The product of a regular form by a polynomial generalized: the case $xu = \lambda x^2 v$

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Abstract

We consider the problem: given a regular form (linear functional) v, find all the regular forms u which satisfy the relation $xu = \lambda x^2 v,\ \lambda \in \mathbb{C} - \{0\}$. We give the second-order recurrence relation of the orthogonal polynomial sequence with respect to u. Some examples are studied.

Introduction

In the present paper, we intend to study the following problem: Let v be a regular form (linear functional), R and D are non-zero polynomials. Find all regular forms u satisfying:

$$Ru = Dv. \tag{1}$$

This problem has been studied in some particular cases. In fact the product of a linear form by a polynomial (R(x)=1) is studied in [5,6,7] and the inverse problem ($D(x) = \lambda$, $\lambda \in \mathbb{C} - \{0\}$) is considered in [12,15,19,21]. More generally, when R and D have non-trivial common factor the authors of [13] found necessary and sufficient conditions for u to be a regular form. The case where $R = D$ is treated in [2,3,12,14]. The aim of this contribution is to analyze the case in which $R(x) = x$ and $D(x) = \lambda x^2$, $\lambda \in \mathbb{C} - \{0\}$. We remark that R and D have a common factor and $R \neq D$. In fact, the inverse problem is studied in [23,24]. On the other hand, this situation generalizes the case treated in [14] (see (1.2) below).
In the first section, we will give the regularity conditions and the coefficients of the

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second-order recurrence relation satisfied by the monic orthogonal sequence (MOPS) with respect to u. We will study the case where v is a symmetric form: the regularity conditions become simpler. The particular case where v is a symmetric positive definite form is analyzed. The second section is devoted to the case where v is a semi-classical. We will prove that u is also semi-classical form and some results concerning the class of u are given. In the last section, some examples will be treated. The regular forms found in theses examples are semi-classical of class $s \in \{1, 2\}$. The integral representations of these regular forms and they coefficients of the second-order recurrence satisfied by the MOPS with respect to u are given. As a result, we also found that the list given in [4] is not complete (see proposition 3.2 below).

1 The problem $\mathbf{xu} = \lambda \mathbf{x^2 v}$

1.1 The main problem

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and $\mathcal{P}'$ its dual. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. Let us recall that a form u is called regular if there exists a monic polynomial sequence $\{P_n\}_{n\geq 0}$, $\deg P_n = n,\ n \geq 0$ such that $\langle u, P_n P_m\rangle = r_n \delta_{n,m},\ n, m \geq 0,\ r_n \neq 0,\ n \geq 0$. The left-multiplication hw of the form w by a polynomial h is defined by $\langle hw, p\rangle := \langle w, hp\rangle$ for all $p \in \mathcal{P}$.
We consider the following problem: given a regular form v, find all regular forms u satisfying

$$xu = \lambda x^2 v\ ,\ \lambda \in \mathbb{C} - \{0\}, \tag{1.1}$$

with the constraints

$$(u)_0 = 1\ ,\ (v)_0 = 1,$$

where $(u)_n := \langle u, x^n\rangle$, $n \geq 0$, are the moments of u. This is equivalent to

$$u = \lambda xv + (1 - \lambda(v)_1)\delta, \tag{1.2}$$

where $\langle \delta, f\rangle = f(0)$.
We see that when $1 - \lambda(v)_1 \neq 0$ and xv is regular, we meet again the problem studied in [14].
We suppose that the form v possesses the following integral representation:

$$\langle v, f\rangle = \int_{-\infty}^{+\infty} V(x)f(x)dx, \quad \text{for each polynomial} f,$$

where V is a locally integrable function with rapid decay. Then the form u is represented by

$$\langle u, f\rangle = \lambda \int_{-\infty}^{+\infty} xV(x)f(x)dx + (1 - \lambda(v)_1)f(0). \tag{1.3}$$

