Inner invariant means on locally compact topological semigroups*  

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Abstract

Let \( S \) be a locally compact semigroup and \( M_a(S) \) be its semigroup algebra. In this paper, we investigate inner invariant means on \( L^\infty(S, M_a(S)) \) of all \( M_a(S) \)-measurable complex-valued bounded functions on \( S \) and its closed subspace \( C_b(S) \), the space of all bounded continuous complex-valued functions on \( S \). We also study topological inner invariant means on certain closed subspaces \( X \) of \( L^\infty(S, M_a(S)) \) and their relation with inner invariant means on \( X \).

1 Introduction

Throughout this paper, \( S \) denotes a locally compact semigroup; i.e., a semigroup with a locally compact Hausdorff topology whose binary operation is jointly continuous. The space of all bounded complex regular Borel measures on \( S \) is denoted by \( M(S) \). This space with the convolution multiplication \( * \) and the total variation norm defines a Banach algebra. The space of all measures \( \mu \in M(S) \) for which the maps \( s \mapsto \delta_s |\mu| \) and \( s \mapsto |\mu| * \delta_s \) from \( S \) into \( M(S) \) are weakly continuous is denoted by \( M_a(S) \) (or \( \tilde{L}(S) \) as in [2]), where \( \delta_s \) denotes the Dirac measure at \( s \). It is well-known that \( M_a(S) \) is a closed two-sided \( L \)-ideal of \( M(S) \); see [2].

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Denote by $L^\infty(S, M\mu(S))$ the set of all complex-valued bounded functions $g$ on $S$ that are $M\mu(S)$-measurable; that is, $\mu$-measurable for all $\mu \in M\mu(S)$. We identify functions in $L^\infty(S, M\mu(S))$ that agree $\mu$-almost everywhere for all $\mu \in M\mu(S)$. For every $g \in L^\infty(S; M\mu(S))$, define
\[
\| g \|_\infty = \sup\{ \| g \|_{\infty, |\mu|} : \mu \in M\mu(S) \},
\]
where $\| . \|_{\infty, |\mu|}$ denotes the essential supremum norm with respect to $|\mu|$. Observe that $L^\infty(S, M\mu(S))$ with the complex conjugation as involution, the pointwise operations and the norm $\| . \|_\infty$ is a commutative $C^*$-algebra. Let $X$ be a subspace of $L^\infty(S, M\mu(S))$ which is left and right translations invariant; that is, $sg$ and $gs$ are in $X$ for all $g \in X$ and $s \in S$, where
\[
(sg)(t) = g(st) \quad \text{and} \quad (gs)(t) = g(ts)
\]
for all $t \in S$. A linear functional $F$ on $X$ is called inner invariant whenever
\[
F(sg) = F(gs)
\]
for all $s \in S$ and $g \in X$. Recall that a bounded linear functional $m$ with norm one on $X$ is said to be a mean if $m(g) \geq 0$ for all $g \in X$ with $g \geq 0$.

The study of inner invariant means was initiated by Effros [10] and pursued by Akemann [1], H. Choda and M. Choda [5], M. Choda [6, 7] for discrete groups, Lau and Paterson [16] and [17], Losert and Rindler [19], Yuan [28] for locally compact groups, and by Ling [18] and the authors [21] for discrete semigroups.

In this paper, we investigate inner invariant means on $L^\infty(S, M\mu(S))$ and its closed subspace $C_b(S)$ of all bounded continuous complex-valued functions on $S$. We also study topological inner invariant means on certain closed subspaces $X$ of $L^\infty(S, M\mu(S))$ and their relation with inner invariant means on $X$.

2 Topological inner invariant means

Given any $\mu \in M\mu(S)$ and $g \in L^\infty(S, M\mu(S))$, define the complex-valued functions $g \circ \mu$ and $\mu \circ g$ on $S$ by
\[
(g \circ \mu)(s) = \mu(sg) \quad \text{and} \quad (\mu \circ g)(s) = \mu(gs)
\]
for all $s \in S$. It is clear that
\[
(g \circ \mu)(s) = (\delta_x \ast \mu)(g) \quad \text{and} \quad (\mu \circ g)(s) = (\mu \ast \delta_x)(g)
\]
and so $g \circ \mu$ and $\mu \circ g$ are in $C_b(S)$ with
\[
\| g \circ \mu \|_\infty \leq \| g \|_\infty \| \mu \| \quad \text{and} \quad \| \mu \circ g \|_\infty \leq \| g \|_\infty \| \mu \|.
\]
A closed subspace $X$ of $L^\infty(S; M\mu(S))$ is called topologically invariant if
\[
X \circ M\mu(S) \subseteq X \quad \text{and} \quad M\mu(S) \circ X \subseteq X.
\]
Let \( LUC(S) \) (resp. \( RUC(S) \)) be the space of all left (resp. right) uniformly continuous functions on \( S \); recall that a function \( f \in C_b(S) \) is called left (resp. right) uniformly continuous if the mapping \( s \mapsto s f \) (resp. \( s \mapsto f_s \)) from \( S \) into \( C_b(S) \) is \( \| \cdot \|_{\infty} \)-continuous. Also, a function \( f \in C_b(S) \) is called uniformly continuous if \( f \) is in \( UC(S) := LUC(S) \cap RUC(S) \).

