# Rotundity, smoothness and drops in Banach spaces

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**Abstract**

In this note we study the geometry of drops in Banach spaces, and we use it to characterize two well known geometrical properties: rotundity and smoothness.

## 1 Introduction

Given a normed space $X$, a drop is a set of the form $\text{co}\,(\{e\} \cup \mathsf{B})$ where $\mathsf{B}$ is a closed ball in $X$ and $e \in X \backslash \mathsf{B}$. Clearly, every drop is a bounded closed convex set with non-empty interior. In [2] it is proved that, in any Banach space $X$, for every closed subset $S$ not containing 0 there exists $x \in S$ so that $\text{co}\,(\{x\} \cup \mathsf{B}_X)$ intersects $S$ at only $x$. This theorem motivated the following definition: a normed space $X$ is said to have the drop property if, for every closed set $S$ disjoint from $\mathsf{B}_X$, there exists $x \in S$ such that the drop $\text{co}\,(\{x\} \cup \mathsf{B}_X)$ intersects $S$ at only $x$. In [4] it is proved that every uniformly rotund Banach space has the drop property, and every Banach space having the drop property is reflexive; Rolewicz then posed the question whether all reflexive Banach spaces can be renormed to have this property. In [3] Montesinos answered Rolewicz's question in the positive, moreover, he proved that a Banach space has the drop property if and only if the space is reflexive and has the Radon-Riesz property. In this note we study the geometry of drops, and we use it to characterize rotundity and smoothness of normed spaces.

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## 2 Drops and rotundity

In this section, we characterize the geometrical property known as rotundity by means of drops. Let us recall that, in a normed space, a bounded closed convex subset with non-empty interior is said to be rotund if its boundary does not contain segments. Equivalently, every point in its boundary is an extreme point. A normed space is said to be rotund if so is its unit ball. As expected, we show in Proposition 2.2 that no drop can be rotund, therefore it is necessary to "modify" the definition of rotundity for drops. Nevertheless, we begin by introducing the following lemma upon which we will base all the results related to rotundity.

**Lemma 2.1.** *Let $X$ be a normed space. Let $\mathsf{D} = \operatorname{co}(\{e\} \cup \mathsf{B})$ be a drop in $X$ an consider $x \in \mathsf{D}$. Then, $x \in \operatorname{int}(\mathsf{D})$ if and only if $x$ can be written in the form $x = \lambda e + (1-\lambda) b$ with $b \in \operatorname{int}(\mathsf{B})$ and $\lambda \in [0,1)$.*

*Proof.* First, assume that $x$ can be written in the form $x = \lambda e + (1-\lambda) b$ with $b \in \operatorname{int}(\mathsf{B})$ and $\lambda \in [0,1)$. Then, there exists $\varepsilon > 0$ such that $\mathsf{B}_X(b, \varepsilon) \subseteq \mathsf{B}$. We will show that, in this situation, $\mathsf{B}_X(x, \varepsilon(1-\lambda)) \subseteq \mathsf{D}$. Let $y \in \mathsf{B}_X(x, \varepsilon(1-\lambda))$ and consider $b' = (y-x)/(1-\lambda) + b$. Then, $b' \in \mathsf{B}_X(b,\varepsilon) \subseteq \mathsf{B}$ and hence $y = \lambda e + (1-\lambda) b' \in \mathsf{D}$.

