Bounded solutions for nonlinear elliptic equations with degenerate coercivity and data in an $L \log L$

A. Benkirane A. Youssfi D. Meskine

Abstract

In this paper, we prove L^∞-regularity for solutions of some nonlinear elliptic equations with degenerate coercivity whose prototype is

$$\begin{cases} -\mathrm{div}\left(\frac{1}{(1+|u|)^{\theta(p-1)}}|\nabla u|^{p-2}\nabla u\right) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded open set in $\mathbb{R}^{\mathrm{N}}$, $N \geq 2$, $1 < p < N$, θ is a real such that $0 \leq \theta \leq 1$ and $f \in L^{\frac{N}{p}} log^\alpha L$ with some $\alpha > 0$.

1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^{\mathrm{N}}$, with $N \geq 2$, and p a real such that $1 < p < N$. We consider the following problem

$$\begin{cases} A(u) := -\mathrm{div}\, a(x, u, \nabla u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (that is, $a(., s, \xi)$ is measurable on Ω for every (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$, and $a(x, ., .)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$

Received by the editors March 2006 - In revised form in November 2006.
Communicated by P. Godin.
2000 *Mathematics Subject Classification :* 35J70, 35J60, 46E30.
Key words and phrases : Zygmund spaces, nonlinear elliptic equations, L^∞-estimates, rearrangements.

for almost every x in Ω), which we assume to satisfy the following assumptions

$$a(x,s,\xi).\xi \geq h^{p-1}(|s|)|\xi|^p \tag{1.2}$$

for almost every x in Ω, for every (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$, where $h:\mathbb{R}^+\to]0,\infty[$ is a decreasing continuous function such that its primitive

$$H(s)=\int_0^s h(t)dt$$

is unbounded;

$$|a(x,s,\xi)| \leq a_0(x)+|s|^{p-1}+|\xi|^{p-1} \tag{1.3}$$

for almost x in Ω, for every $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, where a_0 is a non negative function in $L^{p'}(\Omega)$ with $p'=\frac{p}{p-1}$, and

$$(a(x,s,\xi)-a(x,s,\xi')).(\xi-\xi')>0 \tag{1.4}$$

for almost $x\in\Omega$, for every $s\in\mathbb{R}$ and for every ξ, ξ' in $\mathbb{R}^N$ with $\xi\neq\xi'$.

It is our purpose in this paper, to prove the existence of a weak solution for (1.1) in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ when the data satisfies the assumption

$$f\in L^{\frac{N}{p}}log^\alpha L \tag{1.5}$$

with $\alpha > \frac{N(p-1)}{p}$.

In the literature, many results concerning L^∞ estimate for weak solutions of (1.1) had been obtained. It was shown earlier in the setting of Orlicz spaces (see [7]) that when h in (1.2) is a constant function, every weak solution of (1.1) is bounded provided that f belongs to the Lorentz space $L(m,\infty)$ with $m>\frac{N}{p}$.

Under the assumption (1.2), existence of bounded solutions for (1.1) has been proved first in [1] and [3] when $p=2$ and f belongs to $L^m(\Omega)$ with $m>\frac{N}{2}$, and for more general p, in [2] when the datum f belongs to $L^m(\Omega)$ with $m>\frac{N}{p}$.

Let us recall, as mentioned in [2] and [3], that if the data f belongs to $L^{\frac{N}{p}}(\Omega)$, the problem (1.1) has no bounded solution.

We note that there is a difficulty in dealing with (1.1) under the assumption (1.2), since the operator A is not coercive on $W_0^{1,p}(\Omega)$ and so the classical Leray-Lions surjectivity theorem do not apply even in the case in which the datum f belongs to $W^{-1,p'}(\Omega)$ where $p'=\frac{p}{p-1}$ (see [6]). To overcome this situation, we will proceed by approximation by means of truncatures in $a(x,s,\xi)$ to get a coercive differential operator on $W_0^{1,p}(\Omega)$.

2 Background

It's worth recalling here some definitions and notations that we will use later. Let $p \geq 1$ and $\alpha \in \mathbb{R}$, the Zygmund space $L^p log^\alpha L$, consists of all measurable functions g on Ω for which

$$\int_\Omega |g|^p log^\alpha(e + |g|)dx < \infty.$$

It is an Orlicz space generated by the N-function $\Theta(t) = t^p log^\alpha(e + t)$, $t \geq 0$, equipped with the so called Luxemburg norm

$$\|g\|_\Theta = \inf\{\lambda > 0 : \int_\Omega \Theta(\frac{|g|}{\lambda})dx \leq 1\}.$$

For $p > 1$, the conjugate N-function of Θ is equivalent (see [5]) to $t^{p'} log^{-\alpha\frac{p'}{p}}(e + t)$ where $p' = \frac{p}{p-1}$. Thus, it follows that the dual of $L^p log^\alpha L$ coincides with $L^{p'} log^{-\alpha\frac{p'}{p}} L$. The inverse function Θ^{-1} of Θ is equivalent to $t^{\frac{1}{p}} log^{-\frac{\alpha}{p}}(e + t)$.

