Riemann-Stieltjes operators between different weighted Bergman spaces

Songxiao Li   Stevo Stević

Abstract

Let $g : B \to \mathbb{C}^1$ be a holomorphic map of the unit ball B. We give a complete picture regarding the boundedness and compactness of the following two integral operators

$$T_g f(z) = \int_0^1 f(tz)\Re g(tz)\frac{dt}{t} \quad \text{and} \quad L_g f(z) = \int_0^1 \Re f(tz)g(tz)\frac{dt}{t}, \quad z \in B,$$

between different weighted Bergman spaces.

1 Introduction

Let $B = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball in $\mathbb{C}^n$, $S = \partial B = \{z \in \mathbb{C}^n : |z| = 1\}$ be its boundary, $d\nu$ the normalized Lebesgue measure on B, i.e. $\nu(B) = 1$, and $d\nu_\alpha(z) = c_\alpha(1-|z|^2)^\alpha d\nu(z)$, where $c_\alpha = \Gamma(n+\alpha+1)/(\Gamma(n+1)\Gamma(\alpha+1))$. Let $H(B)$ denote the class of all holomorphic functions on the unit ball. For $f \in H(B)$ with the Taylor expansion $f(z) = \sum_{|\beta|\geq 0} a_\beta z^\beta$, let $\Re f(z) = \sum_{|\beta|\geq 0} |\beta| a_\beta z^\beta$ be the radial derivative of f, where $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ is a multi-index and $z^\beta = z_1^{\beta_1} \cdots z_n^{\beta_n}$. It is well known that $\Re f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)$, (see, for example, [17]).

Let $\beta(z, w)$ be the distance between z and w in the Bergman metric of B. For any $r > 0$ and $z \in B$, we write $E(z, r) = \{w \in B : \beta(z, w) < r\}$. The volume of $E(z, r)$ is given by (see [17])

$$\nu(E(z,r)) = \frac{R^{2n}(1-|z|^2)^{n+1}}{(1-R^2|z|^2)^{n+1}},$$

Received by the editors December 2006 - In revised form in October 2007.
Communicated by F. Bastin.
2000 *Mathematics Subject Classification* : Primary 47B38, Secondary 30H05.
Key words and phrases : Riemann-Stieltjes operator, Bergman space, boundedness, compactness.

where $R = \tanh(r)$. Set $|E(z,r)| = \nu(E(z,r))$. For $w \in E(z,r)$, $r > 0$, we have that (see, for example, [17])

$$(1-|z|^2)^{n+1} \asymp (1-|w|^2)^{n+1} \asymp |1-\langle z,w\rangle|^{n+1} \asymp |E(z,r)|. \tag{1}$$

For any $\zeta \in S$ and $r > 0$, the set $Q_r(\zeta)$ is defined by

$$Q_r(\zeta) = \{z \in B : |1-\langle z,\zeta\rangle| < r\}. \tag{2}$$

A positive Borel measure μ on B is called a γ-Carleson measure if there exists a constant $C > 0$ such that

$$\mu(Q_r(\zeta)) \le Cr^\gamma \tag{3}$$

for all $\zeta \in S$ and $r > 0$. A well-known result about the γ-Carleson measure ([15]), is that μ is a γ-Carleson measure if and only if

$$\sup_{a\in B} \int_B \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^\gamma d\mu(z) < \infty. \tag{4}$$

For $p \in (0,\infty)$ and $\alpha > -1$, the weighted Bergman space $A^p_\alpha(B) = A^p_\alpha$ is defined to be the space of all holomorphic functions f on B such that

$$\|f\|^p_{A^p_\alpha} = \int_B |f(z)|^p d\nu_\alpha(z) = c_\alpha \int_B |f(z)|^p (1-|z|^2)^\alpha d\nu(z) < \infty.$$

When $\alpha = 0$, $A^p_0(B) = A^p(B)$ is the standard Bergman space. It is known that $f \in A^p_\alpha$ if and only if $(1-|z|^2)\Re f(z) \in L^p(B, d\nu_\alpha)$. Moreover

$$\|f\|^p_{A^p_\alpha} \asymp |f(0)|^p + \int_B |\Re f(z)|^p (1-|z|^2)^p d\nu_\alpha(z). \tag{5}$$

See [16, 17] for some basic facts on Bergman spaces.

