows

$$\frac{1}{1-\alpha}\frac{d}{dt}U_i^{1-\alpha}(t)-\delta^2U_i(t)=0, 0\leq i\leq I-1, t\in(0,T_q^h),$$
$$\frac{1}{1-\alpha}\frac{d}{dt}U_I^{1-\alpha}(t)-\delta^2U_I(t)=-\frac{2}{h}U_I^{-\beta}(t), t\in(0,T_q^h),$$

which implies that

$$\frac{1}{1-\alpha}\frac{d\eta_0(t)}{dt}-\frac{Z_1(t)-3Z_0(t)}{h^2}=0,\quad t\in(0,T_q^h), \tag{19}$$
$$\frac{1}{1-\alpha}\frac{d\eta_i(t)}{dt}-\delta^2Z_i(t)=0,\quad 1\leq i\leq I-2,\quad t\in(0,T_q^h), \tag{20}$$
$$\frac{1}{1-\alpha}\frac{d\eta_{I-1}(t)}{dt}-\frac{Z_{I-2}(t)-3Z_{I-1}(t)}{h^2}<0,\quad t\in(0,T_q^h), \tag{21}$$

where $\eta_i(t)=U_{i+1}^{1-\alpha}(t)-U_i^{1-\alpha}(t)$ and $Z_i(t)=U_{i+1}(t)-U_i(t)$, $0\leq i\leq I-1$. We know by hypothesis that

$$U_{i+1}(0)<U_i(0),\quad 0\leq i\leq I-1.$$

Let t_0 be the first $t\in(0,T_q^h)$ such that

$$U_{i+1}(t)<U_i(t)\quad\text{for}\quad t\in[0,t_0),\quad 0\leq i\leq I-1,$$

but $U_{i_0+1}(t_0)=U_{i_0}(t_0)$ for a certain $i_0\in\{0,...,I-1\}$. We observe that

$$\frac{d\eta_{i_0}(t_0)}{dt}=\lim_{k\to0}\frac{\eta_{i_0}(t_0)-\eta_{i_0}(t_0-k)}{k}\geq0,$$

$$\frac{Z_{I-2}(t_0)-3Z_{I-1}(t_0)}{h^2}=\frac{Z_{I-2}(t_0)}{h^2}\leq0\quad\text{if}\quad i_0=I-1.$$

This implies that

$$\frac{1}{1-\alpha}\frac{d\eta_{I-1}(t_0)}{dt}-\frac{Z_{I-2}(t_0)-3Z_{I-1}(t_0)}{h^2}\geq0\quad\text{if}\quad i_0=I-1,$$

which contradicts (21). Without loss of generality, we may suppose that $U_{i_0+1}(t_0)=U_{i_0}(t_0)$ and $U_{i_0+2}(t_0)<U_{i_0+1}(t_0)$ if $i_0\in\{0,...,I-2\}$. We find that

$$\delta^2Z_{i_0}(t_0)=\frac{Z_{i_0+1}(t_0)+Z_{i_0-1}(t_0)}{h^2}<0\quad\text{if}\quad 1\leq i_0\leq I-2,$$

$$\frac{Z_1(t_0)-3Z_0(t_0)}{h^2}=\frac{Z_1(t_0)}{h^2}<0\quad\text{if}\quad i_0=0.$$

These inequalities imply that

$$\frac{1}{1-\alpha}\frac{d\eta_{i_0}(t_0)}{dt}-\delta^2Z_{i_0}(t_0)>0\quad\text{if}\quad 1\leq i_0\leq I-2,$$
$$\frac{1}{1-\alpha}\frac{d\eta_0(t_0)}{dt}-\frac{Z_1(t_0)-3Z_0(t_0)}{h^2}>0\quad\text{if}\quad i_0=0.$$

But we have a contradiction because of (19)–(20) and the proof is complete. ∎

The above lemma says that if the initial data at (11) is decreasing in space, then the semidiscrete solution is also decreasing in space. This property will be used in the following theorem to ensure that the minimum of the semidiscrete solution is attained at the last node x_I.
Now, let us give our result on the quenching time.