It follows from the equalities

\[
s(f \circ \mu) = s f \circ \mu \quad \text{and} \quad (\mu \circ f)_s = \mu \circ f_s
\]

for all \( s \in S, f \in C_b(S) \) and \( \mu \in M_a(S) \) that

\[
LUC(S) \circ M_a(S) \subseteq LUC(S) \quad \text{and} \quad M_a(S) \circ RUC(S) \subseteq RUC(S).
\]

Before we state the following lemma which is needed in the sequel, let us recall that \( S \) is called foundation semigroup if \( \bigcup \{ \text{supp}(\mu) : \mu \in M_a(S) \} \) is dense in \( S \). Foundation semigroups form a large class of locally compact semigroups which includes locally compact groups and discrete semigroups as elementary examples; as another example, consider the semigroup \( S := [0,1] \) with the usual topology of the real line and the operation \( xy = \min\{x+y,1\} \) defines a compact foundation semigroup with identity; indeed,

\[
M_a(S) = L^1([0,1]) \oplus \mathbb{C} \delta_1.
\]

Moreover, the additive semigroup \( S := \mathbb{R}^+ \) of all non-negative real numbers with the usual topology defines a non-compact foundation semigroup with identity; indeed,

\[
M_a(S) = L^1(\mathbb{R}^+).
\]

Also, the multiplicative semigroup \( S := \{0,1,1/2,1/3,...\} \) with the restriction of the usual topology of the real line defines a compact foundation semigroup with identity; indeed,

\[
M_a(S) = l^1(S \setminus \{0\}).
\]

**Lemma 2.1.** Let \( S \) be a foundation semigroup with identity. If \( X \) and \( Y \) are closed subspaces of \( L^\infty(S;M_a(S)) \) such that \( UC(S) \subseteq X \subseteq LUC(S) \) and \( UC(S) \subseteq Y \subseteq RUC(S) \). Then

\[
X \subseteq M_a(S) \circ X \subseteq LUC(S) \quad \text{and} \quad Y \subseteq Y \circ M_a(S) \subseteq RUC(S).
\]

In particular, \( UC(S), LUC(S) \) and \( RUC(S) \) are topologically invariant.

**Proof.** Let \( f \in Y \) and \( \mu \in M_a(S) \). It follows from the hypothesis that the map

\[
x \mapsto \mu * \delta_x
\]

from \( S \) into \( M_a(S) \) is norm continuous; see [9], Theorem 5.6.1. This together with

\[
(f \circ \mu)_x = f \circ (\mu * \delta_x) \quad (x \in S)
\]

imply that \( f \circ \mu \in RUC(S) \). It follows that \( Y \circ M_a(S) \subseteq RUC(S) \), in particular,

\[
RUC(S) \circ M_a(S) \subseteq RUC(S),
\]
and thus \( RUC(S) \) is topologically invariant.

On the other hand, for every \( \varepsilon > 0 \), there is a neighbourhood \( U \) of the identity element of \( S \) such that
\[
\|f_x - f\|_\infty < \varepsilon \quad (x \in U).
\]
Since \( S \) is foundation, there exists a probability measure \( e_0 \) in \( M_\mu(S) \) with \( \text{supp} \, (e_0) \subseteq U \). Then
\[
\|f \circ e_0 - f\|_\infty \leq \varepsilon.
\]
Now, let \( (e_\gamma)_{\gamma \in \Gamma} \) be an approximate identity for \( M_\mu(S) \) bounded by one; see [12], Lemma 2.1. Then for each \( \gamma \in \Gamma \) we have
\[
\|f \circ e_\gamma - f\|_\infty \leq \|f \circ e_\gamma - (f \circ e_0) \circ e_\gamma\|_\infty + \|f \circ e_0 - f\|_\infty
\]
\[
\leq \|f - f \circ e_0\|_\infty + \|f \circ (e_0 \ast e_\gamma - e_0)\|_\infty + \|f \circ e_0 - f\|_\infty
\]
\[
\leq 2\varepsilon + \|f\|_\infty \|e_0 \ast e_\gamma - e_0\|.
\]
It follows that \( \|f \circ e_\gamma - f\|_\infty \to 0 \). This together with the Cohen factorization theorem imply that \( Y \subseteq Y \circ M_\mu(S) \); see [11], Theorem 32.5. The proof of the other inclusions are similar. \( \square \)

Let us point out that the second dual \( M_\mu(S)^{**} \) of \( M_\mu(S) \) is a Banach algebra with the first Arens product \( \circ \) defined by the equations
\[
(F \circ H)(f) = F(Hf),
\]
\[
(Hf)(\mu) = H(f\mu),
\]
\[
(f\mu)(v) = f(\mu \ast v)
\]
for all \( F, H \in M_\mu(S)^{**}, f \in M_\mu(S)^* \), and \( \mu, v \in M_\mu(S) \). In the case where, \( S \) is a foundation semigroup with identity, \( M_\mu(S)^* \) can be identified with \( L^\infty(S, M_\mu(S)) \); in fact, the equation
\[
\tau(g)(\mu) := \mu(g) = \int_S f \, d\mu
\]
defines an isometric isomorphism \( \tau \) of \( L^\infty(S; M_\mu(S)) \) into the continuous dual space \( M_\mu(S)^* \) of \( M_\mu(S) \); see Proposition 3.6 of Sleijpen [27]. Moreover, for each \( g \in L^\infty(S, M_\mu(S)) \) and \( \mu \in M_\mu(S) \),
\[
\tau(g \circ \mu) = \mu \, \tau(g) \quad \text{and} \quad \tau(\mu \circ g) = \tau(g) \, \mu.
\]
Let \( X \) be a topologically invariant closed subspace of \( L^\infty(S; M_\mu(S)) \) containing the constant functions and \( m \) be a mean on \( X \); i.e., \( \|m\| = m(1) = 1 \). Recall from [14] that \( m \) is topological inner invariant on \( X \) whenever
\[
\mu \circ m = m \circ \mu \quad (\mu \in M_\mu(S));
\]
or equivalently
\[
m(\mu \circ g) = m(g \circ \mu) \quad (\mu \in M_\mu(S), g \in X.)
\]
Inner invariant means on locally compact topological semigroups