Given $g \in H(B)$, the Riemann-Stieltjes or Extended-Cesàro operator T_g with symbol g is defined on $H(B)$ as follows

$$T_g f(z) = \int_0^1 f(tz)\frac{dg(tz)}{dt} = \int_0^1 f(tz)\Re g(tz)\frac{dt}{t}, \qquad z \in B,$$

where $f \in H(B)$. This operator was introduced in [2], and studied in [2, 3, 4, 5, 6, 7, 8, 11, 14].

Similarly, we define the operator

$$L_g f(z) = \int_0^1 \Re f(tz) g(tz)\frac{dt}{t}, \qquad z \in B.$$

In [14] Xiao gave the characterization on g for which the Riemann-Stieltjes operator T_g is bounded or compact on the weighted Bergman space A^p_α. Hu considered the boundedness and compactness of T_g on the weighted Bergman space $L^p_{a,\omega}$, see [4].

The purpose of this paper is to study the boundedness and compactness of operators T_g and L_g between different weighted Bergman spaces. This paper can also be considered as a natural continuation of our investigations in [5, 6, 7, 8, 11]. For related results in the case of the unit polydisk see [12, 13].

Throughout this paper, C will stand for a positive constant, whose value may differ from one occurrence to the other. The expression $a \asymp b$ means that there is a positive constant C such that $C^{-1}a \le b \le Ca$.

2 Auxiliary Results

Here we state some auxiliary results which are incorporated in the following lemmas.

Lemma 1. *For every $f, g \in H(B)$ it holds*

$$\Re[T_g(f)](z) = f(z)\Re g(z) \quad \text{and} \quad \Re[L_g(f)](z) = \Re f(z) g(z).$$

The proof of the first identity can be found in [2], while the second identity can be proved similarly (see [6]).

The following criterion for compactness follows from standard arguments similar, for example, to those outlined in Proposition 3.11 of [1].

Lemma 2. *Assume that $g \in H(B)$, $\alpha, \beta > -1$ and $0 < p, q < \infty$. Then the operator T_g (or L_g) : $A^p_\alpha \to A^q_\beta$ is compact if and only if T_g (or L_g) : $A^p_\alpha \to A^q_\beta$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in A^p_α which converges to zero uniformly on compact subsets of B, we have $\|T_g f_k\|_{A^q_\beta} \to 0$ (or $\|L_g f_k\|_{A^q_\beta} \to 0$) as $k \to \infty$.*

Lemma 3. *Assume that $g \in H(B)$, $\alpha, \beta > -1$ and $q \geq p > 0$. Then the following two conditions are equivalent.*

(a)

$$b_g = \sup_{z\in B} |g(z)|(1-|z|^2)^{\frac{n+1+\beta}{q} - \frac{n+1+\alpha}{p}} < \infty; \tag{6}$$

(b)

$$M := \sup_{a\in B} \int_B \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{(n+1+\alpha+p)q/p} |g(z)|^q (1-|z|^2)^q d\nu_\beta(z) < \infty. \tag{7}$$

Proof. Let $t = \frac{n+1+\beta}{q} - \frac{n+1+\alpha}{p}$. By the subharmonicity of the function $|g|^q$ and (1), it follows that

$$\begin{aligned} &\left(|g(z)|(1-|z|^2)^t\right)^q \\ \leq\ & C\frac{(1-|z|^2)^{tq}}{|E(z,r)|}\int_{E(z,r)} |g(w)|^q d\nu(w) \\ \leq\ & C\int_{E(z,r)} \left(\frac{1-|z|^2}{|1-\langle z,w\rangle|^2}\right)^{(n+1+\alpha+p)q/p} |g(w)|^q (1-|w|^2)^q d\nu_\beta(w), \end{aligned} \tag{8}$$

from which easily follows that (7) implies (6).

