Subordinations for domains bounded by conic sections

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Abstract

We establish some conditions under which the differential subordination of the type $p(z) + zp'(z)/p(z) \prec Q(z)$ yields $p \prec q$ in $\mathcal{U}$. Functions Q and q are chosen so that they map the unit disk onto domains enclosed by conic sections. Some applications of obtained results are given.

1 Introduction

We denote by $\mathcal{H}$ the set of functions of the form

$$f(z) = z + a_2 z^2 + \cdots , \tag{1.1}$$

analytic in the unit disk $\mathcal{U}$. By $\mathcal{S}$ we denote the subclass of $\mathcal{H}$ of *univalent* functions. Also let $\mathcal{CV}$, $\mathcal{ST}$, $\mathcal{ST}(\alpha)$ ($\alpha \in [0,1)$), and $\mathcal{U}CV$ denote subclasses of $\mathcal{S}$ consisting of *convex, starlike, starlike of order* α and *uniformly convex* functions, respectively.

Recently, Kanas and Wiśniowska ([7]) introduced subfamilies of univalent functions called *k-uniformly convex* and *k-starlike*, with $0 \le k < \infty$, and denoted k-$\mathcal{UCV}$ and k-$\mathcal{ST}$, respectively. The analytic conditions of those classes are following (cf. [3], [4], [7], [8], [10]):

$$k\text{-}\mathcal{UCV} = \left\{ f \in \mathcal{H} : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > k \left|\frac{zf''(z)}{f'(z)}\right|,\ z \in \mathcal{U} \right\}, \tag{1.2}$$

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and

$$k\text{-}\mathcal{ST} = \{f \in \mathcal{H} : \operatorname{Re}[zf'(z)/f(z)] > k\,|zf'(z)/f(z) - 1|\,,\ z \in \mathcal{U}\}\,. \tag{1.3}$$

Note that k-$\mathcal{UCV}$ and k-$\mathcal{ST}$ are connected by the well known Alexander relation, i.e. $f \in k$-$\mathcal{UCV}$ if and only if $zf'(z) \in k$-$\mathcal{ST}$. Also, we note that geometrically k-$\mathcal{UCV}$ is a subclass of univalent functions that map circular arcs, contained in the unit disk, with centers contained in the disk of the radius $k, (k \in [0, \infty))$, onto convex arcs. The similar class but only for the case $k = 1$ was studied by Goodman ([1]). Observe that in the case $k = 0$ families k-$\mathcal{UCV}$ and k-$\mathcal{ST}$ coincide with well known classes of convex ($\mathcal{C}V$) and starlike ($\mathcal{S}T$) functions, respectively.

Let the functions g and G be analytic in the unit disc $\mathcal{U}$. The function g is said to be *subordinate* to G, written $g \prec G$ (or $g(z) \prec G(z),\ z \in \mathcal{U}$), if G is univalent in $\mathcal{U},\ g(0) = G(0)$ and $g(\mathcal{U}) \subset G(\mathcal{U})$. When $g(z) = \psi(p(z), zp'(z); z)$ with ψ, p analytic and having appropriate normalization, then the subordination is called *the first-order differential subordination.*

The theory of subordination has significant meaning in the theory of univalent functions; the notion of subordination is also often used for defining classes of functions. For example k-$\mathcal{UCV}$ and k-$\mathcal{ST}$ can be rewritten in terms of subordination as $1 + zf''(z)/f'(z) \in \Omega_k$, where

$$\Omega_k = \left\{u + iv :\ u^2 > k^2(u-1)^2 + k^2v^2, u > 0\right\}.$$

The family of plane domains Ω_k consists of domains convex, symmetric about real axis and bounded by conic sections. Denoting by p_k the conformal mapping that maps $\mathcal{U}$ onto Ω_k, we obtain the family of conformal mappings depending on k ($k \in [0, \infty)$). Therefore, for each fixed k, the family k-$\mathcal{UCV}$ is the family of all $f \in \mathcal{H}$ for which $1 + zf''(z)/f'(z) \prec p_k(z)$ and k-$\mathcal{ST}$ consists of all functions $f \in \mathcal{H}$ such that $zf'(z)/f(z) \prec p_k(z), z \in \mathcal{U}$.