Theorem 3.1. *Suppose that there exists a constant $A \in (0,1]$ such that the initial data at (11) satisfies*

$$\varphi_{i+1} < \varphi_i, \quad \delta^2\varphi_i \le 0, 0 \le i \le I-1, \tag{22}$$

$$\delta^2\varphi_I - \frac{2}{h}\varphi_I^{-\beta} \le -A\varphi_I^{-\gamma-\alpha}, \tag{23}$$

where $\gamma \in (0,\beta]$. Then the solution U_h of (9)–(11) quenches in a finite time T_q^h with the following estimation

$$T_q^h \le \frac{1}{A}\frac{\|\varphi_h\|_{\inf}^{\gamma+1}}{(\gamma+1)}.$$

Proof. Since $(0, T_q^h)$ is the maximal time interval on which $\|U_h(t)\|_{\inf} > 0$, our aim is to show that T_q^h is finite and satisfies the above inequality. Introduce the vector $J_h(t)$ defined as follows

$$J_i = \frac{d}{dt}U_i, \quad 0 \le i \le I-1, \quad J_I = \tfrac{d}{dt}U_I + AU_I^{-\gamma}. \tag{24}$$

A straightforward computation reveals that

$$\frac{d}{dt}J_i - U_i^{\alpha}\delta^2 J_i = \frac{d}{dt}(\frac{dU_i}{dt} - U_i^{\alpha}\delta^2 U_i) + \alpha U_i^{\alpha-1}\frac{dU_i}{dt}\delta^2 U_i, \quad 0 \le i \le I-1, \tag{25}$$

$$\begin{aligned}\frac{d}{dt}J_I - U_I^{\alpha}\delta^2 J_I = {} & \frac{d}{dt}(\frac{dU_I}{dt} - U_I^{\alpha}\delta^2 U_I) + \alpha U_I^{\alpha-1}\frac{dU_I}{dt}\delta^2 U_I \\ & -A\gamma U_I^{-\gamma-1}\frac{dU_I}{dt} - AU_I^{\alpha}\delta^2 U_I^{-\gamma}.\end{aligned} \tag{26}$$

From Lemma 3.1, $\delta^2 U_I^{-\gamma} \ge -\gamma U_I^{-\gamma-1}\delta^2 U_I$. Hence the equality (26) implies that

$$\begin{aligned}\frac{d}{dt}J_I - U_I^{\alpha}\delta^2 J_I \le {} & \frac{d}{dt}(\frac{dU_I}{dt} - U_I^{\alpha}\delta^2 U_I) + \alpha U_I^{\alpha-1}\frac{dU_I}{dt}\delta^2 U_I \\ & -A\gamma U_I^{-\gamma-1}(\frac{dU_I}{dt} - U_I^{\alpha}\delta^2 U_I).\end{aligned} \tag{27}$$

It follows from (9), (10), (25), (27) that

$$\frac{d}{dt}J_i - U_i^{\alpha}\delta^2 J_i = \alpha U_i^{\alpha-1}\frac{dU_i}{dt}\delta^2 U_i, \quad 0 \le i \le I-1, \tag{28}$$

$$\begin{aligned}\frac{d}{dt}J_I - U_I^{\alpha}\delta^2 J_I \le {} & -\frac{2}{h}(\alpha-\beta)U_I^{\alpha-\beta-1}\frac{dU_I}{dt} + \alpha U_I^{\alpha-1}\frac{dU_I}{dt}\delta^2 U_I \\ & +\frac{2A\gamma}{h}U_I^{-\gamma-1+\alpha-\beta}.\end{aligned} \tag{29}$$

Multiplying both sides of (9) and (10) by $\frac{dU_i}{dt}$ and $\frac{dU_I}{dt}$ respectively, we find that $U_i^{\alpha-1}\frac{dU_i}{dt}\delta^2 U_i \geq 0$, $0 \leq i \leq I-1$ and $U_I^{\alpha-1}\frac{dU_I}{dt}\delta^2 U_I \geq \frac{2}{h}U_I^{\alpha-\beta-1}\frac{dU_I}{dt}$. Since $\alpha \leq 0$, we deduce from (28), (29) that

$$\frac{d}{dt}J_i - U_i^{\alpha}\delta^2 J_i \leq 0, 0 \leq i \leq I-1, \tag{30}$$

$$\frac{d}{dt}J_I - U_I^{\alpha}\delta^2 J_I \leq \frac{2\beta}{h}U_I^{\alpha-\beta-1}\frac{d}{dt}U_I + \frac{2A\gamma}{h}U_I^{-\gamma-1+\alpha-\beta}. \tag{31}$$