Now assume that (6) holds. Then, from (6) and by a well-known estimate (see, for example, Theorem 1.12 in [17]), we have

$$\begin{aligned} &\sup_{a\in B\setminus B(0,1/2)} \int_B \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{(n+1+\alpha+p)q/p} |g(z)|^q (1-|z|^2)^q d\nu_\beta(z) \\ \leq\ & b_g^q \sup_{a\in B\setminus B(0,1/2)} (1-|a|^2)^{(n+1+\alpha+p)q/p} \int_B \frac{(1-|z|^2)^{q-qt+\beta}}{|1-\langle z,a\rangle|^{2(n+1+\alpha+p)q/p}} d\nu(z) \\ \leq\ & C. \end{aligned} \tag{9}$$

On the other hand, since $q \geq p$ we have that

$$\begin{aligned} &\sup_{a\in B(0,1/2)} \int_B \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{(n+1+\alpha+p)q/p} |g(z)|^q(1-|z|^2)^q d\nu_\beta(z) \\ \leq\;& C\int_B (1-|z|^2)^{\frac{q(p+\alpha)+(n+1)(q-p)}{p}} d\nu(z) < \infty. \end{aligned} \tag{10}$$

From (9) and (10) the result follows.

Lemma 4. *Assume that $g \in H(B)$, $\alpha, \beta > -1$ and $q \geq p > 0$. Then the following two conditions are equivalent.*

(a)

$$\lim_{|z|\to 1} |g(z)|(1-|z|^2)^{\frac{n+1+\beta}{q}-\frac{n+1+\alpha}{p}} = 0; \tag{11}$$

(b)

$$\lim_{|a|\to 1} \int_B \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{(n+1+\alpha+p)q/p} |g(z)|^q(1-|z|^2)^q d\nu_\beta(z) = 0. \tag{12}$$

Proof. That (12) implies (11), follows from estimate (8).

On the other hand, if (11) holds, then for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|g(z)|(1-|z|^2)^{\frac{n+1+\beta}{q}-\frac{n+1+\alpha}{p}} < \varepsilon,$$

whenever $\delta < |z| < 1$. From this we have that

$$\begin{aligned} &\int_B \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{(n+1+\alpha+p)q/p} |g(z)|^q(1-|z|^2)^q d\nu_\beta(z) \\ \leq\;& \varepsilon^q(1-|a|^2)^{(n+1+\alpha+p)q/p}\int_{B\setminus\delta B} \frac{(1-|z|^2)^{q-qt+\beta}}{|1-\langle z,a\rangle|^{2(n+1+\alpha+p)q/p}} d\nu(z) \\ &+\int_{\delta B} \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{(n+1+\alpha+p)q/p} |g(z)|^q(1-|z|^2)^q d\nu_\beta(z) \\ \leq\;& C\varepsilon^q + C\frac{\max_{|z|=\delta}|g(z)|^q}{(1-\delta)^{2(n+1+\alpha+p)q/p}}(1-|a|^2)^{(n+1+\alpha+p)q/p}. \end{aligned} \tag{13}$$

Letting $|a| \to 1$ in (13) and using the fact that ε is an arbitrary positive number the result follows.

3 Main Results

In this section we formulate and prove the main results of this paper.