We briefly recall one of the problem that characterize the theory of differential subordinations; for more details the reader should consult the monograph due to Miller and Mocanu ([13]). Assume that p is a function with positive real part and such that $p(0) = a$, $p(z) \not\equiv a$. Also, let $\psi : \mathbb{C}^2 \times \mathcal{U} \longrightarrow \mathcal{U}$ be analytic and such that $\psi(p(0), 0; 0) = a$. Given ψ and univalent function q, $q(0) = p(0)$, find the "largest" function Q, such that $Q(0) = a$, and the relation

$$\psi(p(z), zp'(z); z) \prec Q(z) \Longrightarrow p(z) \prec q(z) \tag{1.4}$$

holds. The second problem of this theory is to find the "smallest" function q for given Q, and so that (1.4) holds.

The above mentioned problems were investigated in a numerous papers (cf. e.g. [2], [11], [12], [13], [14], [15]), for various choice of Q and q. However, authors mostly consider cases when functions q and Q map the unit disk onto half-plane, sector or a disk. Thus they can not be strictly applied for k-$\mathcal{UCV}$ and k-$\mathcal{ST}$. The aim of the paper is the study of the relation (1.4) when Q or q is p_k. We choose function ψ such that it provides the containment results for k-$\mathcal{UCV}$ and k-$\mathcal{ST}$ and related subclasses of univalent functions. Finally, we compare obtained conclusions with the classical results from the univalent function theory.

Now, we recall the lemma basic in the theory of differential subordinations.

Lemma 1.1 ([12]). *Let h be an analytic function on $\overline{\mathcal{U}}$ except for at most one pole on $\partial\mathcal{U}$, and univalent on $\overline{\mathcal{U}}$, p be an analytic function in $\mathcal{U}$ with $p(0) = h(0)$ and $p(z) \not\equiv p(0)$, $z \in \mathcal{U}$. If p is not subordinate to h, then there exist points $z_0 \in \mathcal{U}$, $\zeta_0 \in \partial\mathcal{U}$ and $m \geq 1$ for which*

$$p(|z| < |z_0|) \subset h(\mathcal{U}), \quad p(z_0) = h(\zeta_0), \quad z_0\, p'(z_0) = m\, \zeta_0\, h'(\zeta_0).$$

2 Main results

Theorem 2.1. *Let $k \in [0, \infty)$. Also, let p be an function analytic in the unit disk such that $p(0) = 1$. If*

$$\mathrm{Re}\ \left(p(z) + \frac{zp'(z)}{p(z)}\right) - k\left|p(z) - 1 + \frac{zp'(z)}{p(z)}\right| > 0, \tag{2.1}$$

then

$$p(z) \prec \frac{1 + (1-2\alpha)z}{1-z} =: h(z) \tag{2.2}$$

where $\alpha \geq \alpha(k)$, and $\alpha(k)$ is given by

$$\alpha(k) = \frac{1}{4}\left[\sqrt{\left(\frac{1-k}{1+k}\right)^2 + 8} - \frac{1-k}{1+k}\right]. \tag{2.3}$$

Proof. We may assume that $\alpha \geq 1/2$ since the condition $\mathrm{Re}\ (p + zp'(z)/p(z)) > 0$ implies at least $\mathrm{Re}\ p(z) > 1/2$ (c.f., e.g. [13], p. 60).

Suppose now, on the contrary, that $p \not\prec h$. Then, by Lemma 1.1 and results of Miller and Mocanu ([12]), there exists $z_0 \in \mathcal{U}$, $\zeta_0 \in \partial\mathcal{U}$, and $m \geq 1$ such that

$$p(z_0) = \alpha + ix, \quad z_0 p'(z_0) = my, \quad \text{where} \quad y \leq -\frac{(1-\alpha)^2 + x^2}{2(1-\alpha)}, \quad x, y \in \mathbb{R}.$$