We recall that for a subset E of Ω, the Luxemburg norm, associated to an N-function M, of the characteristic function χ_E of E is (see [5])

$$\|\chi_E\|_M = \frac{1}{M^{-1}(\frac{1}{|E|})} \tag{2.1}$$

where $|E|$ denotes the Lebesgue measure of E.

The decreasing rearrangement of a measurable function $w : \Omega \to \mathbb{R}$ is defined as

$$w^*(s) = \inf\{t \in \mathbb{R} : \mu_w(t) \leq s\} \quad \text{for} \quad s \in (0, |\Omega|),$$

where

$$\mu_w(t) = |\{x \in \Omega : |w(x)| > t\}|$$

is the distribution function of w. Hence, w^* is the generalized inverse function of μ_w and

$$w^*(0) = \|w\|_\infty.$$

For more details, one can see [7, 8].

Throughout the paper, T_k, the truncation at level $k > 0$, and G_k are functions defined by $T_k(s) = \max(-k, \min(s, k))$ and $G_k(s) = s - T_k(s)$.

3 Main result

Our main result is the following,

Theorem 3.1. *Under the assumptions* (1.2), (1.3), (1.4) *and* (1.5), *the problem* (1.1) *has at least a weak solution* u *in* $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *in the sense that*

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla v dx = \int_\Omega f v dx \tag{3.1}$$

for all v *in* $\mathcal{D}(\Omega)$.

Remark 3.1. *The space* $L^{\frac{N}{p}} log^\alpha L$ *is contained in* $L^{\frac{N}{p}}(\Omega)$ *and contains the spaces* $L^m(\Omega)$ *with* $m > \frac{N}{p}$, *in this sense our result is a refinement of the one given in* [2].

4 Proof of theorem 3.1

Step1: L^∞-bound

Remark that the operator A in (1.1) is not coercive, this is due to the assumption (1.2). To get rid of this situation, we consider the differential operator

$$A_n(u) = -\mathrm{div}\, a(x, T_n(u), \nabla u), \quad n \in \mathbb{N}$$

which turns out to be pseudo-monotone from $W_0^{1,p}(\Omega)$ to its dual $W^{-1,p'}(\Omega)$. Moreover, by (1.2), we have

$$\begin{aligned} < A_n(u), u > &= \int_\Omega a(x, T_n(u), \nabla u)\nabla u dx \\ &\geq \int_\Omega h^{p-1}(|T_n(u)|)|\nabla u|^p dx \\ &\geq h^{p-1}(n)\int_\Omega |\nabla u|^p dx. \end{aligned}$$

Hence, A_n is coercive on $W_0^{1,p}(\Omega)$.
Let $(f_n)_n$ be a sequence of L^∞-functions such that

$$f_n \to f \quad \text{in} \quad L^1(\Omega) \quad \text{and} \quad |f_n| \leq |f|.$$

It is known, thanks to the Leray-Lions existence theorem (see [6]), that there exists a function u_n in $W_0^{1,p}(\Omega)$ such that

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla \phi \, dx = \int_\Omega f_n \phi \, dx \tag{4.1}$$

holds for every ϕ in $W_0^{1,p}(\Omega)$.
For $t > 0$ and $\epsilon > 0$, we use $T_\epsilon(G_t(u_n))$ as test function in (4.1), obtaining

$$\int_{\{t<|u_n|\leq t+\epsilon\}} a(x, T_n(u_n), \nabla u_n)\nabla u_n \, dx \leq \epsilon \int_{\{|u_n|>t\}} |f| dx.$$

Dividing both sides by ϵ and using (1.2), we get

$$\frac{1}{\epsilon}\int_{\{t<|u_n|\leq t+\epsilon\}} h^{p-1}(|u_n|)|\nabla u_n|^p dx \leq \int_{\{|u_n|>t\}} |f| dx.$$

Since h is a nonnegative and decreasing function, one has

$$\frac{1}{\epsilon}h^{p-1}(t+\epsilon)\int_{\{t<|u_n|\leq t+\epsilon\}} |\nabla u_n|^p dx \leq \int_{\{|u_n|>t\}} |f| dx.$$

Then, letting ϵ tends to 0^+ we have

$$h^{p-1}(t)\left(-\frac{d}{dt}\int_{\{|u_n|>t\}} |\nabla u_n|^p dx\right) \leq \int_{\{|u_n|>t\}} |f| dx. \tag{4.2}$$