Theorem 1. *Suppose that $g \in H(B)$, $0 < p \leq q < \infty$, $\alpha, \beta > -1$. Then*

(a) $T_g : A^p_\alpha \to A^q_\beta$ is bounded if and only if

$$\sup_{a\in B} |\Re g(a)|(1-|a|^2)^{1+\frac{n+1+\beta}{q}-\frac{n+1+\alpha}{p}} < \infty. \tag{14}$$

(b) $L_g : A^p_\alpha \to A^q_\beta$ *is bounded if and only if* (6) *holds.*

Proof. (a) It is easy to see that $T_g f(0) = 0$. By (5) and Lemma 1, we have

$$\begin{aligned} \|T_g f\|^q_{A^q_\beta} &\asymp \int_B |\Re(T_g f)(z)|^q (1-|z|^2)^q d\nu_\beta(z) \\ &= \int_B |\Re g(z)|^q |f(z)|^q (1-|z|^2)^q d\nu_\beta(z) = \int_B |f(z)|^q d\mu_1(z), \end{aligned} \tag{15}$$

where

$$d\mu_1(z) = |\Re g(z)|^q (1-|z|^2)^q d\nu_\beta(z). \tag{16}$$

By Theorem 50 of [16], we see that $T_g : A^p_\alpha \to A^q_\beta$ is bounded if and only if

$$\mu_1(Q_r(\zeta)) \le C r^{(n+1+\alpha)q/p}.$$

From this and (4), we have that

$$\sup_{a\in B} \int_B \left(\frac{1-|a|^2}{|1-\langle z, a\rangle|^2}\right)^{(n+1+\alpha)q/p} |\Re g(z)|^q (1-|z|^2)^q d\nu_\beta(z) < \infty.$$

From Theorem 2.1 of [9], we find that the above inequality is equivalent to (14).

(b) Similar to the previous case, we have that

$$\|L_g f\|^q_{A^q_\beta} \asymp \int_B |\Re f(z)|^q d\mu_2(z), \tag{17}$$

where

$$d\mu_2(z) = |g(z)|^q (1-|z|^2)^q d\nu_\beta(z). \tag{18}$$

From (17) and by Theorem 50 of [16], we find that $L_g : A^p_\alpha \to A^q_\beta$ is bounded if and only if

$$\mu_2(Q_r(\zeta)) \le C r^{(n+1+\alpha+p)q/p}.$$

By (4), we obtain (7). From this and by employing Lemma 3 the result follows.

Theorem 2. *Suppose that* $g \in H(B)$, $0 < p \le q < \infty$, $\alpha, \beta > -1$. *Then*

(a) $T_g : A^p_\alpha \to A^q_\beta$ *is compact if and only if*

$$\lim_{|a|\to 1} |\Re g(a)|(1-|a|^2)^{1+\frac{n+1+\beta}{q}-\frac{n+1+\alpha}{p}} = 0. \tag{19}$$

(b) $L_g : A^p_\alpha \to A^q_\beta$ *is compact if and only if* (11) *holds.*

Proof. (a) Suppose that $T_g : A^p_\alpha \to A^q_\beta$ is compact. Assume that $(a_k)_{k\in\mathbb{N}}$ is a sequence in B such that $\lim_{k\to\infty} |a_k| = 1$. Set

$$f_k(z) = \left(\frac{1-|a_k|^2}{(1-\langle z, a_k\rangle)^2}\right)^{\frac{n+1+\alpha}{p}}, \qquad k \in \mathbb{N}. \tag{20}$$

By Theorem 1.12 of [17], we see that there exists a constant C such that $\sup_{k\in\mathbb{N}} \|f_k\|_{A^p_\alpha} \le C$. Also, it is easy to see that the sequence $(f_k)_{k\in\mathbb{N}}$ converges to 0 uniformly on compact subsets of B. By Lemma 2, we have that $\|T_g f_k\|_{A^q_\beta} \to 0$ as $k \to \infty$. Hence, in view of Lemma 1, from (5) and by letting $k \to \infty$, we have that

$$\begin{aligned} &\lim_{k\to\infty} \int_B \left(\frac{1-|a_k|^2}{|1-\langle z, a_k\rangle|^2}\right)^{(n+1+\alpha)q/p} |\Re g(z)|^q (1-|z|^2)^q d\nu_\beta(z) \\ =\ &\lim_{k\to\infty} \int_B |\Re(T_g f_k)(z)|^q (1-|z|^2)^q d\nu_\beta(z) \\ \asymp\ &\lim_{k\to\infty} \|T_g f_k\|^q_{A^q_\beta} = 0. \end{aligned}$$

This implies

$$\lim_{|a|\to 1} \int_B \left(\frac{1-|a|^2}{|1-\langle z, a\rangle|^2}\right)^{(n+1+\alpha)q/p} |\Re g(z)|^q (1-|z|^2)^q d\nu_\beta(z) = 0. \tag{21}$$

From Theorem 3.1 of [9], we see that (21) is equivalent to (19), as desired.