Making use of the above relations, we thus have

$$\begin{aligned} & \mathrm{Re}\ \left(p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)}\right) - k\left|p(z_0) - 1 + \frac{z_0 p'(z_0)}{p(z_0)}\right| \\ = \ & \mathrm{Re}\ \left(\alpha + ix + \frac{my}{\alpha + ix}\right) - k\left|\alpha - 1 + ix + \frac{my}{\alpha + ix}\right| \\ = \ & \alpha + \frac{\alpha my}{\alpha^2 + x^2} - k\frac{\sqrt{[\alpha(1-\alpha) + x^2 - my]^2 + x^2(2\alpha - 1)^2}}{\sqrt{\alpha^2 + x^2}} \end{aligned}$$

$$\begin{aligned}
&\leq \ \alpha - \frac{\alpha}{2(1-\alpha)} \frac{(1-\alpha)^2 + x^2}{\alpha^2 + x^2} \\
&\quad - k\frac{\sqrt{\left[\alpha(1-\alpha) + x^2 + \frac{1-\alpha}{2} + \frac{x^2}{2(1-\alpha)}\right]^2 + x^2(2\alpha-1)^2}}{\sqrt{\alpha^2+x^2}} \\
&= \ \alpha - \frac{\alpha}{2(1-\alpha)} \frac{(1-\alpha)^2 + x^2}{\alpha^2 + x^2} \\
&\quad - k\frac{\sqrt{\left[\frac{(1-\alpha)(2\alpha+1)}{2} + x^2\frac{3-2\alpha}{2(1-\alpha)}\right]^2 + x^2(2\alpha-1)^2}}{\sqrt{\alpha^2+x^2}} =: r(x).
\end{aligned}$$

The function $r(x)$ is even as regards x, and attains its maximum at $x = 0$ when $\alpha \geq 1/2$. Indeed, denoting

$$A = \frac{3-2\alpha}{2(1-\alpha)}, \quad B = \frac{(1-\alpha)(2\alpha+1)}{2}, \quad C = 2\alpha - 1,$$

we have

$$\begin{aligned}
r'(x) \ &= \ \frac{-x}{(\alpha^2+x^2)^{3/2}} \left[\frac{\alpha(2\alpha-1)}{(1-\alpha)\sqrt{\alpha^2+x^2}}\right. \\
&\quad + \left. k\frac{A^2x^4 + 2A^2\alpha^2x^2 + B(2A^2\alpha^2 - B) + C^2\alpha^2}{\sqrt{(B+Ax^2)^2 + C^2x^2}}\right].
\end{aligned}$$

Then $r'(x) = 0$ if and only if $x = 0$, since the expression in the square brackets is positive ($B(2A\alpha^2 - B) = \frac{2\alpha+1}{4}(-6\alpha^3+9\alpha^2-1) = \frac{2\alpha+1}{4}[3\alpha(1-\alpha)(2\alpha-1)+3\alpha-1] > 0$ when $\alpha \geq 1/2$), for $x \in \mathbb{R}$. Thus $r(x)$ attains its maximum at $x = 0$, and we have

$$r(x) \leq r(0) = \alpha - \frac{1-\alpha}{2\alpha} - k\frac{(1-\alpha)(2\alpha+1)}{2\alpha} = 0, \tag{2.4}$$

for $\alpha = \alpha(k)$, as given by (2.3), that contradicts the assumption. ∎

Now, we consider other form of the function $\psi(p(z), zp'(z); z)$ that provides various containment results for classes k-$\mathcal{UCV}$ and k-$\mathcal{ST}$.

Theorem 2.2. *Let $\delta \in (0, 2]$ and $k \in [0, \infty)$. Also, let p be the function analytic in the unit disk such that $p(0) = 1$. If*

$$\operatorname{Re}\left(1 + \delta\frac{zp'(z)}{p(z)}\right) - k\delta\left|\frac{zp'(z)}{p(z)}\right| > 0, \tag{2.5}$$

then $p(z) \prec h(z)$, where h is given by (2.2), and $\alpha \geq \alpha(k, \delta)$ where

$$\alpha(k, \delta) = \frac{\delta(k+1)}{2 + \delta(k+1)}. \tag{2.6}$$

Proof. As in the proof of Theorem 2.1 we may assume that $\alpha \geq 1/2$. Supposing, on the contrary that $p \not\prec h$, and making use of Lemma 1.1, we obtain