The notion of topological inner invariant means was introduced and studied by the second author [23] for a large class of Banach algebras known as Lau algebras. The subject of Lau algebras originated with the paper [15] published in 1983 by Lau in which he referred to them as $F$-algebras. Later on, in his useful monograph, Pier [25] introduced the name Lau algebra. Let us remark from [22] that $M_{a}(S)$ is a Lau algebra for all foundation semigroups $S$ with identity; in this case, any mixed identity with norm one in $M_{a}(S)^{**}$ is a topological inner invariant mean on $L^\infty(S,M_{a}(S))$.

Proposition 2.2. Let $S$ be a foundation semigroup with identity. Then any topological inner invariant mean on $UC(S)$, $LUC(S)$, or $RUC(S)$ is inner invariant.

Proof. Let $m$ be a topological inner invariant mean on $LUC(S)$. By Lemma 2.1, for each $f$ in $LUC(S)$ we have $f = \mu \circ g$ for some $\mu \in M_{a}(S)$ and $g \in LUC(S)$. Since for each $s \in S$,

$$s(\mu \circ g) = (\mu * \delta_s) \circ g$$
$$g \circ (\mu * \delta_s) = g \circ \mu$$
$$\mu \circ g_s = (\mu \circ g)_s$$

we conclude

$$m(sf) = m(s(\mu \circ g))$$
$$= m((\mu * \delta_s) \circ g)$$
$$= m(g \circ (\mu * \delta_s))$$
$$= m(g \circ \mu)$$
$$= m(\mu \circ g_s)$$
$$= m((\mu \circ g)_s)$$
$$= m(f_s).$$

That is, $m$ is inner invariant on $LUC(S)$. Similar arguments hold for $RUC(S)$ and $UC(S)$.

As a consequence of Proposition 2.2 we have the following improvement of Theorem 3.1 of [20] from locally compact groups to a large class of locally compact semigroups; see also [13] and [14].

Corollary 2.3. Let $S$ be a foundation semigroup with identity and $m$ be a mean on $UC(S)$. Then $m$ is inner invariant if and only if it is topological inner invariant.

Proof. The “if” part follows from Proposition 2.2. To prove the converse, let $m$ be an inner invariant mean on $UC(S)$. Then there is a net $(m_{\gamma})_{\gamma \in \Gamma}$ in $UC(S)^{*}$ such that $m_{\gamma} \to m$ in the weak* topology of $UC(S)^{*}$ and

$$m_{\gamma} = \sum_{i=1}^{n_{\gamma}} c_{i,\gamma} \delta_{h_{i,\gamma}} \quad (\gamma \in \Gamma),$$
where \( s_{i,\gamma} \in \mathcal{S}, c_{i,\gamma} \) are complex numbers with
\[
\sum_{i=1}^{n_\gamma} |c_{i,\gamma}| \leq 1;
\]
see for example [8], page 417, Theorem 10. Now, let \( f \in \text{UC}(\mathcal{S}) \) and \( \mu \in \text{Ma}(\mathcal{S}) \) be a measure with compact support \( C \). Then the sets
\[
\{sf : s \in C\} \quad \text{and} \quad \{fs : s \in C\}
\]
are norm compact in \( \text{UC}(\mathcal{S}) \), and therefore
\[
m_\gamma(sf) \to m(sf) \quad \text{and} \quad m_\gamma(fs) \to m(fs)
\]
uniformly on \( C \) by the Makey-Arens theorem. We thus have
\[
m(f \circ \mu) = \lim_\gamma m_\gamma(f \circ \mu)
\]
\[
= \lim_\gamma \sum_{i=1}^{n_\gamma} c_{i,\gamma} \delta_{s_{i,\gamma}}(f \circ \mu)
\]
\[
= \lim_\gamma \int_{\mathcal{S}} c_{i,\gamma} \delta_{s_{i,\gamma}}(fs) \, d\mu(s)
\]
\[
= \int_{\mathcal{S}} m(fs) \, d\mu(s)
\]
\[
= \lim_\gamma \int_{\mathcal{S}} m_\gamma(fs) \, d\mu(s)
\]
\[
= \lim_\gamma \sum_{i=1}^{n_\gamma} c_{i,\gamma} \int_{\mathcal{S}} \delta_{s_{i,\gamma}}(sf) \, d\mu(s)
\]
\[
= \lim_\gamma \int_{\mathcal{S}} m_\gamma(sf) \, d\mu(s)
\]
\[
= \int_{\mathcal{S}} \mu \circ f \, d\mu.
\]
Since measures with compact supports are norm dense in \( \text{Ma}(\mathcal{S}) \), it follows that \( m \) is topological inner invariant on \( \text{UC}(\mathcal{S}) \). \( \blacksquare \)

Let us remark that an element \( E \in \text{Ma}(\mathcal{S})^{**} \) is called a mixed identity if
\[
\mu \circ E = E \circ \mu \quad (\mu \in \text{Ma}(\mathcal{S})).
\]
It well-known from [3], page 146, that an element \( E \in \text{Ma}(\mathcal{S})^{**} \) is a mixed identity with norm one if and only if it is a weak* cluster point of an approximate identity.
bounded by one in $M_d(S)$; see [14], Theorem 2.3, for other descriptions of mixed identities with norm one in $M_d(S)^{**}$.