Conversely, suppose that (19) holds, that is, (21) holds. Then for any fixed $\varepsilon > 0$, there exists $\eta_0 \in (0,1)$ such that

$$\int_B \left(\frac{1-|a|^2}{|1-\langle z, a\rangle|^2}\right)^{\frac{q(n+1+\alpha)}{p}} d\mu_1(z) < \varepsilon \tag{22}$$

for all $a \in B$ with $\eta_0 < |a| < 1$, where μ_1 is defined in (16). Let $r_0 = 1 - \eta_0$. For $\zeta \in S$ and $r \in (0, r_0)$, let $a = (1-r)\zeta$. Then $a \in B$, $\eta_0 < |a| < 1$,

$$|1 - \langle z, a\rangle| < 2r \quad \text{and} \quad 1 - |a|^2 \geq r,$$

for each $z \in Q_r(\zeta)$. Hence

$$\left(\frac{1-|a|^2}{|1-\langle z, a\rangle|^2}\right)^{\frac{q(n+1+\alpha)}{p}} \geq \left(\frac{r}{(2r)^2}\right)^{\frac{q(n+1+\alpha)}{p}} = \frac{1}{(4r)^{\frac{q(n+1+\alpha)}{p}}} \tag{23}$$

for each $z \in Q_r(\zeta)$. From (22) and (23), we obtain

$$\begin{aligned} \frac{\mu_1(Q_r(\zeta))}{4^{\frac{q(n+1+\alpha)}{p}} r^{\frac{q(n+1+\alpha)}{p}}} &\leq \int_{Q_r(\zeta)} \left(\frac{1-|a|^2}{|1-\langle z, a\rangle|^2}\right)^{\frac{q(n+1+\alpha)}{p}} d\mu_1(z) \\ &\leq \int_B \left(\frac{1-|a|^2}{|1-\langle z, a\rangle|^2}\right)^{\frac{q(n+1+\alpha)}{p}} d\mu_1(z) < \varepsilon \end{aligned}$$

for all $r \in (0, r_0)$ and $\zeta \in S$. Let $\varepsilon > 0$ be fixed and $\widetilde{\mu_1} \equiv \mu_1 \mid_{B\setminus(1-r_0)\overline{B}}$. As in the proof of [10, Theorem 1.1], we obtain that there exists a constant $C > 0$ such that

$$\widetilde{\mu_1}(Q_r(\zeta)) \leq C\varepsilon r^{\frac{q(n+1+\alpha)}{p}}, \tag{24}$$

for every $r > 0$. Suppose that $(f_k)_{k\in\mathbb{N}}$ is a sequence in A^p_α which converges to 0 uniformly on compact subsets of B and satisfies $\sup_{k\in\mathbb{N}} \|f_k\|_{A^p_\alpha} \leq L$. By Lemma 1, we have

$$\begin{aligned} \|T_g f_k\|^q_{A^q_\beta} &\asymp \int_B |\Re g(z)|^q |f_k(z)|^q (1-|z|^2)^q d\nu_\beta(z) \\ &= \int_B |f_k(z)|^q d\widetilde{\mu_1}(z) + \int_{(1-r_0)\overline{B}} |f_k(z)|^q d\mu_1(z). \end{aligned} \tag{25}$$

