A Subordination Result with Generalized Sakaguchi Univalent Functions Related to Complex Order

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Abstract

In the present paper, we obtain an interesting subordination relation for a family of analytic functions of complex order by using subordination theorem.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also, let $\mathcal{C}$ denote

familiar class of functions $f(z) \in \mathcal{A}$ which are convex in $\mathbb{U}$.A function $f(z)$ in $\mathcal{A}$ is said to be starlike of complex order b if and only if it satisfies

$$Re\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0 \; z \in \mathbb{U}.$$

This class is denoted by $S^*(b)$.

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A function $f(z)$ in $\mathcal{A}$ is said to be convex of complex order b if and only if it satisfies

$$Re\left\{1+\frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right)\right\}>0 \ z\in\mathbb{U}.$$

This class is denoted by $K(b)$.

A function $f(z)$ in $\mathcal{A}$ is called to be Sakaguchi function if and only if it satisfies

$$Re\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\}>0\ , z\in\mathbb{U}.$$

This class is denoted by S_S [1].

A function $f(z)$ in $\mathcal{A}$ is called to be Sakaguchi function of order α if and only if it satisfies

$$Re\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\}>\alpha\ ;0\leq\alpha<\frac{1}{2}\ z\in\mathbb{U}.$$

This class is denoted by $S_S(\alpha)$ [2].However, the class $T_S(\alpha)$ is defined by $f(z)\in T_S(\alpha)\Leftrightarrow zf'(z)\in S_S(\alpha)$.

A function $f(z)$ in $\mathcal{A}$ is said to be Generalized Sakaguchi function of order α if and only if it satisfies

$$Re\left\{\frac{(1-t)zf'(z)}{f(z)-f(tz)}\right\}>\alpha\ ;0\leq\alpha<1\, , t\in\mathbb{C}\, ,|t|\leq 1\, , t\neq 1\, , z\in\mathbb{U}.$$

This class is denoted by $S(\alpha,t)$ [3]. However,the class $T(\alpha,t)$ is defined by $f(z)\in T(\alpha,t)\Leftrightarrow zf'(z)\in S(\alpha,t)$.

A function $f(z)\in A$ is said to be in the class $S(b,\lambda,t)$ if and only if it satisfies the inequality

$$Re\left\{1+\frac{1}{b}(\frac{(1-t)(zf'(z)+\lambda z^2f''(z))}{(1-\lambda)(f(z)-f(tz))+\lambda(zf'(z)-tzf'(tz))}-1)\right\}>0$$

where $t\in\mathbb{C},\ |t|\leq 1,\ t\neq 1,\ b\in\mathbb{C}-\{0\}\ 0\leq\lambda\leq 1$ and $z\in\mathbb{U}$.

Güney [4] proved that if the function $f(z)$ defined by (1.1) and

$$\sum_{n=2}^{\infty}(1+\lambda(n-1))(|n-u_n|+|b||u_n|)|a_n|\leq|b|, \tag{1.2}$$

then $f(z)\in S(b,\lambda,t)$ where $t\in\mathbb{C},\ |t|\leq 1,\ t\neq 1,\ u_n=\sum_{j=0}^{n-1}t^j,\ \ b\in\mathbb{C}-\{0\}$, $0\leq\lambda\leq 1$ and $z\in\mathbb{U}$. We obtain the various subclasses of $\mathcal{A}$ can be represented as $S(b,\lambda,t)$ for suitable choices of b,λ, and t. For example,

$$S(b,0,0)\equiv S^*(b),$$

$$S(b,1,0)\equiv K(b),$$

$$S(1-\alpha,0,-1)\equiv S_S(\alpha)$$

and

$$S(1-\alpha,0,t)\equiv S(\alpha,t).$$

Let $S[b,\lambda,t]$ denote the class of functions $f(z)\in\mathcal{A}$ whose coefficients satisfy the condition (1.2).

We note that

$$S[b,\lambda,t]\subseteq S(b,\lambda,t).$$

If we consider

$$f(z)=z+\sum_{n=2}^{\infty}\frac{b}{n(n-1)(1+\lambda(n-1))(|n-u_n|+|b||u_n|)}z^n$$

with

$$a_n=\frac{b}{n(n-1)(1+\lambda(n-1))(|n-u_n|+|b||u_n|)},$$

then we have that

$$\begin{aligned}&\sum_{n=2}^{\infty}(1+\lambda(n-1))(|n-u_n|+|b||u_n|)|a_n|\\ &\qquad=|b|\sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n}\right)\\ &\qquad=|b|.\end{aligned}$$

Therefore, we see that $f(z)$ is in the class $S[b,\lambda,t]$.