Moreover, note that if $S$ is a foundation semigroup with identity, then for every $F \in M_d(S)^{**}$ and $n \in LUC(S)^*$, the functional $F \circ n$ can be defined as an element of $M_d(S)^{**}$ in a way similar to the first Arens product; this is because $\mu \circ g \in LUC(S)$ for all $\mu \in M_d(S)$ and $g \in L^\infty(S, M_d(S))$; see Lemma 2.1 of [12].

The next proposition should be compared with the corresponding result concerning topological left invariant means; see Theorem 4.2.4 of [9]. It should be noted that the standard technic used in [9] does not work in our setting.

**Proposition 2.4.** Let $S$ be a foundation semigroup with identity. If $m$ is a topological inner invariant mean on $LUC(S)$, then $E \circ m$ is a topological inner invariant mean on $L^\infty(S, M_d(S))$ for all mixed identities $E$ in $M_d(S)^{**}$ with norm one.

**Proof.** Let $E \in M_d(S)^{**}$ be a mixed identity with norm one, and $(e_\gamma)$ be an approximate identity for $M_d(S)$ bonded by one such that $e_\gamma$ converges to $E$ in the weak$^*$ topology of $M_d(S)^{**}$. Then for $\mu \in M_d(S)$ and $g \in L^\infty(S, M_d(S))$ we have

$$\|e_\gamma \circ (\mu \circ g) - \mu \circ g\|_\infty = \|(e_\gamma \ast \mu) \circ g\|_\infty \to 0$$

and

$$\|\mu \circ (e_\gamma \circ g) - \mu \circ g\|_\infty = \|(\mu \ast e_\gamma - \mu) \circ g\|_\infty \to 0.$$ 

Now, suppose that $m$ is a topological inner invariant mean on $LUC(S)$. Then

$$\lim_{\gamma} m(e_\gamma \circ (\mu \circ g)) = \lim_{\gamma} m(\mu \circ (e_\gamma \circ g)).$$ 

Since $M_d(S) \circ L^\infty(S, M_d(S)) \subseteq LUC(S)$, it follows that $e_\gamma \circ g \in LUC(S)$ for all $\gamma$ and thus

$$m(\mu \circ (e_\gamma \circ g)) = m((e_\gamma \circ g) \circ \mu) = m(e_\gamma \circ (g \circ \mu)) = (e_\gamma \circ m)(g \circ \mu).$$

This shows that

$$\lim_{\gamma} (e_\gamma \circ m)(\mu \circ g) = \lim_{\gamma} m(e_\gamma \circ (\mu \circ g)) = \lim_{\gamma} m(\mu \circ (e_\gamma \circ g)) = \lim_{\gamma} (e_\gamma \circ m)(g \circ \mu).$$

Since $e_\gamma \circ m$ converges to $E \circ m$ in the weak$^*$ topology of $M_d(S)^{**}$, we get

$$(E \circ m)(\mu \circ g) = \lim_{\gamma} (e_\gamma \circ m)(\mu \circ g) = \lim_{\gamma} (e_\gamma \circ m)(g \circ \mu) = (E \circ m)(g \circ \mu).$$
This implies that $E \odot m$ is a topological inner invariant mean on $L^\infty(S, M_a(S))$ and the proof is complete.

The following result is of independent interest.

**Proposition 2.5.** Let $S$ be a foundation semigroup with identity. If $m$ is a topological inner invariant mean on $C_b(S)$, then $m(\mu \circ g) = m(g \circ \mu)$ for all $\mu \in M_a(S)$ and $g \in L^\infty(S, M_a(S))$. In particular, any extension of $m$ to a mean on $L^\infty(S, M_a(S))$ is topological inner invariant.

**Proof.** Let $\mu, \nu \in M_a(S)$ and $g \in L^\infty(S, M_a(S))$. Then $\nu \circ g, g \circ \mu \in C_b(S)$, and hence we have

\[
m((\nu \ast \mu) \circ g) = m(\nu \circ (\nu \circ g)) = m((\nu \circ g) \circ \mu) = m(\nu \circ (g \circ \mu) = m((g \circ \mu) \circ \nu) = m(g \circ (\nu \ast \mu)).
\]

Now, let $(e_\gamma)$ be an approximate identity for $M_a(S)$. Then for each $\gamma$,

\[
\lim_{\gamma} m((e_\gamma \ast \mu) \circ g) = \lim_{\gamma} m(g \circ (e_\gamma \ast \mu)).
\]

Also,

\[
\| (e_\gamma \ast \mu) \circ g - \mu \circ g \|_{\infty} \to 0
\]

and

\[
\| g \circ (e_\gamma \ast \mu) - g \circ \mu \|_{\infty} \to 0.
\]

It follows that

\[
m(\mu \circ g) = \lim_{\gamma} m((e_\gamma \ast \mu) \circ g) = \lim_{\gamma} m(g \circ (e_\gamma \ast \mu) = m(g \circ \mu).
\]

Therefore, if $M$ is an extension of $m$ from $C_b(S)$ to a mean on $L^\infty(S, M_a(S))$, then $M(\mu \circ g) = M(g \circ \mu)$. Thus $M$ defines a topological inner invariant mean on $L^\infty(S, M_a(S))$.