The inequality (31) implies that

$$\frac{d}{dt}J_I - U_I^{\alpha}\delta^2 J_I \leq \frac{2\beta}{h}U_I^{\alpha-\beta-1}J_I \tag{32}$$

because $\gamma \in (0,\beta]$. We observe from (22)–(23) that

$$J_i(0) = \varphi_i^{\alpha}\delta^2\varphi_i \leq 0, 0 \leq i \leq I-1,$$

$$J_I(0) = \varphi_I^{\alpha}(\delta^2\varphi_I - \frac{2}{h}\varphi_I^{-\beta} + A\varphi_I^{-\gamma-\alpha}) \leq 0.$$

It follows from Lemma 2.1 that $J_h(t) \leq 0$ for $t \in (0, T_q^h)$, which implies that

$$\frac{dU_I}{dt} + AU_I^{-\gamma} \leq 0 \quad \text{for} \quad t \in (0, T_q^h).$$

This estimate may be rewritten in the following manner

$$U_I^{\gamma}dU_I \leq -Adt \quad \text{for} \quad t \in (0, T_q^h).$$

Integrating the above inequality over (t, T_q^h) to obtain

$$T_q^h - t \leq \frac{1}{A}\frac{(U_I(t))^{\gamma+1}}{(\gamma+1)}. \tag{33}$$

From Lemma 3.2, $U_I(0) = \|\varphi_h\|_{\inf}$. Replacing t by 0 in (33) and taking into account the fact that $U_I(0) = \|\varphi_h\|_{\inf}$, we get

$$T_q^h \leq \frac{1}{A}\frac{\|\varphi_h\|_{\inf}^{\gamma+1}}{(\gamma+1)}.$$

Use the fact that the quantity on the right hand side of the above inequality is finite to complete the rest of the proof. ∎

Remark 3.1. *Thanks to Lemma 3.2, $U_I(t) = \|U_h(t)\|_{\inf}$. Using the inequality (33), we get*

$$T_q^h - t_0 \leq \frac{1}{A}\frac{\|U_h(t_0)\|_{\inf}^{\gamma+1}}{(\gamma+1)} \quad for \quad t_0 \in (0, T_q^h)$$

and

$$\|U_h(t)\|_{\inf} \geq (A(\gamma+1)(T_q^h - t))^{\frac{1}{\gamma+1}} \quad for \quad t \in (0, T_q^h).$$

Remark 3.2. *Let us notice that the assumptions (22)–(23) are the discrete version of those given in (4)–(5) for the continuous solution. These conditions allow us to establish an upper bound of the semidiscrete quenching time and also to obtain the estimate of Remark 3.1 which is crucial for the convergence of the semidiscrete quenching time to the theoretical one.*

4 Convergence of the semidiscrete quenching time

In this section, under some conditions, we prove that the semidiscrete solution quenches in a finite time and its quenching time goes to the real one when the mesh size tends to zero. Firstly, we show that for each fixed time interval $[0, T]$ where the continuous solution u obeys $\|u(\cdot,t)\|_{\inf} > 0$, the semidiscrete solution U_h approximates u when the mesh parameter h goes to zero. We denote

$$u_h(t) = (u(x_0,t), ..., u(x_I,t))^T \quad \text{and} \quad \|U_h(t)\|_\infty = \sup_{0\le i\le I} |U_i(t)|.$$

Our first result is the following.