Evidently, we have

$$S[b,0,0]\equiv S^*[b],$$

$$S[b,1,0]\equiv K[b],$$

$$S[1-\alpha,0,-1]\equiv S_S[\alpha]$$

and

$$S[1-\alpha,0,t]\equiv S[\alpha,t].$$

In this paper, we prove an interesting subordination result for the class $S[b,\lambda,t]$. In our proposed investigation of functions in the class $S[b,\lambda,t]$, we need the following definitions and lemma.

Definition 1. Given two functions $f,g\in\mathcal{A}$ where $f(z)$ is given by (1.1) and $g(z)$ is defined by

$$g\left(z\right)=z+\sum_{n=2}^{\infty}c_nz^n.$$

The Hadamard product $f*g$ is defined by

$$\left(f*g\right)\left(z\right)=z+\sum_{n=2}^{\infty}a_nc_nz^n\ ,\ z\in\mathbb{U}.$$

Definition 2. (Subordination Principle) For two functions f and g analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ and write $f(z)\prec$

$g(z)$, $z \in \mathbb{U}$, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, $z \in \mathbb{U}$. In particular, if the function $g(z)$ is univalent in $\mathbb{U}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$.

Definition 3. (Subordinating Factor Sequence) A sequence $\{c_n\}_{n=1}^{\infty}$ of complex numbers is said to be a *Subordinating Factor Sequence* if for the function $f(z)$ of the form (1.1) is analytic, univalent and convex in $\mathbb{U}$, we have the subordination given by

$$\sum_{n=1}^{\infty} a_n c_n z^n \prec f(z)\,; \qquad z \in \mathbb{U}, a_1 = 1. \tag{1.3}$$

Lemma. The sequence $\{b_n\}_{n=1}^{\infty}$ is Subordinating factor sequence iff

$$Re\left\{1 + 2\sum_{n=1}^{\infty} b_n z^n\right\} > 0. \tag{1.4}$$

The above lemma due to Wilf [5].

2 Main Theorem

Theorem . *If $f(z) \in S[b, \lambda, t]$ then*

$$\frac{(1+\lambda)(|2-u_2|+|b||u_2|)}{2(|b|((1+\lambda)|u_2|+1)+(1+\lambda)|2-u_2|)}(f*g)(z) \prec g(z) \tag{2.1}$$

for every function $g(z) \in \mathcal{C}$ and

$$Ref(z) > -1 - \frac{|b|}{(1+\lambda)(|2-u_2|+|b||u_2|)}. \tag{2.2}$$

The constant $\frac{(1+\lambda)(|2-u_2|+|b||u_2|)}{2(|b|((1+\lambda)|u_2|+1)+(1+\lambda)|2-u_2|)}$ is the best estimate.

Proof . Let $f(z) \in S[b, \lambda, t]$ and $g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{C}$. Then

$$\frac{(1+\lambda)(|2-u_2|+|b||u_2|)}{2(|b|((1+\lambda)|u_2|+1)+(1+\lambda)|2-u_2|)}(f*g)(z)$$

$$= \frac{(1+\lambda)(|2-u_2|+|b||u_2|)}{2(|b|((1+\lambda)|u_2|+1)+(1+\lambda)|2-u_2|)}(z + \sum_{n=2}^{\infty} a_n c_n z^n).$$

Thus, by Definition 3, (2.1) will hold if

$$\left\{\frac{(1+\lambda)(|2-u_2|+|b||u_2|)}{2(|b|((1+\lambda)|u_2|+1)+(1+\lambda)|2-u_2|)}a_n\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence with $a_1 = 1$. In view of Lemma, this is equivalent to

$$Re\left\{1 + 2\sum_{n=1}^{\infty} \frac{(1+\lambda)(|2-u_2|+|b||u_2|)}{2(|b|((1+\lambda)|u_2|+1)+(1+\lambda)|2-u_2|)}a_n z^n\right\} > 0. \tag{2.3}$$