### 3 Inner invariant means on $L^\infty(S, M_a(S))$

In this section we shall be concerned with the inner invariant means on $L^\infty(S, M_a(S))$ for a locally compact semigroup $S$. Before, we give our first result, let us recall that a subset $A$ of $S$ is called $M_a(S)$-measurable if it is $\mu$-measurable for all $\mu \in M_a(S)$.
Lemma 3.1. Let $S$ be a left (resp. right) cancellative locally compact semigroup such that $xA$ (resp. $Ax$) are $M_a(S)$-measurable for all $M_a(S)$-measurable subset $A$ of $S$. Then the space of inner invariant functionals on $L^\infty(S, M_a(S))$ is the linear span of inner invariant means.

Proof. Let $F$ be an inner invariant functional on $L^\infty(S, M_a(S))$. We have to show that $F$ is a linear span of some inner invariant means on $L^\infty(S, M_a(S))$. Without loss of generality we may assume that $F$ is nonzero and self-adjoint. In view of 1.14.3 of [26], there are unique positive functionals $F^+$ and $F^-$ on $L^\infty(S, M_a(S))$ such that

$$F = F^+ - F^- \quad \text{and} \quad \|F\| = \|F^+\| + \|F^-\|.$$ 

The result then will follow if we show that $F^+$ and $F^-$ are inner invariant functionals. This is because that if $F^+$ (resp. $F^-$) is nonzero, then the mean $F^+(1)^{-1}F^+$ (resp. $F^-(1)^{-1}F^-$) is inner invariant.

To this end, let $s \in S$, and $sF$ be the linear functional on $L^\infty(S, M_a(S))$ defined by

$$(sF)(g) = F(sg) \quad (g \in L^\infty(S, M_a(S))).$$

Then there are unique positive functionals $(sF)^+$ and $(sF)^-$ on $L^\infty(S, M_a(S))$ such that

$$sF = (sF)^+ - (sF)^- \quad \text{and} \quad \|sF\| = \|(sF)^+\| + \|(sF)^-\|.$$ 

We show that $(sF)^+ = sF^+$ and $(sF)^- = sF^-$. By uniqueness and that $sF = sF^+ - sF^-$ we only need to prove that

$$\|(sF)^+\| = \|sF^+\| \quad \text{and} \quad \|(sF)^-\| = \|sF^-\|.$$ 

Using the fact that

$$\|(sF)^+\| = (sF)^+(1) = \sup \{F(sg) : g \in L^\infty(S, M_a(S)), \ 0 \leq g \leq 1\}$$

$$\|(sF)^-\| = (sF)^-(1) = - \inf \{F(sg) : g \in L^\infty(S, M_a(S)), \ 0 \leq g \leq 1\}$$

and

$$\|sF^+\| = F^+(1) = \sup \{F(g) : g \in L^\infty(S, M_a(S)), \ 0 \leq g \leq 1\}$$

$$\|sF^-\| = F^-(1) = - \inf \{F(g) : g \in L^\infty(S, M_a(S)), \ 0 \leq g \leq 1\}.$$ 

By the hypothesis it suffices to show that the two sets

$$\{F(sg) : g \in L^\infty(S, M_a(S)), \ 0 \leq g \leq 1\}$$

and

$$\{F(g) : g \in L^\infty(S, M_a(S)), \ 0 \leq g \leq 1\}$$

are the same. To see this, let $g \in L^\infty(S, M_a(S))$ with $0 \leq g \leq 1$. If $S$ is left cancellative, then for each $s \in S$ and $t \in sS$, let $s^{-1}t$ denote the unique element $y$ of $S$ for which $t = sy$. Now, we may define $g' : S \to \mathbb{C}$ for each $t \in S$ by

$$g'(t) = \begin{cases} g(s^{-1}t) & t \in sS \\ 0 & t \not\in sS \end{cases}$$
Then $g'$ is well-defined. Moreover, for any open subset $V$ of $\mathbb{C}$ we have
\[
(g')^{-1}(V) = \begin{cases} 
  s(S \cap g^{-1}(V)) & 0 \not\in V \\
  s(S \cap g^{-1}(V)) \cup (S \setminus sS) & 0 \in V 
\end{cases}
\]
Thus $g'$ is $M_a(S)$-measurable by the hypothesis and that $g \in L^\infty(S, M_a(S))$. It follows that
\[
g' \in L^\infty(S, M_a(S)), \quad \text{with} \quad 0 \leq g' \leq 1 \quad \text{and} \quad g = s g'.
\]
In the case where $S$ is right cancellative, in a similar way it can be proved that there is
\[
g' \in L^\infty(S, M_a(S)), \quad \text{with} \quad 0 \leq g' \leq 1 \quad \text{and} \quad g = g' s.
\]
In both cases, since $F$ is inner invariant, we have $F(s g') = F(g')$.

By a similar argument we have $(F s)^+ = F^+.s$ and $(F s)^- = F^- .s$ for all $s \in S$, where $F.s$ is the linear functional on $L^\infty(S, M_a(S))$ defined by
\[
(F s)(g) = F(g s) \quad (g \in L^\infty(S, M_a(S))).
\]
Therefore
\[
s F^+ = (s F)^+ = (F s)^+ = F^+.s,
\]
and
\[
s F^- = (s F)^- = (F s)^- = F^- .s.
\]
That is $F^+$ and $F^-$ are inner invariant as required.