By (24) and utilizing the method of Theorem 1.1 of [10], it follows that there exists a positive constant C such that

$$\int_B |f_k|^q d\widetilde{\mu_1} \leq C\varepsilon \|f_k\|^q_{A^p_\alpha} \leq CL^q\varepsilon, \tag{26}$$

for each $k \in \mathbb{N}$. Moreover, $f_k \to 0$ uniformly on $(1-r_0)\overline{B}$ implies that the second term in (25) can be made small enough for sufficiently large k. From this and since μ_1 is finite, it follows that

$$\lim_{k\to\infty} \int_{(1-r_0)\overline{B}} |f_k(z)|^q d\mu_1(z) = 0. \tag{27}$$

Estimate (25), together with (26) and (27) gives that $\|T_g f_k\|_{A^q_\beta} \to 0$ as $k \to \infty$. Employing Lemma 2, the result follows.

(b) Suppose that $L_g : A^p_\alpha \to A^q_\beta$ is compact. Further assume that $(a_k)_{k\in\mathbb{N}}$ is a sequence in B such that $\lim_{k\to\infty} |a_k| = 1$. Set

$$h_k(z) = (1-|a_k|^2)^{\frac{n+1+\alpha+p}{p}} \int_0^1 \left(\frac{1}{(1-\langle tz, a_k\rangle)^{\frac{2(n+1+\alpha+p)}{p}}} - 1 \right) \frac{dt}{t}. \tag{28}$$

By using (5), the fact that $h_k(0) = 0$, and Theorem 1.12 of [17], we obtain

$$\begin{aligned} \|h_k\|^p_{A^p_\alpha} &\asymp \int_B |\Re h_k(z)|^p (1-|z|^2)^{\alpha+p} d\nu(z) \\ &= \int_B \frac{(1-|a_k|^2)^{n+1+\alpha+p}}{|1-\langle z, a_k\rangle|^{2(n+1+\alpha+p)}} (1-|z|^2)^{\alpha+p} d\nu(z) \\ &\leq C. \end{aligned} \tag{29}$$

Hence $\sup_{k\in\mathbb{N}} \|h_k\|_{A^p_\alpha} \leq C$. Clearly $h_k \to 0$ uniformly on compact subsets of B. Therefore, by Lemma 2 we have that $\|L_g h_k\|_{A^q_\beta} \to 0$ as $k \to \infty$. Hence

$$\begin{aligned} &\lim_{k\to\infty} \int_B \left(\frac{1-|a_k|^2}{|1-\langle z, a_k\rangle|^2} \right)^{(n+1+\alpha+p)q/p} |g(z)|^q (1-|z|^2)^q d\nu_\beta(z) \\ &= \lim_{k\to\infty} \int_B |\Re(L_g h_k)(z)|^q (1-|z|^2)^q d\nu_\beta(z) \\ &\asymp \lim_{k\to\infty} \|L_g h_k\|^q_{A^q_\beta} = 0. \end{aligned} \tag{30}$$

From (30) we see that (12) holds, and by Lemma 4 that (11) holds. The remainder of the proof is similar to the proof of part (a) and will be omitted.

Remark 1. Note that when $1 + \frac{n+1+\beta}{q} - \frac{n+1+\alpha}{p} \leq 0$, then the symbol g in (14) and (19) is a constant, while when $t := \frac{n+1+\beta}{q} - \frac{n+1+\alpha}{p} < 0$, then the symbol g in (6) and (11) is a constant. Note also that if $t > 0$ then conditions (6) and (14) are equivalent.

Theorem 3. *Suppose that $g \in H(B)$, $0 < q < p < \infty$, $\alpha, \beta > -1$. Then the following statements are equivalent.*

(a) $T_g : A^p_\alpha \to A^q_\beta$ is bounded;

(b) $T_g : A^p_\alpha \to A^q_\beta$ is compact;

(c) $g \in A^r_\gamma$, where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, and $\frac{\gamma}{r} = \frac{\beta}{q} - \frac{\alpha}{p}$.