Conversely, assume that $x \in \operatorname{int}(\mathsf{D})$. First, we will show that $x \neq e$. In order to see this, we will prove that $e \in \operatorname{bd}(\mathsf{D})$. Assume, on the contrary, that there exists a ball centered at $e$ and contained in $\mathsf{D}$, call it $\mathsf{B}'$. Then, we can find two different points $u, v$ such that $e = (u+v)/2$ and the segment $[u,v]$ is contained in $\mathsf{B}'$. Thus, $u, v \in \mathsf{D}$ so we can find $b_u, b_v \in \mathsf{B}$ such that $u \in [b_u, e)$ and $v \in (e, b_v]$. It follows that $e \in (b_u, b_v) \subseteq \mathsf{B}$, which is impossible. Now, we can write $x = \delta e + (1-\delta) b'$ with $b' \in \mathsf{B}$ and $\delta \in [0,1)$. Keeping in mind that $x \in \operatorname{int}(\mathsf{D})$, we can suppose that $x \neq b'$ and so $\delta > 0$. Next, we will reach a contradiction by assuming that $\gamma e + (1-\gamma) b' \notin \operatorname{int}(\mathsf{B})$ for every $\gamma \in \mathbb{R}$. Indeed, in that situation the straight line determined by $e$ and $b'$ does not intersect $\operatorname{int}(\mathsf{B})$. This means that, according to the Hahn-Banach separation theorem, we can find $f \in X^*$, a non-zero functional which is constant on this straight line, such that $f(u) > f(v)$ for every $u$ in the straight line and every $v \in \operatorname{int}(\mathsf{B})$. Now, there is $\varepsilon > 0$ such that $\mathsf{B}_X(x,\varepsilon) \subseteq \mathsf{D}$, and we can also find $z \in \mathsf{B}_X(x,\varepsilon)$ with $f(z) > f(x) = f(e)$. By hypothesis, $z = \alpha e + (1-\alpha) b''$ for some $\alpha \in [0,1]$ and $b'' \in \mathsf{B}$. Thus, $f(z) = \alpha f(e) + (1-\alpha) f(b'') \leq f(e)$, which is a contradiction. Consequently, there must exist $\gamma \in \mathbb{R}$ so that $\gamma e + (1-\gamma) b' \in \operatorname{int}(\mathsf{B})$. Next, we have three possibilities:

a. $\gamma \geq 1$. In this case, we have that $e \in \mathsf{B}$, which is impossible.

b. $\delta < \gamma < 1$. In this case, we deduce that $x \in \operatorname{int}(\mathsf{B}) \subset \operatorname{int}(\mathsf{D})$, so we take $\lambda = 0$.

c. $\gamma \leq \delta$. In this case, it suffices to take $\lambda = \frac{\delta(1-\gamma)-\gamma(1-\delta)}{1-\gamma}$ and $b = \gamma e + (1-\gamma) b'$, since with these choices of $\lambda$ and $b$ we have $x = \lambda e + (1-\lambda) b$. ∎

A consequence of this lemma is the proposition we referred to at the beginning of this section, which is related to the non-rotundity of drops.

**Proposition 2.2.** *Let $X$ be a normed space. Let $\mathsf{D} = \mathrm{co}\left(\{e\} \cup \mathsf{B}\right)$ be a drop in $X$. Then, $\mathsf{D}$ is not rotund.*

*Proof.* Take any $x \in \mathrm{bd}\left(\mathsf{D}\right) \cap \mathsf{B}_X\left(e, \frac{\mathrm{dist}(e,\mathsf{B})}{2}\right)$ different from $e$. We will show that the segment $[e, x]$ is entirely contained in the boundary of $\mathsf{D}$. If not, then we can find $\alpha \in (0,1)$ with $\alpha e + (1-\alpha)\, x \in \mathrm{int}\left(\mathsf{D}\right)$. By Lemma 2.1, there are $\lambda \in [0,1)$ and $b \in \mathrm{int}\left(\mathsf{B}\right)$ such that $\alpha e + (1-\alpha)\, x = \lambda e + (1-\lambda)\, b$. Now, observe that $b \notin [e,x]$ since $[e,x] \in \mathsf{B}_X\left(e, \frac{\mathrm{dist}(e,\mathsf{B})}{2}\right)$ and $\mathsf{B}_X\left(e, \frac{\mathrm{dist}(e,\mathsf{B})}{2}\right) \cap \mathsf{B} = \varnothing$. Therefore, it necessarily happens to be that $\lambda > \alpha$ and $x = \frac{\lambda-\alpha}{1-\alpha} e + \frac{1-\lambda}{1-\alpha} b$. Finally, by applying again Lemma 2.1, we deduce that $x \in \mathrm{int}\left(\mathsf{D}\right)$, which is a contradiction. ∎

Proposition 2.2 suggests a change in the definition of rotundity for drops.