In the next theorem, we denote by $\mathcal{H}(S)$ (resp. $\mathcal{H}_R(S)$) the complex (resp. real) linear span of functions of the form $s g - g_s$ for some $s \in S$ and complex-valued (resp. real-valued) functions $g \in L^\infty(S, M_a(S))$.

**Theorem 3.2.** Let $S$ be a locally compact semigroup and consider the following statements.

(a) There is an inner invariant mean on $L^\infty(S, M_a(S))$.

(b) $\sup \{h(s) : s \in S\} \geq 0$ for all $h \in \mathcal{H}_R(S)$.

(c) $\inf \{\|1 - h\|_\infty : h \in \mathcal{H}(S)\} = 1$.

(d) $\mathcal{H}(S)$ is not norm dense in $L^\infty(S, M_a(S))$.

Then $(a) \iff (b) \iff (c) \implies (d)$. If $S$ is as in Lemma 3.1, then (a)-(d) are equivalent.

**Proof.** (a)$\implies$(b). If $m$ is an inner invariant mean on $L^\infty(S, M_a(S))$, then for each $h \in \mathcal{H}_R(S)$ we have
\[
\sup \{h(s) : s \in S\} \geq m(h) = 0.
\]

(b)$\implies$(c). Suppose on the contrary that $\inf \{\|1 - h\|_\infty : h \in \mathcal{H}(S)\} < 1$. Then
\[
\sup \{-\text{Re} \ h(s) : s \in S\} < 0
\]
for some $h \in \mathcal{H}(S)$. This together with that $-\text{Re} \ h \in \mathcal{H}_R(S)$ contradict (b). Now, (c) follows from that $0 \in \mathcal{H}(S)$.
The implications \((c) \implies (d)\) and \((c) \implies (a)\) follow from the fact that by the Hahn-Banach theorem, there is \(n \in L^\infty(S, M_a(S))^*\) with norm one such that \(n(H(S)) = \{0\}\), and
\[
n(1) = \inf\{\|1 - h\|_\infty : h \in H(S)\}.
\]
The rest of the proof follows at once from Lemma 3.1.

In the following result, let \(L^\infty(G)\) be the usual Lebesgue space of all essentially bounded measurable functions on a locally compact group \(G\), and note that \(L^\infty(G) = L^\infty(G, M_a(G))\).

**Corollary 3.3.** Let \(G\) be a locally compact group. Then the following statements are equivalent

(a) There is an inner invariant mean on \(L^\infty(G)\).

(b) \(\sup\{h(x) : x \in G\} \geq 0\) for all \(h \in H_R(G)\).

(c) \(\inf\{\|1 - h\|_\infty : h \in H(G)\} = 1\).

(d) \(H(G)\) is not norm dense in \(L^\infty(G)\).

Before we give the next result, let us recall that a family \((A_\gamma)_{\gamma \in D}\) of sets is upward directed if \(D\) is a directed set and \(A_\gamma \subseteq A_\beta\) when \(\gamma \leq \beta\).

**Proposition 3.4.** Let \((S_\gamma)_{\gamma \in D}\) be an upward directed family of locally compact subsemigroups of a locally compact semigroup \(S\). If for each \(\gamma \in D\), there exists an inner invariant mean on \(L^\infty(S_\gamma, M_a(S_\gamma))\), then there exists an inner invariant mean on \(L^\infty(\bigcup_{\gamma \in D} S_\gamma, M_a(\bigcup_{\gamma \in D} S_\gamma))\).

**Proof.** By Theorem 3.2, we only need to note that if \(h \in H_R(\bigcup_{\gamma \in D} S_\gamma)\), then \(h \in H_R(S_\gamma)\) for some \(\gamma \in D\).

As an immediate consequence of Proposition 3.4, we obtain

**Corollary 3.5.** Let \(S\) be a locally compact semigroup. If there is an inner invariant mean on \(L^\infty(S_0, M_a(S_0))\) for all finitely generated subsemigroups \(S_0\) of \(S\), then there is an inner invariant mean on \(L^\infty(S, M_a(S))\).

Let \(S_0\) be a subset of a locally compact semigroup \(S\). We say that a mean \(m\) on \(L^\infty(S, M_a(S))\) is inner \(S_0\)-invariant if \(m(\chi g) = m(g x)\) for all \(x \in S_0\) and \(g \in L^\infty(S, M_a(S))\).

**Proposition 3.6.** Suppose that \(S_0\) is a closed subsemigroup of a locally compact semigroup \(S\). Then there exists an inner invariant mean on \(L^\infty(S_0, M_a(S_0))\) if and only if there is an inner \(S_0\)-invariant mean \(m\) on \(L^\infty(S, M_a(S))\) with \(m(\chi_{S_0}) = 1\).