Proof. From the proof of Theorem 1 we know that

$$\|T_g f\|^q_{A^q_\beta} \asymp \int_B |f(z)|^q d\mu_1(z),$$

where $d\mu_1$ is defined by (16). By Theorem 54 of [16], we know that (a) and (b) are equivalent and both are equivalent to the following condition

$$\int_B \frac{(1-|a|^2)^{n+1+\alpha}}{|1-\langle z,a\rangle|^{2(n+1+\alpha)}} d\mu_1(z) \in L^{p/(p-q)}(\nu_\alpha),$$

which is the same as

$$\int_B |\Re g(z)|^q (1-|z|^2)^q \frac{(1-|a|^2)^{n+1+\alpha}}{|1-\langle z,a\rangle|^{2(n+1+\alpha)}} d\nu_\beta(z) \in L^{p/(p-q)}(\nu_\alpha). \tag{31}$$

By the subharmonicity of $|\Re g|^q$, using Lemma 24, p.59 in [17] and (1),

$$\begin{aligned}
&\int_B |\Re g(z)|^q (1-|z|^2)^q \frac{(1-|a|^2)^{n+1+\alpha}}{|1-\langle z,a\rangle|^{2(n+1+\alpha)}} d\nu_\beta(z) \\
\geq\ & \int_{E(a,\rho)} |\Re g(z)|^q (1-|z|^2)^q \frac{(1-|a|^2)^{n+1+\alpha}}{|1-\langle z,a\rangle|^{2(n+1+\alpha)}} d\nu_\beta(z) \\
\geq\ & (1-|a|^2)^{q-n-1-\alpha} \int_{E(a,\rho)} |\Re g(z)|^q d\nu_\beta(z) \\
\geq\ & (1-|a|^2)^{q+\beta-\alpha} |\Re g(a)|^q.
\end{aligned} \tag{32}$$

Therefore (31) implies that

$$(1-|a|^2)^{q+\beta-\alpha} |\Re g(a)|^q \in L^{p/(p-q)}(\nu_\alpha),$$

which is the same as

$$\int_B |\Re g(a)|^r (1-|a|^2)^{r+\gamma} d\nu(a) < \infty.$$

By (5) we get that $g \in A^r_\gamma$.

Conversely, if $g \in A^r_\gamma$, then by Hölder's inequality, we get

$$\begin{aligned}
&\|T_g f\|^q_{A^q_\beta} \asymp \int_B |f(z)|^q |\Re g(z)|^q (1-|z|^2)^q d\nu_\beta(z) \\
\leq\ & \left(\int_B |\Re g(z)|^r (1-|z|^2)^{(q+\frac{q\gamma}{r})\frac{r}{q}} d\nu(z)\right)^{\frac{q}{r}} \left(\int_B |f(z)|^{\frac{qr}{r-q}} (1-|z|^2)^{(\beta-\frac{q\gamma}{r})\frac{r}{r-q}} d\nu(z)\right)^{1-\frac{q}{r}}.
\end{aligned}$$

From this, since

$$\left(q+\frac{q\gamma}{r}\right)\frac{r}{q} = r+\gamma, \quad \frac{qr}{r-q} = p, \quad 1-\frac{q}{r} = \frac{q}{p}$$

and

$$\left(\beta - \frac{q\gamma}{r}\right)\frac{r}{r-q} = \frac{r\beta - q\gamma}{r-q} = \frac{\frac{\beta}{q} - \frac{\gamma}{r}}{\frac{1}{q} - \frac{1}{r}} = \alpha,$$

and by using (5), it follows that

$$\|T_g f\|^q_{A^q_\beta} \le C\|g\|^q_{A^r_\gamma}\|f\|^q_{A^p_\alpha},$$

which means that the operator $T_g : A^p_\alpha \to A^q_\beta$ is bounded, finishing the proof of the theorem.

Theorem 4. *Suppose that* $g \in H(B)$, $0 < q < p < \infty$, $\alpha, \beta > -1$. *Then the following statements are equivalent.*
(a) $L_g : A^p_\alpha \to A^q_\beta$ *is bounded;*
(b) $L_g : A^p_\alpha \to A^q_\beta$ *is compact;*
(c) $g \in A^r_\gamma$, *where* $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, *and* $\frac{\gamma}{r} = \frac{\beta}{q} - \frac{\alpha}{p}$.