Theorem 4.1. *Assume that (6)-(8) has a solution $u \in C^{4,1}([0,1]\times[0,T])$ such that $\inf_{t\in[0,T]} \|u(\cdot,t)\|_{\inf} = \varrho > 0$ and the initial data at (11) satisfies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad \text{as} \quad h \to 0. \tag{34}$$

Then, for h sufficiently small, the problem (9)–(11) has a unique solution $U_h \in C^1([0,T], \mathbb{R}^{I+1})$ such that

$$\max_{0\le t\le T} \|U_h(t) - u_h(t)\|_\infty = O(\|\varphi_h - u_h(0)\|_\infty + h) \quad \text{as} \quad h \to 0. \tag{35}$$

Proof . The problem (9)–(11) has for each h, a unique solution $U_h \in C^1([0, T_q^h), \mathbb{R}^{I+1})$. Let $t(h)$ be the greatest value of $t > 0$ such that

$$\|U_h(t) - u_h(t)\|_\infty < \frac{\varrho}{2} \quad \text{for} \quad t \in (0, t(h)). \tag{36}$$

The relation (34) implies that $t(h) > 0$ for h sufficiently small. Let $t^*(h) = \min\{t(h), T\}$. We have $U_i(t) = u(x_i,t) + U_i(t) - u(x_i,t)$, $0 \le i \le I$, which implies that

$$U_i(t) \ge u(x_i,t) - \|U_h(t) - u_h(t)\|_\infty, \quad 0 \le i \le I.$$

Let $i_0 \in \{0, ..., I\}$ be such that $U_{i_0}(t) = \|U_h(t)\|_{\inf}$. We deduce

$$\|U_h(t)\|_{\inf} \ge u(x_{i_0},t) - \|U_h(t) - u_h(t)\|_\infty.$$

Hence, we get

$$\|U_h(t)\|_{\inf} \ge \|u(\cdot,t)\|_{\inf} - \|U_h(t) - u_h(t)\|_\infty \quad \text{for} \quad t \in (0, t^*(h)), \tag{37}$$

which implies that

$$\|U_h(t)\|_{\inf} \ge \varrho - \frac{\varrho}{2} = \frac{\varrho}{2} \quad \text{for} \quad t \in (0, t^*(h)). \tag{38}$$

Since $u \in C^{4,1}$, taking the derivative in x on both sides of (6) and due to the fact that u_x and u_{xt} vanish at $x = 0$, we observe that u_{xxx} also vanishes at $x = 0$. Using Taylor's expansion, we find that

$$\delta^2 u(x_i,t) = u_{xx}(x_i,t) + \frac{h^2}{12} u_{xxxx}(\widetilde{x}_i,t), \quad 0 \le i \le I-1, \tag{39}$$

$$\delta^2 u(x_I,t) = u_{xx}(x_I,t) + \frac{2}{h}u^{-\beta}(x_I,t) - \frac{h}{3}u_{xxx}(x_I,t) + \frac{h^2}{12}u_{xxxx}(\widetilde{x}_I,t). \tag{40}$$

To establish the above equality for $i = 0$, we have used the fact that u_{xxx} and u_x vanish at $x = 0$. Taking into account (6), we deduce that

$$\frac{d}{dt}u(x_i,t) = u^{\alpha}(x_i,t)\delta^2 u(x_i,t) - \frac{h^2}{12}u^{\alpha}(x_i,t)u_{xxxx}(\widetilde{x}_i,t),\ 0 \le i \le I-1, \tag{41}$$

$$\frac{d}{dt}u(x_I,t) = u^{\alpha}(x_I,t)\delta^2 u(x_I,t) - \frac{h^2}{12}u^{\alpha}(x_I,t)u_{xxxx}(\widetilde{x}_I,t) - \frac{2}{h}u^{\alpha-\beta}(x_I,t) + \frac{h}{3}u^{\alpha}(x_I,t)u_{xxx}(x_I,t). \tag{42}$$

Let $e_h(t) = U_h(t) - u_h(t)$ be the error of discretization. Applying the mean value theorem, we have for $t \in (0, t^*(h))$,

$$\frac{d}{dt}e_i(t) - U_i^{\alpha}\delta^2 e_i(t) = \alpha\xi_i^{\alpha-1}\delta^2 u(x_i,t)e_i(t) + \frac{h^2}{12}u^{\alpha}(x_i,t)u_{xxxx}(\widetilde{x}_i,t),\quad 0 \le i \le I-1, \tag{43}$$