**Proof.** Suppose that \(n\) is an inner invariant mean on \(L^\infty(S_0, M_a(S_0))\). Since \(g|_{S_0}\), the restriction of \(g\) to \(S_0\), belongs to \(L^\infty(S_0, M_a(S_0))\) for all \(g\) in \(L^\infty(S, M_a(S))\), the map
\[
m : g \mapsto n(g|_{S_0})
\]
defines a mean on \(L^\infty(S, M_a(S))\). Moreover, \(m(\chi_{S_0}) = 1\) trivially, and also \(m\) is inner \(S_0\)-invariant. Indeed, for each \(t \in S_0\) and \(g \in L^\infty(S, M_a(S))\) we have
\[
m(t g - g t) = n((t g - g t)|_{S_0}) = n(t(g|_{S_0}) - (g|_{S_0}) t).
\]
Conversely, suppose $m$ is an inner $S_0$-invariant mean on $L^\infty(S, M_a(S))$ with $m(\chi_{S_0}) = 1$. For every $f \in L^\infty(S_0, M_a(S_0))$, let $\tilde{f} : S \rightarrow C$ be the function which is equal to $f$ on $S_0$ and zero on $S \setminus S_0$. Since the restriction of $\mu$ to $S_0$ is in $M_a(S_0)$ for all $\mu \in M_a(S)$, it follows easily that $\tilde{f}$ is $M_a(S)$-measurable. That is $\tilde{f} \in L^\infty(S, M_a(S))$. Thus the linear functional

$$n : f \mapsto m(\tilde{f})$$

defines a mean on $L^\infty(S_0, M_a(S_0))$. Furthermore, $(f_s\tilde{\chi}) = \tilde{f}_s$ on $S_0$ for all $s \in S_0$ and $f \in L^\infty(S_0, M_a(S_0))$, and therefore,

$$|(f_s\tilde{\chi}) - \tilde{f}_s| \leq \|f\|_\infty \chi_{S \setminus S_0}.$$ 

It follows that $n((f_s\tilde{\chi})) = n(\tilde{f}_s)$. Similarly, $n((s f \tilde{\chi})) = n(s \tilde{f})$. That is $n$ is an inner invariant mean on $L^\infty(S_0, M_a(S_0))$ as required. 

4 Inner invariant means on $C_b(S)$

Let $S$ be a locally compact semigroup. In the case where $S$ has an identity $e$, $C_b(S)$ has always an inner invariant mean; in fact, $\delta_e$ is an inner invariant mean on $C_b(S)$. However, this is not true in general; for example, consider a left zero semigroup with at least two elements. In this section, we study the existence of inner invariant means on $C_b(S)$. Before we give our first result of this section, let us remind that a mean $m$ on $C_b(S)$ is called two-sided invariant if

$$m(s f) = m(f_s) = m(f) \quad (s \in S, f \in C_b(S)).$$

**Proposition 4.1.** Let $S_1$ and $S_2$ be two locally compact semigroups. If $C_b(S_1)$ has a two-sided invariant mean and $C_b(S_2)$ has an inner invariant mean, then $C_b(S_1 \times S_2)$ has an inner invariant mean.

**Proof.** Let $m$ be a two-sided invariant mean on $C_b(S_1)$ and $n$ be an inner invariant mean on $C_b(S_2)$. For each $f \in C_b(S_1 \times S_2)$, define the function $f_2 \in C_b(S_2)$ by

$$f_2(t) = m(f_1^t) \quad (t \in S_2),$$

where $f_1^t \in C_b(S_1)$ is defined by

$$f_1^t(s) = f(s, t) \quad (s \in S_1).$$

It follows that

$$(f \chi x)_1^t = x^{f_1^t} \quad \text{and} \quad (f \chi x)_1^t = (f_1^t y)_x$$

for all $x \in S_1$ and $y, t \in S_2$. Moreover,

$$f_2(t) = m((x, y) f)_1^t = m((x, y) f)_2(t)$$

$$= m((x, y) f)_2(t)$$

$$= m(f_1^t y)$$

$$= y(f_2)(t),$$
Proposition 4.3. Let \( S_1 \) and \( S_2 \) be non-trivial locally compact semigroups with identities \( e_1 \) and \( e_2 \) respectively. Suppose that \( C_b(S_1) \) has a two-sided invariant mean and \( C_b(S_2) \) has an inner invariant mean. Then there is an inner invariant mean on \( C_b(S_1 \times S_2) \).

Proof. Let \( m \) be a two-sided invariant mean on \( C_b(S_1) \), and \( M \) be the inner invariant mean on \( C_b(S_1 \times S_2) \) defined as in the proof of Proposition 4.1. Since \( C_b(S_1) \) separates the points of \( S_1 \), \( m(f) \neq m(\theta) \) for some \( \theta \in C_b(S_1) \). Now, define the function \( f \in C_b(S_1 \times S_2) \) by \( f(s_1, s_2) = \theta(s_1) \) for all \( s_1 \in S_1 \) and \( s_2 \in S_2 \). Then

\[
M(f) = m(f) \neq m(\theta) = f(e_1, e_2).
\]

Therefore \( M \neq \delta_{(e_1, e_2)} \) as required.

Corollary 4.2. Let \( S_1 \) and \( S_2 \) be non-trivial locally compact semigroups with identities \( e_1 \) and \( e_2 \) respectively. Suppose that \( C_b(S_1) \) has a two-sided invariant mean and \( C_b(S_2) \) has an inner invariant mean. Then there is an inner invariant mean on \( C_b(S_1 \times S_2) \) not equal to \( \delta_{(e_1, e_2)} \).