Proof. By Theorem 54 of [16], we know that $L_g : A^p_\alpha \to A^q_\beta$ is bounded if and only if

$$\int_B |g(z)|^q \frac{(1-|a|^2)^{n+1+\alpha+p}}{|1-\langle z,a\rangle|^{2(n+1+\alpha+p)}}(1-|z|^2)^q d\nu_\beta(z) \in L^{p/(p-q)}(\nu_{\alpha+p}).$$

The remainder of the proof is similar to the proof of Theorem 3, therefore is omitted.

Acknowledgments. The authors would like to thank the referee for many helpful suggestions. The first author of this paper is supported by the NSF of Guangdong Province (No.7300614).

References

[1] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, 1995.

[2] Z. Hu, Extended Cesàro operators on mixed norm spaces, *Proc. Amer. Math. Soc.* **131** (7) (2003), 2171-2179.

[3] Z. Hu, Extended Cesàro operators on the Bloch space in the unit ball of $\mathbb{C}^n$, *Acta Math. Sci. Ser. B Engl. Ed.* **23** (4)(2003), 561-566.

[4] Z. Hu, Extended Cesàro operators on Bergman spaces, *J. Math. Anal. Appl.* **296** (2004), 435-454.

[5] S. Li, Riemann-Stieltjes operators from $F(p, q, s)$ to Bloch space on the unit ball, *J. Inequal. Appl.* Vol. 2006, Article ID 27874, (2006), 14 pages.

[6] S. Li and S. Stević, Riemann-Stieltjes type integral operators on the unit ball in $\mathbb{C}^n$, *Complex Variables Elliptic Functions* **52** (6) (2007), 495-517.

[7] S. Li and S. Stević, Compactness of Riemann-Stieltjes operators between $F(p,q,s)$ and α-Bloch spaces, *Publ. Math. Debrecen* **72** (1-2) (2008), 111-128.

[8] S. Li and S. Stević, Riemann-Stieltjes operators on Hardy spaces in the unit ball of $\mathbb{C}^n$, *Bull. Belg. Math. Soc. Simon Stevin*, **14** (2007), 621-628.

[9] B. Li and C. H. Ouyang, Higher radial derivative of Bloch type functions, *Acta Math Scientia*, **22B** (4) (2002), 433-445.

[10] B. D. MacCluer, Compact composition operators on $H^p(B_N)$, *Michigan Math. J.* **32** (1985), 237-248.

[11] S. Stević, On an integral operator on the unit ball in $\mathbb{C}^n$, *J. Inequal. Appl.* **1** (2005), 81-88.

[12] S. Stević, Boundedness and compactness of an integral operator on a weighted space on the polydisc, *Indian J. Pure Appl. Math.* **37** (6) (2006), 343-355.

[13] S. Stević, Boundedness and compactness of an integral operator on mixed norm spaces on the polydisc, *Sibirsk. Mat. Zh.* **48** (3) (2007), 694-706.

[14] J. Xiao, Riemann-Stieltjes operators on weighted Bloch and Bergman spaces of the unit ball, *J. London. Math. Soc.* **70** (2) (2004), 199-214.

[15] W. S. Yang, Carleson type measure characterizations of Q_p spaces, *Analysis*, **18** (1998), 345-349.

[16] R. Zhao and K. Zhu, Theory of Bergman spaces in the unit ball of $\mathbb{C}^n$, *Mém. Soc. Math. Fr.* (to appear) 2007.

[17] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Graduate Texts in Mathematics, Springer, New York, 2005.

Department of Mathematics, JiaYing University,
514015, Meizhou, GuangDong, China
E-mail address: jyulsx@163.com, lsx@mail.zjxu.edu.cn

Mathematical Institute of the Serbian Academy of Science,
Knez Mihailova 35/I, 11000 Beograd, Serbia
E-mail address: sstevic@ptt.rs