$$\begin{aligned}
&\mathrm{Re}\,\left(1+\delta\frac{z_0p'(z_0)}{p(z_0)}\right) - k\delta\left|\frac{z_0p'(z_0)}{p(z_0)}\right| \\
=\;& \mathrm{Re}\,\left(1+\frac{m\delta y}{\alpha+ix}\right) - k\delta\left|\frac{my}{\alpha+ix}\right| \\
=\;& 1+\frac{\alpha\delta my}{\alpha^2+x^2} - k\delta\frac{m|y|}{\sqrt{\alpha^2+x^2}} \\
\leq\;& 1-\frac{m\alpha\delta}{2(1-\alpha)}\frac{(1-\alpha)^2+x^2}{\alpha^2+x^2} - \frac{k\,m\,\delta}{2(1-\alpha)}\frac{(1-\alpha)^2+x^2}{\sqrt{\alpha^2+x^2}} \\
\leq\;& 1-\frac{\alpha\delta}{2(1-\alpha)}\frac{(1-\alpha)^2+x^2}{\alpha^2+x^2} - \frac{k\delta}{2(1-\alpha)}\frac{(1-\alpha)^2+x^2}{\sqrt{\alpha^2+x^2}} =: s(x).
\end{aligned}$$

The function $s(x)$ attains its maximum at the point $x=0$. Indeed,

$$s'(x) = \frac{-\delta\,x}{2(1-\alpha)}\left[\frac{2\alpha(2\alpha-1)}{(\alpha^2+x^2)^2} + k\frac{x^2+\alpha^2+2\alpha-1}{(\alpha^2+x^2)^{3/2}}\right] = 0$$

if and only if $x=0$ since, for $\alpha \geq 1/2$, the expressions $\alpha^2+2\alpha-1$ and $2\alpha-1$ are nonnegative, and

$$s''(0) = \frac{-\delta}{2(1-\alpha)}\left[\frac{2(2\alpha-1)+k(\alpha^2+2\alpha-1)}{\alpha^3}\right] < 0.$$

Thus, we have

$$s(x) \leq s(0) = 1-\frac{\delta(1-\alpha)}{2\alpha} - k\frac{\delta(1-\alpha)}{2\alpha} = 0$$

for $\alpha = \alpha(k,\delta)$ where $\alpha(k,\delta)$ is given by (2.7). ∎

The above Theorems are key results for obtaining some inclusions for k-uniformly convex and k-starlike functions and its connections with other well known subclasses of univalent functions. Applying Theorem 2.1 we may formulate the following:

Theorem 2.3. *Let $k \in [0,\infty)$. If $f \in k$-$\mathcal{UCV}$, then $f \in \mathcal{ST}(\alpha)$, where $\alpha \geq \alpha(k)$, with $\alpha(k)$ is given in* (2.3).

Proof. Assume $f \in k$-$\mathcal{UCV}$. Then, by (1.2),

$$\mathrm{Re}\,\left(1+\frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right| \quad z \in \mathcal{U}.$$

Setting $p(z) = zf'(z)/f(z),\ p(0)=1$, the above condition can be rewritten as

$$\mathrm{Re}\,\left(p(z)+\frac{zp'(z)}{p(z)}\right) > k\left|p(z)-1+\frac{zp'(z)}{p(z)}\right|$$

or

$$\mathrm{Re}\,\left(p(z)+\frac{zp'(z)}{p(z)}\right) - k\left|p(z)-1+\frac{zp'(z)}{p(z)}\right| > 0.$$

Now, applying the Theorem 2.1, we obtain the assertion. ∎

Remark. Observe, that in the case, when $k = 0$, we recover the classical sharp result that each convex function is at least $1/2$ starlike. Also, we have $\alpha(1) = \sqrt{2}/2 \approx 0.705$, that gives related result for uniformly convex functions. It improves the result concerning the order of starlikeness for uniformly convex functions ([9]).