Proof. Let \( m \) be a two-sided invariant mean on \( C_b(S_1) \), and \( M \) be the inner invariant mean on \( C_b(S_1 \times S_2) \) defined as in the proof of Proposition 4.1. Since \( C_b(S_1) \) separates the points of \( S_1 \), \( m(f) \neq m(\theta) \) for some \( \theta \in C_b(S_1) \). Now, define the function \( f \in C_b(S_1 \times S_2) \) by \( f(s_1, s_2) = \theta(s_1) \) for all \( s_1 \in S_1 \) and \( s_2 \in S_2 \). Then

\[
M(f) = m(f) \neq m(\theta) = f(e_1, e_2).
\]

Therefore \( M \neq \delta_{(e_1, e_2)} \) as required.

Let \( S_0 \) be a subset of a locally compact semigroup \( S \). We say that a mean \( m \) on \( C_b(S) \) is inner \( S_0 \)-invariant if \( m(gs) = m(g(s)) \) for all \( s \in S_0 \) and \( g \in C_b(S) \).

Proposition 4.3. Suppose that \( S_1 \) and \( S_2 \) are two locally compact semigroups and \( \theta \) is a continuous homomorphism from \( S_1 \) into \( S_2 \). If there is an inner invariant mean on \( C_b(S_1) \), then there is an inner \( \theta(S_1) \)-invariant mean \( m \) on \( C_b(S_2) \).

Proof. First note that if \( s_1 \in S_1 \), then

\[
(s_2 g - g s_2) \circ \theta = s_1 (g \circ \theta) - (g \circ \theta) s_1,
\]

where \( s_2 = \theta(s_1) \). Indeed, for each \( s \in S_1 \) we have

\[
[(s_2 g - g s_2) \circ \theta](s) = g(\theta(s_1)\theta(s)) - g(\theta(s)\theta(s_1)) = g(\theta(s_1s)) - g(\theta(ss_1)) = (g \circ \theta)(s_1s) - (g \circ \theta)(ss_1) = s_1(g \circ \theta)(s) - (g \circ \theta)s_1(s).
\]

Therefore \( m \neq \delta_{(s_1, s_2)} \) as required.
Now, suppose \( n \) is an inner invariant mean on \( C_b(S_1) \). Then the mean
\[
m \colon g \mapsto n(g \circ \theta)
\]
is inner \( \theta(S_1) \)-invariant on \( C_b(S_2) \). In fact, for each \( s_2 \in S_2 \) with \( s_2 = \theta(s_1) \) for some \( s_1 \in S_1 \), we have
\[
m(s_2 g - g s_2) = n(s_1 (g \circ \theta) - (g \circ \theta) s_1) = 0.
\]
This establishes the proof. \( \blacksquare \)

Let \( C \) be a congruence relation on \( S \); that is, an equivalence relation such that \( x C y \) implies both \( xs C ys \) and \( sx C sy \) \((x, y, s \in S)\). We denote by \( S/C \) the semigroup of all equivalence classes \( x/C \) \((x \in S)\) induced by \( C \) with the usual operation
\[
(x/C) (y/C) = xy/C \quad (x, y \in S).
\]
The quotient space \( S/C \) endowed with the quotient topology is in general not a locally compact semigroup in our sense; see [4], pages 46-50. Observe that if \( C \) is a congruence relation on a locally compact semigroup \( S \) such that \( S/C \) is a locally compact semigroup, then the natural map \( \phi : S \mapsto S/C \) is a continuous homomorphism.

**Corollary 4.4.** Let \( C \) be a closed congruence relation such that \( S/C \) is a locally compact semigroup. If there is an inner invariant mean on \( C_b(S) \), then there is an inner invariant mean on \( C_b(S/C) \).

**Proof.** The canonical map \( s \mapsto s/C \) from \( S \) onto \( S/C \) is a continuous homomorphism. So the result follows from Proposition 4.3 \( \blacksquare \)

**Corollary 4.5.** Let \( S \) be a \( \sigma \)-compact locally compact semigroup and \( S_0 \) be a closed ideal of \( S \). If there is an inner invariant mean on \( C_b(S) \), then there is an inner invariant mean on \( C_b(S/S_0) \).

**Proof.** From Theorem 1.57 of [4], it follows that \( S/S_0 \) is a locally compact semigroup. So, the result follows from Proposition 4.3 and Corollary 4.4. \( \blacksquare \)

Let \( \{S_i : i \in I\} \) be a family of locally compact semigroups. The full direct product \( \Pi_{i \in I} S_i \) of \( \{S_i : i \in I\} \) is the set of all functions \( \phi \) defined on \( I \) with \( \phi(i) \in S_i \) for \( i \in I \). Note that \( \Pi_{i \in I} S_i \) equipped with the binary operation \((\phi, \psi) \mapsto \phi \cdot \psi \) defined by
\[
(\phi \cdot \psi)(i) = \phi(i) \psi(i) \quad (i \in I)
\]
is a semigroup. Moreover, if every \( S_i \) is a locally compact semigroup, then \( \Pi_{i \in I} S_i \) together with the product topology is also a locally compact semigroup.

**Corollary 4.6.** Suppose that \( \{S_i : i \in I\} \) is a family of locally compact semigroups. If there is an inner invariant mean on \( C_b(\Pi_{i \in I} S_i) \), then for each \( i \in I \), there is an inner invariant mean on \( C_b(S_i) \).
Proof. Set $S := \prod_{i \in I} S_i$ and note that for each $i \in I$, the projection map $\phi \mapsto \phi(i)$ from $S$ onto $S_i$ is a continuous homomorphism. So, the result follows from Proposition 4.3.

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References