Now because $(1+\lambda(n-1))(|n-u_n|+|b||u_n|)$ is increasing function of n, we have

$$Re\left\{1+\frac{(1+\lambda)(|2-u_2|+|b||u_2|)}{|b|((1+\lambda)|u_2|+1)+(1+\lambda)|2-u_2|}\sum_{n=1}^{\infty}a_nz^n\right\}$$
$$=Re\left\{1+\frac{(1+\lambda)(|2-u_2|+|b||u_2|)}{|b|((1+\lambda)|u_2|+1)+(1+\lambda)|2-u_2|}z+\right.$$
$$\left.+\frac{1}{|b|((1+\lambda)|u_2|+1)+(1+\lambda)|2-u_2|}\sum_{n=2}^{\infty}(1+\lambda)(|2-u_2|+|b||u_2|)a_nz^n\right\}$$
$$\geq 1-\frac{(1+\lambda)(|2-u_2|+|b||u_2|)}{|b|((1+\lambda)|u_2|+1)+(1+\lambda)|2-u_2|}r-$$
$$-\frac{1}{|b|((1+\lambda)|u_2|+1)+(1+\lambda)|2-u_2|}\sum_{n=2}^{\infty}(1+\lambda(n-1))(|n-u_n|+|b||u_n|)a_nr^n$$
$$>1-\frac{(1+\lambda)(|2-u_2|+|b||u_2|)}{|b|((1+\lambda)|u_2|+1)+(1+\lambda)|2-u_2|}r-\frac{|b|}{|b|((1+\lambda)|u_2|+1)+(1+\lambda)|2-u_2|}r$$
$$=0.$$

Hence, (2.3) holds true in $\mathbb{U}$ and also the subordination result (2.1) asserted by Theorem . The inequality (2.2) follows by taking $g(z)=\frac{z}{1-z}=\sum_{n=1}^{\infty}z^n\in\mathcal{C}$ in (2.1).

Now, consider the function

$$t(z)=z-\frac{|b|}{(1+\lambda)(|2-u_2|+|b||u_2|)}z^2$$

which is a member of the class $S[b,\lambda,t]$. Then by using (2.1), we have

$$\frac{(1+\lambda)(|2-u_2|+|b||u_2|)}{2(|b|((1+\lambda)|u_2|+1)+(1+\lambda)|2-u_2|)}t(z)\prec\frac{z}{1-z}\qquad;\qquad z\in\mathbb{U}.$$

It is easily verified that

$$minRe\left\{\frac{(1+\lambda)(|2-u_2|+|b||u_2|)}{2(|b|((1+\lambda)|u_2|+1)+(1+\lambda)|2-u_2|)}t(z)\right\}=-\frac{1}{2}\qquad;\qquad z\in\mathbb{U}.$$

Then the constant $\frac{(1+\lambda)(|2-u_2|+|b||u_2|)}{2(|b|((1+\lambda)|u_2|+1)+(1+\lambda)|2-u_2|)}$ cannot be replaced by a larger one, which completes the proof of Theorem.

Corollary 1. *Let $f(z)\in S^*[b]$, then for every function $g\in\mathcal{C}$*

$$\frac{(1+|b|)}{2(2|b|+1)}(f*g)(z)\prec g(z)\tag{1}$$

and

$$Ref(z)>-1-\frac{|b|}{1+|b|}.\tag{2}$$

The constant $\frac{(1+|b|)}{2(2|b|+1)}$ is the best estimate.

Corollary 2. *Let $f(z) \in K[b]$, then for every function $g \in \mathcal{C}$*

$$\frac{1+|b|}{3|b|+2}(f*g)(z) \prec g(z) \tag{3}$$

and

$$Ref(z) > -1 - \frac{|b|}{2(1+|b|)} \tag{4}$$

The constant $\frac{1+|b|}{3|b|+2}$ is the best estimate.

Corollary 3. *Let $f(z) \in S_S[\alpha, t]$, then for every function $g \in \mathcal{C}$*

$$\frac{(|2-u_2|+(1-\alpha)|u_2|)}{2((1-\alpha)(|u_2|+1)+|2-u_2|)}(f*g)(z) \prec g(z) \tag{5}$$

and

$$Ref(z) > -1 - \frac{(1-\alpha)}{(|2-u_2|+(1-\alpha)|u_2|)} \tag{6}$$

The constant $\frac{(|2-u_2|+(1-\alpha)|u_2|)}{2((1-\alpha)(|u_2|+1)+|2-u_2|)}$ is the best estimate.

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