Using Theorem 2.2 for the case, when $\delta = 1$ we have:

Theorem 2.4. *Let* $0 \le k < \infty$*. If* $f \in k\text{-}\mathcal{ST}$*, then*

$$\operatorname{Re} f(z)/z > \frac{k+1}{k+3}.$$

Similarly, if $f \in k\text{-}\mathcal{UCV}$*, then*

$$\operatorname{Re} f'(z) > \frac{k+1}{k+3}.$$

Proof. Setting $p(z) = f(z)/z$ (or $p(z) = f'(z)$)), under the assumption that $f \in k\text{-}\mathcal{ST}$ (or $f \in k\text{-}\mathcal{UCV}$, respectively) and making use of Theorem 2.2 the assertion follows. ∎

Setting $\delta = 2$ in Theorem 2.2 we obtain:

Theorem 2.5. *Let* $0 \le k < \infty$*. If* $f \in k\text{-}\mathcal{UCV}$*, then*

$$\operatorname{Re} \sqrt{f'(z)} > \frac{k+1}{k+2},$$

and if $f \in k\text{-}\mathcal{ST}$*, then*

$$\operatorname{Re} \sqrt{f(z)/z} > \frac{k+1}{k+2}.$$

Proof. By putting $p(z) = \sqrt{f'(z)}$ with $f \in k\text{-}\mathcal{UCV}$ (or $f \in k\text{-}\mathcal{ST}$, respectively) and using the Theorem 2.2 we conclude the assertion. ∎

Next, we prove some theorems that will be helpful for comparing the obtained results with the classical ones.

Theorem 2.6. *Let* p *be an analytic function in the unit disk, such that* $p(0) = 1$*. If, for* $\alpha \in [1/2, 1]$*,*

$$\operatorname{Re}\left(1 + \frac{zp'(z)}{p(z)}\right) > \alpha, \quad \text{then} \quad \operatorname{Re} p(z) > \beta,$$

where $\beta \ge \beta(\alpha)$*, and*

$$\beta(\alpha) = \frac{1}{3 - 2\alpha}. \tag{2.7}$$

Proof. Reasoning along the same line as in the proof of Theorem 2.1 and 2.2, and supposing $\operatorname{Re} p \leq \beta$, we have

$$\begin{aligned}\operatorname{Re}\left(1+\frac{zp'(z)}{p(z)}\right) &= 1+m\frac{\beta y}{\beta^2+x^2}\\ &\leq 1-\frac{\beta[(1-\beta)^2+x^2]}{2(1-\beta)(\beta^2+x^2)}\end{aligned}$$

where $x \in \mathbb{R}$, $m \geq 1$ and $y \leq -\frac{(1-\beta)^2+x^2}{2(1-\beta)}$.

It is easy to check that he function $u(x) = 1 - \frac{\beta[(1-\beta)^2+x^2]}{2(1-\beta)(\beta^2+x^2)}$ attains its only maximum at $x = 0$, thus

$$\operatorname{Re}\left(1+\frac{zp'(z)}{p(z)}\right) \leq \frac{3}{2}-\frac{1}{2\beta}$$

that equals to α when $\beta = 1/(3-2\alpha)$. This contradicts the assumption, so that the assertion follows. $\square$

Setting $p(z) = f(z)/z$ in Theorem 2.6 we obtain the following containment result.

Theorem 2.7. *If for some $\alpha \in [1/2, 1]$ $f \in ST(\alpha)$, then*

$$\operatorname{Re}\frac{f(z)}{z} > \frac{1}{3-2\alpha}.$$

Theorem 2.8. *Let p be an analytic function in the unit disk, such that $p(0) = 1$. If, for $\alpha \in \left[\frac{1}{2}, 1\right]$*

$$\operatorname{Re}\sqrt{p(z)+zp'(z)} > \alpha, \quad \textit{then} \quad \operatorname{Re} p(z) > \delta,$$

where $\delta \geq \delta(\alpha)$, and

$$\delta(\alpha) = \frac{2\alpha^2+1}{3}. \tag{2.8}$$

Proof. Supposing, on the contrary, that $\operatorname{Re} p \leq \delta$, we have

$$\begin{aligned}\operatorname{Re}^2\sqrt{p(z)+zp'(z)} &= \frac{1}{2}\left[\delta+my+\sqrt{x^2+(\delta+my)^2}\right]\\ &\leq \frac{3}{4}\delta-\frac{1}{2}-\frac{x^2}{4(1-\delta)}+\frac{1}{2}\sqrt{x^2+\left(\frac{3\delta-1}{2}-\frac{x^2}{2(1-\delta)}\right)^2},\end{aligned}$$

where $x \in \mathbb{R}, m \geq 1$ and $y \leq -\frac{(1-\delta)^2+x^2}{2(1-\delta)}$.