$$\frac{d}{dt}e_I(t) - U_I^{\alpha}\delta^2 e_I(t) = -\frac{2}{h}(\alpha-\beta)\xi_I^{\alpha-\beta-1}e_I(t) + \alpha\theta_I^{\alpha-1}\delta^2 u(x_I,t)e_I(t) - \frac{h}{3}u^{\alpha}(x_I,t)u_{xxx}(x_I,t) + \frac{h^2}{12}u^{\alpha}(x_I,t)u_{xxxx}(\widetilde{x}_I,t), \tag{44}$$

where for $i \in \{0,...,I\}$, ξ_i is an intermediate value between $U_i(t)$ and $u(x_i,t)$ and θ_I the one between $U_I(t)$ and $u(x_I,t)$. Since $u \in C^{4,1}$ and $\inf_{t\in[0,T]} \|u(\cdot,t)\|_{\inf} = \varrho > 0$, we observe from (39)–(40) that $\delta^2 u(x_i,t)$, $0 \le i \le I-1$ and $h\delta^2 u(x_I,t)$ are bounded. Using (38), (43) and (44), there exists a constant $M > 0$ such that

$$\frac{d}{dt}e_i(t) - U_i^{\alpha}(t)\delta^2 e_i(t) \le M|e_i(t)| + Mh^2, 0 \le i \le I-1,$$

$$\frac{de_I(t)}{dt} - U_I^{\alpha}(t)\delta^2 e_I(t) \le \frac{M|e_I(t)|}{h} + hM.$$

Introduce the function $W(x,t)$ defined as follows

$$W(x,t) = e^{(Dt+Ex^2)}(\|\varphi_h - u_h(0)\|_{\infty} + Fh) \quad \text{in} \quad [0,1]\times[0,T], \tag{45}$$

where D, E, F are positive constants which will be determined later. A direct calculation yields

$$W_t = DW,$$
$$W_x = 2ExW,$$
$$W_{xx} = (2E + 4E^2x^2)W,$$
$$W_{xxx} = (12xE^2 + 8x^3E^3)W,$$
$$W_{xxxx} = (12E^2 + 48x^2E^3 + 16x^4E^4)W.$$

We observe that

$$W_t(x_i,t) - U_i^\alpha W_{xx}(x_i,t) = (D - (2E + 4E^2x_i^2)U_i^\alpha)W(x_i,t), \quad 0 \le i \le I.$$

Since U_i^α, $0 \le i \le I$ are bounded because of (38), we may choose D and E so that

$$W_t(x_i,t) - U_i^\alpha W_{xx}(x_i,t) \ge \frac{D}{2}W(x_i,t), \quad 0 \le i \le I.$$

We also have

$$W_x(x_0,t) = 0, \quad W_x(x_I,t) = 2EW(x_I,t),$$
$$W(x_i,0) = e^{Ex_i^2}(\|\varphi_h - u_h(0)\|_\infty + Fh), 0 \le i \le I.$$

Let us notice that $W_{xxx}(x_0,t) = 0$. Apply Taylor's expansion to obtain

$$\delta^2 W(x_i,t) = W_{xx}(x_i,t) + \frac{h^2}{12}W_{xxxx}(\widetilde{x}_i,t), 0 \le i \le I-1, \tag{46}$$
$$\delta^2 W(x_I,t) = W_{xx}(x_I,t) - \frac{2E}{h}W(x_I,t) - \frac{h}{3}W_{xxx}(\widetilde{x}_I,t). \tag{47}$$

To establish the above equality for $i = 0$, we have used the fact that $W_x(x_0,t) = 0$ and $W_{xxx}(x_0,t) = 0$. Taking into account (46) and (47), we deduce that

$$W_t(x_i,t) - U_i^\alpha \delta^2 W(x_i,t) \ge \frac{D}{2}W(x_i,t) - \frac{h^2}{12}W_{xxxx}(\widetilde{x}_i,t)U_i^\alpha, 0 \le i \le I-1,$$
$$W_t(x_I,t) - U_I^\alpha \delta^2 W(x_I,t) \ge \frac{D}{2}W(x_I,t) + \frac{2E}{h}W(x_I,t)U_I^\alpha + \frac{h}{3}W_{xxx}(\widetilde{x}_I,t)U_I^\alpha.$$