On the other hand, Hölder's inequality implies that

$$\begin{aligned}\int_{\{t<|u_n|\le t+\epsilon\}}|\nabla u_n|dx &\le |\{t<|u_n|\le t+\epsilon\}|^{\frac{1}{p'}}\left(\int_{\{t<|u_n|\le t+\epsilon\}}|\nabla u_n|^p dx\right)^{\frac{1}{p}}\\ &\le (\mu_n(t)-\mu_n(t+\epsilon))^{\frac{1}{p'}}\left(\int_{\{t<|u_n|\le t+\epsilon\}}|\nabla u_n|^p dx\right)^{\frac{1}{p}},\end{aligned}$$

where μ_n denotes the distribution function of u_n, that is

$$\mu_n(t)=|\{x\in\Omega:|u_n(x)|>t\}|.$$

Then, dividing both sides of the last inequality by ϵ we obtain

$$\frac{1}{\epsilon}\int_{\{t<|u_n|\le t+\epsilon\}}|\nabla u_n|dx\le\left(-\frac{1}{\epsilon}(\mu_n(t+\epsilon)-\mu_n(t))\right)^{\frac{1}{p'}}\left(\frac{1}{\epsilon}\int_{\{t<|u_n|\le t+\epsilon\}}|\nabla u_n|^p dx\right)^{\frac{1}{p}}.$$

Letting $\epsilon\to 0^+$, we get

$$-\frac{d}{dt}\int_{\{|u_n|>t\}}|\nabla u_n|dx\le(-\mu'(t))^{\frac{1}{p'}}\left(-\frac{d}{dt}\int_{\{|u_n|>t\}}|\nabla u_n|^p dx\right)^{\frac{1}{p}}. \tag{4.3}$$

Now, let us recall the wellknown inequality (see [7])

$$NC_N^{\frac{1}{N}}(\mu_n(t))^{1-\frac{1}{N}}\le-\frac{d}{dt}\int_{\{|u_n|>t\}}|\nabla u_n|dx \tag{4.4}$$

where C_N denotes the measure of the unit ball in $\mathbb{R}^N$. Combining (4.2), (4.3) and (4.4), we obtain

$$h(t)\le\frac{-\mu_n'(t)}{N^{p'}C_N^{\frac{p'}{N}}(\mu_n(t))^{p'(1-\frac{1}{N})}}\left(\int_{\{|u_n|>t\}}|f|dx\right)^{\frac{p'}{p}}.$$

Using Hölder's inequality in Orlicz spaces, the above inequality becomes

$$h(t)\le\frac{-\mu_n'(t)}{N^{p'}C_N^{\frac{p'}{N}}(\mu_n(t))^{p'(1-\frac{1}{N})}}\,2^{\frac{p'}{p}}\|f\|^{\frac{p'}{p}}_{L^{\frac{N}{p}}log^{\alpha}L}\|\chi_{|u_n|>t}\|^{\frac{p'}{p}}_{L^{\frac{N}{N-p}}log^{-\alpha\frac{p}{N-p}}L}.$$

Thanks to (2.1), we get

$$h(t)\le\frac{2^{\frac{p'}{p}}\|f\|^{\frac{p'}{p}}_{L^{\frac{N}{p}}log^{\alpha}L}}{N^{p'}C_N^{\frac{p'}{N}}}\frac{-\mu_n'(t)}{\mu_n(t)log^{\alpha\frac{p'}{N}}(e+\frac{1}{\mu_n(t)})}.$$

Thus, integrating both sides of the above inequality between 0 and τ gives

$$H(\tau)\le\frac{2^{\frac{p'}{p}}\|f\|^{\frac{p'}{p}}_{L^{\frac{N}{p}}log^{\alpha}L}}{N^{p'}C_N^{\frac{p'}{N}}}\int_0^{\tau}\frac{-\mu_n'(t)}{\mu_n(t)log^{\alpha\frac{p'}{N}}(e+\frac{1}{\mu_n(t)})}dt,$$

and a change of variables yields

$$H(\tau) \le \frac{2^{\frac{p'}{p}} \|f\|_{L^{\frac{N}{p}} log^{\alpha} L}^{\frac{p'}{p}}}{N^{p'} C_N^{\frac{p'}{N}}} \int_{\frac{1}{|\Omega|}}^{\frac{1}{\mu_n(\tau)}} \frac{ds}{s \, log^{\alpha \frac{p'}{N}}(e+s)}.$$