The right hand function $v(x)$ attains its maximum at $x = 0$. Indeed

$$v'(x) = \frac{x}{2}\left[\frac{\frac{3-5\delta}{2(1-\delta)}+\frac{x^2}{2(1-\delta)^2}}{\sqrt{x^2+\left(\frac{3\delta-1}{2}-\frac{x^2}{2(1-\delta)}\right)^2}}-\frac{1}{1-\delta}\right],$$

and $v'(x) = 0$ if and only if

$$x = 0 \quad \text{or} \quad \frac{\frac{3-5\delta}{2(1-\delta)} + \frac{x^2}{2(1-\delta)^2}}{\sqrt{x^2 + \left(\frac{3\delta-1}{2} - \frac{x^2}{2(1-\delta)}\right)^2}} - \frac{1}{1-\delta} = 0.$$

The second equality is satisfied for no $x \in \mathbb{R}$, since after some computations, it reduces to

$$(2\delta - 1)(\delta - 1) = 0.$$

Then $v'(x) = 0$ if and only if $x = 0$ and

$$v''(0) = \frac{5\delta^2 - 8\delta + 2}{(3\delta - 1)(1 - \delta)} < 0$$

for $\delta \in [1/2, 1)$. Thus

$$\mathrm{Re}^{\,2} \sqrt{p(z) + zp'(z)} = \alpha^2$$

when $\delta = (2\alpha^2 + 1)/3$, that contradicts the assumption and the assertion follows. ■

Substituting $p(z) = f(z)/z$ in Theorem 2.8 gives immediately the following result.

Theorem 2.9. *If* $\mathrm{Re}\sqrt{f'(z)} > \alpha$ *for some* $\alpha \in [1/2, 1]$, *then*

$$\mathrm{Re}\,\frac{f(z)}{z} > \frac{2\alpha^2 + 1}{3}.$$

Having in view the above results we can compare the classical results concerning the half-plane with these obtained for domains bounded by conic sections. This comparison shows that such a substitution provides "the step to the right". For instance classical results give

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \implies \mathrm{Re}\,\frac{zf'(z)}{f(z)} > \frac{1}{2} \implies \mathrm{Re}\,\frac{f(z)}{z} > \frac{1}{2},$$

whereas, by Theorems 2.1 and 2.7 we obtain

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right| \implies \mathrm{Re}\,\frac{zf'(z)}{f(z)} > \alpha \implies \mathrm{Re}\,\frac{f(z)}{z} > \beta,$$

where $\alpha = \alpha(k) = (1/4)\left[\sqrt{((1-k)/(1+k))^2 + 8} - (1-k)/(1+k)\right]$, and $\beta = 1/(3 - 2\alpha)$.

Setting $k = 1$, for example, we obtain from the above the following:

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right| \implies \mathrm{Re}\,\frac{zf'(z)}{f(z)} > \frac{\sqrt{2}}{2}$$

$$\implies \mathrm{Re}\,\frac{f(z)}{z} > \frac{3 + \sqrt{2}}{7} \approx 0.63.$$

Also, we know that

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0 \implies \operatorname{Re}\sqrt{f'(z)}>\frac{1}{2} \implies \operatorname{Re}\frac{f(z)}{z}>\frac{1}{2},$$

whereas, applying Theorems 2.5 and 2.9 we conclude

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>k\left|\frac{zf''(z)}{f'(z)}\right| \implies \operatorname{Re}\sqrt{f'(z)}>\frac{k+1}{k+2} \implies \operatorname{Re}\frac{f(z)}{z}>\delta,$$

with $\delta=[2(k+1)^2/(k+2)^2+1]/3$, and when $k=1$ we obtain

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\left|\frac{zf''(z)}{f'(z)}\right| \implies \operatorname{Re}\sqrt{f'(z)}>\frac{2}{3} \implies \operatorname{Re}\frac{f(z)}{z}>\frac{17}{27}.$$

Moreover

$$\operatorname{Re}\frac{zf'(z)}{f(z)}>k\left|\frac{zf'(z)}{f(z)}-1\right| \implies \operatorname{Re}\frac{f(z)}{z}>\frac{k+1}{k+3}.$$

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