**Definition 2.3.** *Let $X$ be a normed space. Let $\mathsf{D} = \mathrm{co}\left(\{e\} \cup \mathsf{B}\right)$ be a drop in $X$. We will say that the drop $\mathsf{D}$ is rotund if, for every $x, y \in \mathrm{bd}\left(\mathsf{D}\right) \setminus \{e\}$ such that $x, y, e$ are not colinear, we have that $[x,y] \cap \mathrm{bd}\left(\mathsf{D}\right) = \{x, y\}$.*

With this definition, we are ready to state the characterization of rotundity in terms of drops. However, we will need one more lemma.

**Lemma 2.4.** *Let $X$ be a normed space. Let $\mathsf{D} = \mathrm{co}\left(\{e\} \cup \mathsf{B}_X\left(b, r\right)\right)$ be a drop in $X$. Then:*

1. *If $x \in \mathsf{S}_X\left(b, r\right) \cap \mathrm{int}\left(\mathsf{D}\right)$ then $2b - x \in \mathrm{bd}\left(\mathsf{D}\right)$.*
2. *If $x \in \mathsf{S}_X\left(b, r\right) \cap \mathrm{bd}\left(\mathsf{D}\right)$ and $[x, e] \subseteq \mathrm{bd}\left(\mathsf{D}\right)$, then there exists $f \in X^*$ such that $f\left(x\right) = f\left(e\right) = \sup f\left(\mathsf{B}_X\left(b,r\right)\right)$; as a consequence, $2b - x \in \mathrm{bd}\left(\mathsf{D}\right)$.*

*Proof.* To simplify the proof we will assume, without loss of generality, that $\mathsf{B}_X\left(b, r\right) = \mathsf{B}_X$, i.e. $b = 0$ and $r = 1$.

1. By applying Lemma 2.1, there exist $\lambda \in (0,1)$ and $u \in \mathsf{U}_X$ such that $x = \lambda e + (1-\lambda)\, u$. If $-x \in \mathrm{int}\left(\mathsf{D}\right)$, then by applying again Lemma 2.1, we can write $-x = \gamma e + (1-\gamma)\, v$ with $\gamma \in (0,1)$ and $v \in \mathsf{U}_X$. Now, consider a functional $f \in \mathsf{S}_{X^*}$ such that $f\left(x\right) = 1$. Then, we have that $1 = \lambda f\left(e\right) + (1-\lambda)\, f\left(u\right)$, and since $f\left(u\right) < 1$, it necessarily happens to be that $f\left(e\right) > 1$. Then, $-1 = \gamma f\left(e\right) + (1-\gamma)\, f\left(v\right)$, and since $f\left(e\right) > 1 > -1$, it necessarily happens to be that $f\left(v\right) < -1$. This contradicts the fact that $v \in \mathsf{U}_X$.

2. Observe that, since the non-trivial segment $[e,x] \subseteq \mathrm{bd}\left(\mathsf{D}\right)$, the straight line determined by $e$ and $x$ does not intersect the interior of $\mathsf{D}$ and, therefore, does not intersect the interior of $\mathsf{B}_X\left(b, r\right)$ either. Thus, according to the Hahn-Banach separation theorem, there exists $f \in X^*$, a non-zero functional which is constant on this straight line, such that $f\left(u\right) > f\left(v\right)$ for every $u$ in the straight line and every $v \in \mathrm{int}\left(\mathsf{B}_X\left(b, r\right)\right)$. Now, if $-x \in \mathrm{int}\left(\mathsf{D}\right)$ then by applying again Lemma 2.1, we can write $-x = \gamma e + (1-\gamma)\, v$ with $\gamma \in (0,1)$ and $v \in \mathsf{U}_X$. Now, consider a functional $f \in \mathsf{S}_{X^*}$ such that $f\left(x\right) = f\left(e\right) = 1$. Then, we have that $-1 = \lambda + (1-\lambda)\, f\left(v\right)$, and hence
$$f\left(v\right) = -\frac{1+\lambda}{1-\lambda} < -1.$$
This contradicts the fact that $v \in \mathsf{U}_X$. ∎