Since U_i^α, $0 \le i \le I$, W_{xxx} and W_{xxxx} are bounded and $W(x_i,t) \ge Fh$, $0 \le i \le I$, $U_I^\alpha \ge \frac{\varrho}{2}$, we may choose D, E and F large enough that

$$\frac{d}{dt}W(x_i,t) - U_i^\alpha \delta^2 W(x_i,t) \ge M|W(x_i,t)| + Mh^2, 0 \le i \le I-1,$$
$$\frac{d}{dt}W(x_I,t) - U_I^\alpha \delta^2 W(x_I,t) \ge \frac{M}{h}|W(x_I,t)| + Mh,$$
$$W(x_i,0) > e_i(0), \quad 0 \le i \le I.$$

It follows from Comparison Lemma 2.2 that

$$W(x_i,t) > e_i(t) \quad \text{for} \quad t \in (0, t^*(h)), \quad 0 \le i \le I.$$

By the same way, we also prove that

$$W(x_i,t) > -e_i(t) \quad \text{for} \quad t \in (0, t^*(h)), \quad 0 \le i \le I,$$

which implies that

$$\|U_h(t) - u_h(t)\|_\infty \le e^{(Dt+E)}(\|\varphi_h - u_h(0)\|_\infty + Fh), \quad t \in (0, t^*(h)). \tag{48}$$

Let us show that $t^*(h) = T$. Suppose that $T > t(h)$. From (36), we obtain

$$\frac{\varrho}{2} = \|U_h(t(h)) - u_h(t(h))\|_\infty \leq e^{(DT+E)}(\|\varphi_h - u_h(0)\|_\infty + Fh). \tag{49}$$

Since the term on the right hand side of the above inequality goes to zero as h tends to zero, we deduce that $\frac{\varrho}{2} \leq 0$, which is impossible. Consequently $t^*(h) = T$ and the proof is complete. ∎

We shall see in the sequel that to establish the convergence of the semidiscrete quenching time, Theorem 4.1 is crucial but the accuracy is not important. Only the fact that $\max_{0\leq t\leq T} \|U_h(t) - u_h(t)\|_\infty$ tends to zero as h goes to zero is taken into account.
Now, we are in a position to prove the main result of this section.

Theorem 4.2. *Suppose that the solution u of (6)–(8) quenches in a finite time T_q such that $u \in C^{4,1}([0,1] \times [0, T_q))$. Assume that the initial data at (11) satisfies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad as \quad h \to 0.$$

Under the assumptions of Theorem 3.1, the problem (9)–(11) has a solution U_h which quenches in a finite time T_q^h and we have

$$\lim_{h\to 0} T_q^h = T_q.$$

Proof. Setting $\varepsilon > 0$, there exists $\varrho > 0$ such that

$$\frac{1}{A}\frac{y^{\gamma+1}}{(\gamma+1)} \leq \frac{\varepsilon}{2} \quad \text{for} \quad y \in [0, \varrho]. \tag{50}$$

Since $u(x,t)$ quenches in a finite time T_q, there exists a time $T_0 \in (T_q - \frac{\varepsilon}{2}, T_q)$ such that $0 < \|u(\cdot,t)\|_{\inf} \leq \frac{\varrho}{2}$ for $t \in [T_0, T_q)$. Setting $T_1 = \frac{T_0+T_q}{2}$, it is not hard to see that $\|u(\cdot,t)\|_{\inf} > 0$ for $t \in [0, T_1]$. From Theorem 4.1, the problem (9)–(11) has a solution U_h and we have $\|U_h(t) - u_h(t)\|_\infty \leq \frac{\varrho}{2}$ for $t \in [0, T_1]$, which leads us to $\|U_h(T_1) - u_h(T_1)\|_\infty \leq \frac{\varrho}{2}$. Obviously, $U_i(T_1) = U_i(T_1) - u(x_i, T_1) + u(x_i, T_1)$, $0 \leq i \leq I$, which implies that

$$U_i(T_1) \leq \|U_h(T_1) - u_h(T_1)\|_\infty + u(x_i, T_1), \quad 0 \leq i \leq I.$$