The definition of the rearrangement allows us to have

$$H(u_n^*(\sigma)) \le \frac{2^{\frac{p'}{p}} \|f\|_{L^{\frac{N}{p}} log^{\alpha} L}^{\frac{p'}{p}}}{N^{p'} C_N^{\frac{p'}{N}}} \int_{\frac{1}{|\Omega|}}^{\frac{1}{\sigma}} \frac{ds}{s \, log^{\alpha \frac{p'}{N}}(e+s)},$$

hence follows the inequality

$$H(\|u_n\|_\infty) \le \frac{2^{\frac{p'}{p}} \|f\|_{L^{\frac{N}{p}} log^{\alpha} L}^{\frac{p'}{p}}}{N^{p'} C_N^{\frac{p'}{N}}} \int_{\frac{1}{|\Omega|}}^{\infty} \frac{ds}{s \, log^{\alpha \frac{p'}{N}}(e+s)}. \tag{4.5}$$

Since $\alpha \frac{p'}{N} > 1$, the integral in (4.5) converges, and the assumptions made on the function H ensures that $\|u_n\|_\infty$ is uniformly bounded, indeed one has

$$\|u_n\|_\infty \le H^{-1}\left(\frac{2^{\frac{p'}{p}} \|f\|_{L^{\frac{N}{p}} log^{\alpha} L}^{\frac{p'}{p}}}{N^{p'} C_N^{\frac{p'}{N}}} \int_{\frac{1}{|\Omega|}}^{\infty} \frac{ds}{s \, log^{\alpha \frac{p'}{N}}(e+s)} \right), \tag{4.6}$$

where H^{-1} denotes the inverse function of H.
In what follows, let us denote by λ the constant on the right of (4.6), that is

$$\|u_n\|_\infty \le \lambda. \tag{4.7}$$

Step2: $W_0^{1,p}$-estimate
Thanks to (1.2) and (4.7), it is easy to get an estimation in $W_0^{1,p}(\Omega)$. Taking u_n as test function in (4.1), we get

$$h^{p-1}(\lambda) \int_\Omega |\nabla u_n|^p dx \le c \|f\|_{L^{\frac{N}{p}} log^{\alpha} L},$$

where c is a constant not depending on n.
Therefore, we can deduce that there exist a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and a function u in $W_0^{1,p}(\Omega)$ such that

$$u_n \to u \quad \text{weakly in } W_0^{1,p}(\Omega) \text{ and a.e in } \Omega. \tag{4.8}$$

Step3: Passage to the limit
In order to pass to the limit in the equation (4.1), we need to prove the almost everywhere convergence of the gradients of solutions, that is

$$\nabla u_n \to \nabla u \quad \text{ a.e in } \Omega. \tag{4.9}$$

This can be proved as previously done in [4], indeed this is easy since u_n is a bounded function. For $n \ge \lambda$, thanks to (4.7), (4.8) and (4.9) one can pass to the limit in (4.1), for all ϕ in $\mathcal{D}(\Omega)$, and conclude, since f belongs at least to $L^1(\Omega)$, that u is a bounded solution to problem (1.1) in the sense of (3.1).

References

[1] Alvino, A., Ferone, V., Trombetti, G.: *A priori estimates for a class of non uniformly elliptic equations*, Atti Semin. Mat. Fis. Univ. Modena **46**-suppl., 381-391(1998)

[2] Alvino, A., Boccardo, L., Ferone, V., Orsina, L., Trombetti, G.: *Existence results for nonlinear elliptic equations with degenerate coercivity*, Annali di Mat. **182**, 53-79(2003)

[3] Boccardo, L., Dall'Aglio, A., Orsina, L.: *Existence and regularity results for some nonlinear equations with degenerate coercivity*, Atti Sem. Mat. Fis. Univ. Modena **46**-suppl., 51-81(1998)

[4] Del Vecchio, T.: *Nonlinear elliptic equations with measure data*, Potential Analysis **4**, 185-203(1995)

[5] Krasnosel'skii, M., Rutikii, Ya.: *Convex functions and Orlicz spaces*, Groningen, Nordhooff (1969)

[6] Leray, J., Lions, J. L.: *Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder*, Bull. Soc. Mat. France **93**, 97-107(1965)

[7] Talenti, G.: *Nonlinear elliptic equations, Rearrangements of functions and Orlicz spaces*, Ann. Mat. Pura Appl.,IV. Ser. **120**, 159-184(1979)

[8] Talenti, G.: *Linear elliptic P.D.E's: Level sets, rearrangements and a priori estimates of solutions*, Boll. Un. Mat. Ital. **4**-B(6), 917-949(1985)

Département de Mathématiques et Informatique
Faculté des Sciences Dhar-Mahraz
B.P 1796 Atlas, Fès, Morocco.
emails : a.benkirane@menara.ma, ahmed.youssfi@caramail.com, driss.meskine@laposte.net