**Theorem 2.5.** *Let $X$ be a normed space. Let $\mathsf{D} = \mathrm{co}\left(\{e\} \cup \mathsf{B}\right)$ be a drop in $X$. Then, $\mathsf{B}$ is rotund if and only if $\mathsf{D}$ is rotund.*

*Proof.* Assume that $\mathsf{B}$ is rotund. Let us take two elements $u$ and $v$ in $\mathrm{bd}\left(\mathsf{D}\right) \setminus \{e\}$ such that $u, v, e$ are not colinear and write $u = \lambda_u e + (1 - \lambda_u) b_u$ and $v = \lambda_v e + (1 - \lambda_v) b_v$ where $\lambda_u, \lambda_v \in [0, 1)$ and $b_u, b_v \in \mathsf{B}$. Note that, by Lemma 2.1, we have $b_u, b_v \in \mathrm{bd}\left(\mathsf{B}\right)$. Now

$$\frac{u+v}{2} = \frac{\lambda_u + \lambda_v}{2} e + \left(1 - \frac{\lambda_u + \lambda_v}{2}\right) \left(\frac{1 - \lambda_u}{2 - \lambda_u - \lambda_v} b_u + \frac{1 - \lambda_v}{2 - \lambda_u - \lambda_v} b_v\right).$$

Since $u, v, e$ are not colinear we obtain that $b_u \neq b_v$. Now, from of the rotundity of $\mathsf{B}$,

$$\frac{1 - \lambda_u}{2 - \lambda_u - \lambda_v} b_u + \frac{1 - \lambda_v}{2 - \lambda_u - \lambda_v} b_v \in \mathrm{int}\left(\mathsf{B}\right).$$

As a consequence of this, together with Proposition 2.1, we have

$$\frac{u+v}{2} \in \mathrm{int}\left(\mathsf{D}\right).$$

Conversely, assume that $\mathsf{D}$ is rotund and suppose, on the contrary, that $\mathsf{B} = \mathsf{B}_X\left(b, r\right)$ is not rotund. Consider any non-trivial segment $[u, v] \subset \mathrm{bd}\left(\mathsf{B}\right)$. We have two possibilities:

a. $[u, v] \subset \mathrm{bd}\left(\mathsf{D}\right)$. In this case, by hypothesis, we obtain that the segment $[u, v]$ and the point $e$ are aligned. Therefore, by Lemma 2.4, paragraph *2*, the segment $[2b - u, 2b - v]$ is contained in $\mathrm{bd}\left(\mathsf{D}\right)$. Again, by hypothesis, $[2b - u, 2b - v]$ and $e$ are colinear, which is impossible.

b. $[u, v] \cap \mathrm{int}\left(\mathsf{D}\right) \neq \varnothing$. In this case, there is an "smaller" non-trivial segment $[u', v'] \subset [u, v] \cap \mathrm{int}\left(\mathsf{D}\right)$. According to Lemma 2.4, paragraph *1*, the segment $[2b - u', 2b - v']$ is contained in $\mathrm{bd}\left(\mathsf{D}\right)$. Now, we can proceed as in paragraph a in order to reach a contradiction. ∎

## 3 Drops and smoothness

In this section we study smoothness and its relation with drops. Let us recall that, in a normed space, a point in its unit sphere is said to be a smooth point of its unit ball if there is only one functional attaining its norm at the point. A normed space is said to be smooth if all the points in its unit sphere are smooth points. Here we will provide a characterization of smoothness in terms of the geometry of the drops. However, as in the previous section, we will begin by stating (without proof) the following necessary lemma, which already appears in [1].

**Lemma 3.1.** *Let $X$ be a 2-dimensional Banach space. If $x \in \mathsf{S}_X$ is not a smooth point of $\mathsf{B}_X$, then there are infinitely many functionals $f \in \mathsf{S}_{X^*}$ verifying that $f^{-1}(1) \cap \mathsf{B}_X = \{x\}$.*

Now we are ready to state and proof the main theorem in this section.