Let $i_0 \in \{0, ..., I\}$ be such that $u(x_{i_0}, T_1) = \|u_h(T_1)\|_{\inf}$. We get

$$U_{i_0}(T_1) \leq \|U_h(T_1) - u_h(T_1)\|_\infty + \|u_h(T_1)\|_{\inf}.$$

It follows that

$$\|U_h(T_1)\|_{\inf} \leq \|U_h(T_1) - u_h(T_1)\|_\infty + \|u_h(T_1)\|_{\inf} \leq \frac{\varrho}{2} + \frac{\varrho}{2} = \varrho.$$

From Theorem 3.1, U_h quenches in a finite time T_q^h. We deduce from Remark 3.1 and (50) that

$$|T_q^h - T_q| \leq |T_q^h - T_1| + |T_1 - T_q| \leq \frac{1}{A}\frac{\|U_h(T_1)\|_{\inf}^{\gamma+1}}{(\gamma+1)} + \frac{\varepsilon}{2} \leq \varepsilon,$$

and the proof is complete. ∎

5 Numerical results

In this section, we present some numerical approximations to the quenching time of (6)–(8) with $\beta = 1$, $\alpha = -\frac{1}{2}$ and $u_0(x) = \cos(\frac{\pi}{2}x) + \frac{2}{\pi}$. We consider the following explicit scheme

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = (U_0^{(n)})^{\alpha}(\frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2}),$$
$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = (U_i^{(n)})^{\alpha}(\frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2}), 1 \le i \le I-1,$$
$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = (U_I^{(n)})^{\alpha}(\frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{h^2}) - \frac{2}{h}(U_I^{(n)})^{\alpha-\beta-1}U_I^{(n+1)},$$
$$U_i^{(0)} = \cos(\frac{\pi}{2}ih) + \frac{2}{\pi}, 0 \le i \le I,$$

and the implicit scheme below

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = (U_0^{(n)})^{\alpha}(\frac{2U_1^{(n+1)} - 2U_0^{(n+1)}}{h^2}),$$
$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = (U_i^{(n)})^{\alpha}(\frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2}), \quad 1 \le i \le I-1,$$
$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = (U_I^{(n)})^{\alpha}(\frac{2U_{I-1}^{(n+1)} - 2U_I^{(n+1)}}{h^2}) - \frac{2}{h}(U_I^{(n)})^{\alpha-\beta-1}U_I^{(n+1)},$$
$$U_i^{(0)} = \cos(\frac{\pi}{2}ih) + \frac{2}{\pi}, 0 \le i \le I,$$

where $n \ge 0$. Here $\Delta t_n = h^2\|U_h^{(n)}\|_{\inf}^{\beta+1-\alpha}$ for the implicit scheme and $\Delta t_n = \min\{\frac{h^2}{2}, h^2\|U_h^{(n)}\|_{\inf}^{\beta+1-\alpha}\}$ for the explicit scheme.
We need the following definition.

Definition 5.1. *We say that the discrete solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme quenches in a finite time if $\lim_{n\to+\infty} \|U_h^{(n)}\|_{\inf} = 0$ and the series $\sum_{n=0}^{+\infty} \Delta t_n$ converges. The quantity $\sum_{n=0}^{+\infty} \Delta t_n$ is called the numerical quenching time of the solution $U_h^{(n)}$.*

For the explicit scheme, let us notice that the restriction on the time step ensures the positivity of the discrete solution. For the implicit scheme, the positivity of the discrete solution is also guaranteed using standard methods (see [3]).
In the tables 1 and 2, in rows, we present the numerical quenching times, the numbers of iterations, CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256. We take for the numerical quenching time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when

$$|T^{n+1} - T^n| \le 10^{-16}.$$

The order(s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Table 1: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	*CPU time*	s
16	0.106438	250	-	-
32	0.104944	816	0.7	-
64	0.104534	2907	2	1.87
128	0.104423	10959	23	1.87
256	0.104321	42603	20	0.28

Table 2: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	*CPU time*	s
16	0.109576	268	-	-
32	0.105784	835	1	-
64	0.104750	2928	4	1.88
128	0.104478	10981	112	1.93
256	0.104408	42626	2883	1.96

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