**Theorem 3.2.** *Let $X$ be a normed space. The following conditions are equivalent:*

1. *$X$ is smooth.*
2. *For every drop $\mathsf{D} = \operatorname{co}(\{e\} \cup \mathsf{B}_X(b,r))$ in $X$ and every $x \in \mathsf{S}_X(b,r)$ such that $[e,x] \subseteq \operatorname{bd}(\mathsf{D})$, the segment $[e, 2b-x]$ is not contained in $\operatorname{bd}(\mathsf{D})$.*

*Proof.* Let us begin by assuming that $X$ is smooth and, on the contrary, suppose that there exists a drop $\mathsf{D} = \operatorname{co}(\{e\} \cup \mathsf{B}_X(b,r))$ in $X$ having an element $x \in \mathsf{S}_X(b,r)$ such that $[e,x], [e,2b-x] \subseteq \operatorname{bd}(\mathsf{D})$. Observe that, without loss of generality, we can suppose that $b = 0$ and $r = 1$. By applying twice Lemma 2.4, paragraph *2*, there are two functionals $f, g \in \mathsf{S}_{X^*}$ such that $f(e) = f(-x) = g(e) = g(x) = 1$. Now, $g$ and $-f$ are two different functionals of norm 1 attaining their norm at $x$. This contradicts the fact that $x$ is a smooth point of $\mathsf{B}_X$.

Conversely, assume that $X$ is not smooth. Then, we can suppose, without loss, that there exists a 2-dimensional subspace $Y$ of $X$ which is not smooth. Let $y \in \mathsf{S}_Y$ be a non-smooth point of $\mathsf{B}_Y$ and consider, by Lemma 3.1, $f \neq g \in \mathsf{S}_{Y^*}$ such that $f^{-1}(1) \cap \mathsf{B}_Y = g^{-1}(1) \cap \mathsf{B}_Y = \{y\}$. Since the lines $f^{-1}(1)$ and $(-g)^{-1}(1)$ are not parallel, we can find a point $e \in f^{-1}(1) \cap (-g)^{-1}(1)$. Finally, if we consider the drop in $X$ given by $\mathsf{D} = \operatorname{co}(\{e\} \cup \mathsf{B}_X)$, we obtain that the segments $[e,y]$ and $[e,-y]$ are both contained in $\operatorname{bd}(\mathsf{D})$. ■

**Remark 3.3.** *Observe that there is a slight difference between Theorem 2.5 and Theorem 3.2, which consists of the following: In Theorem 2.5, in order to obtain the rotundity of the space, it suffices to find one rotund drop; however, in Theorem 3.2 it is needed that all drops verify the condition in paragraph 2 to obtain the smoothness of the space. Note that $\ell_\infty^2$ is not smooth but the drop given by the convex hull of the unit square and the point $(0,2)$ does verify the condition in paragraph 2 of Theorem 3.2.*

To finish this section, we will see that Theorem 3.2 yields to a characterization of smoothness in 2-dimensional spaces.

**Lemma 3.4.** *Let $X$ be a 2-dimensional Banach space. Let $\mathsf{D} = \operatorname{co}(\{e\} \cup \mathsf{B}_X(b,r))$ be a drop in $X$. Then, there are two different and unique elements $x, y \in \mathsf{S}_X(b,r)$ such that $[e,x], [e,y] \subseteq \operatorname{bd}(\mathsf{D})$ and $[e,x), [e,y) \cap \mathsf{B}_X(b,r) = \varnothing$. In particular, $\overline{\operatorname{co}}(\mathsf{D} \setminus \mathsf{B}_X(b,r)) = \operatorname{co}\{e,x,y\}$.*

*Proof.* It is not difficult to see the existence and uniqueness of both $x$ and $y$. By hypothesis, $\mathsf{D} \setminus \mathsf{B}_X(b,r) \supseteq [e,x) \cup [e,y)$, therefore $\overline{\operatorname{co}}(\mathsf{D} \setminus \mathsf{B}_X(b,r)) \supseteq \overline{\operatorname{co}}([e,x) \cup [e,y))$ $= \operatorname{co}\{e,x,y\}$. Finally, in virtue of Lemma 2.4, paragraph *2*, it can be seen that $\mathsf{D} \setminus \mathsf{B}_X(b,r) \subseteq \operatorname{co}\{e,x,y\}$, which finishes the proof. ■

**Corollary 3.5.** *Let $X$ be a 2-dimensional normed space. The following conditions are equivalent:*

1. *$X$ is smooth.*
2. *For every drop $\mathsf{D} = \operatorname{co}(\{e\} \cup \mathsf{B}_X(b,r))$ in $X$ and every $x \in \mathsf{S}_X(b,r)$ such that $[e,x] \subseteq \operatorname{bd}(\mathsf{D})$, the segment $[e, 2b-x]$ is not contained in $\operatorname{bd}(\mathsf{D})$.*

3. *For every drop* $\mathsf{D} = \mathrm{co}\left(\{e\} \cup \mathsf{B}_X(b,r)\right)$ *in* $X$, $b \notin \overline{\mathrm{co}}\left(\mathsf{D} \setminus \mathsf{B}_X(b,r)\right)$.

*Proof.* Let us begin by showing that *3* implies *1*. Suppose that $X$ is not smooth. Let $x \in \mathsf{S}_X$ be a non-smooth point of $\mathsf{B}_X$ and consider, by Lemma 3.1, $f \neq g \in \mathsf{S}_{X^*}$ such that $f^{-1}(1) \cap \mathsf{B}_X = g^{-1}(1) \cap \mathsf{B}_X = \{x\}$. Since the lines $f^{-1}(1)$ and $(-g)^{-1}(1)$ are not parallel, we can find a point $e \in f^{-1}(1) \cap (-g)^{-1}(1)$. Finally, if we consider the drop in $X$ given by $\mathsf{D} = \mathrm{co}\left(\{e\} \cup \mathsf{B}_X\right)$, we obtain, by Lemma 3.4, that $0 \in \mathrm{co}\{e, x, -x\} = \overline{\mathrm{co}}\left(\mathsf{D} \setminus \mathsf{B}_X\right)$.

Theorem 3.2 shows that *1* implies *2*, therefore it remains to show that *2* implies *3*. Suppose that there exists a drop $\mathsf{D} = \mathrm{co}\left(\{e\} \cup \mathsf{B}_X(b,r)\right)$ in $X$ verifying that $b \in \overline{\mathrm{co}}\left(\mathsf{D} \setminus \mathsf{B}_X(b,r)\right)$. Here we will assume that $\mathsf{B}_X(b,r) = \mathsf{B}_X$. According to lemma 3.4, there are two different elements $x, y \in \mathsf{S}_X$ such that $[e,x], [e,y] \subseteq \mathrm{bd}(\mathsf{D})$ and $[e,x), [e,y) \cap \mathsf{B}_X = \varnothing$. Therefore, $0 \in \overline{\mathrm{co}}\left(\mathsf{D} \setminus \mathsf{B}_X\right) = \mathrm{co}\{e, x, y\}$, which means that we can find $\alpha, \beta, \gamma \in [0,1]$ with $1 = \alpha + \beta + \gamma$ and $0 = \alpha x + \beta y + \gamma e$. Now, according to Lemma 2.4, paragraph *2*, there are $f, g \in \mathsf{S}_{X^*}$ such that $f(e) = f(x) = g(e) = g(y) = 1$. Then, $1 \geq |f(y)| = \frac{\alpha+\gamma}{\beta}$, that is, $\beta \geq \alpha + \gamma = 1 - \beta$ so $\beta \geq 1/2$. By applying the same argument to $|g(x)|$, we deduce that $\alpha \geq 1/2$ too, therefore $\gamma = 0$ and $\alpha = \beta = 1/2$, which means that $x = -y$. ■

**Remark 3.6.** *Observe that, in the previous theorem, the proof of the fact that 3 implies 1 can be adapted to fit for any normed space. Therefore, the* 2*-dimensional hypothesis is only needed to prove that 2 implies 3. Also, notice that every drop satisfying 2 also verifies 3. However, in* $\ell_\infty^2$*, the drop given by the convex hull of the unit square and the point* $(-1,2)$ *does verify 3 but not 2.*

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