Let $\{S_n\}_{n\geq 0}$ denote the sequence of monic orthogonal polynomials with respect to v, we have

$$\begin{aligned} &S_0(x) = 1\ ,\ S_1(x) = x - \xi_0\ , \\ &S_{n+2}(x) = (x - \xi_{n+1})S_{n+1}(x) - \sigma_{n+1}S_n(x),\ n \geq 0, \end{aligned} \tag{1.4}$$

with

$$\xi_n = \frac{\langle v, xS_n^2(x)\rangle}{\langle v, S_n^2\rangle} \quad , \quad \sigma_{n+1} = \frac{\langle v, S_{n+1}^2\rangle}{\langle v, S_n^2\rangle}, \; n \geq 0. \tag{1.5}$$

When u is regular, let $\{Z_n\}_{n\geq 0}$ be the corresponding monic orthogonal sequence

$$\begin{aligned} &Z_0(x) = 1 \;\;,\;\; Z_1(x) = x - \beta_0 \;, \\ &Z_{n+2}(x) = (x - \beta_{n+1})Z_{n+1}(x) - \gamma_{n+1}Z_n(x), \; n \geq 0, \end{aligned} \tag{1.6}$$

where $\gamma_{n+1} \neq 0$ for all $n \geq 0$.
From (1.1), we know that the existence of the sequence $\{Z_n\}_{n\geq 0}$ is among all the strictly quasi-orthogonal sequences of order one with respect to $\lambda x^2 v = w$, (w is not necessarily a regular form) [8,16,18,20]. This is

$$\begin{aligned} &x^2 Z_0(x) = S_2(x) + c_1 S_1(x) + b_0, \\ &\quad x^2 Z_{n+1}(x) = S_{n+3}(x) + c_{n+2}S_{n+2}(x) + b_{n+1}S_{n+1}(x) + a_n S_n(x), \; n \geq 0, \end{aligned} \tag{1.7}$$

with $a_n \neq 0$, $n \geq 0$.
By virtue of (1.7), we can deduce

$$S_{n+3}(0) + c_{n+2}S_{n+2}(0) + b_{n+1}S_{n+1}(0) + a_n S_n(0) = 0, \; n \geq 0. \tag{1.8}$$

$$xZ_{n+1}(x) = (\theta_0 S_{n+3})(x) + c_{n+2}(\theta_0 S_{n+2})(x) + b_{n+1}(\theta_0 S_{n+1})(x) + a_n(\theta_0 S_n)(x), \quad n \geq 0. \tag{1.9}$$

$$Z_{n+1}(x) = (\theta_0^2 S_{n+3})(x) + c_{n+2}(\theta_0^2 S_{n+2})(x) + b_{n+1}(\theta_0^2 S_{n+1})(x) + a_n(\theta_0^2 S_n)(x), \quad n \geq 0, \tag{1.10}$$

with in general $(\theta_c f)(x) := \frac{f(x)-f(c)}{x-c}$, $c \in \mathbb{C}$, $f \in \mathcal{P}$.

Lemma 1.1. *Let $\{Z_n\}_{n\geq 0}$ be a sequence of polynomials satisfying* (1.7) *where a_n, b_n and c_n are complex numbers such that $a_n \neq 0$ for all $n \geq 0$. The sequence $\{Z_n\}_{n\geq 0}$ is orthogonal with respect to u if and only if*

$$\langle u, Z_{n+1}\rangle = 0, \; n \geq 0. \tag{1.11}$$

Proof.
The condition (1.11) is necessary from the definition of the orthogonality of $\{Z_n\}_{n\geq 0}$ with respect to u.
For $0 \leq k \leq n$ we have

$$\begin{aligned} \langle u, x^{k+1}Z_{n+1}(x)\rangle &= \langle xu, x^k Z_{n+1}(x)\rangle \\ &= \lambda\langle v, x^{k+2}Z_{n+1}(x)\rangle, \; n \geq 0 \text{ (by (1.1))}. \end{aligned}$$

Taking the relation (1.7) into account, we get

$$\begin{aligned} \langle u, x^{k+1}Z_{n+1}(x)\rangle =& \lambda\langle v, x^k S_{n+3}(x)\rangle + \lambda c_{n+2}\langle v, x^k S_{n+2}(x)\rangle \\ &+ \lambda b_{n+1}\langle v, x^k S_{n+1}(x)\rangle + \lambda a_n\langle v, x^k S_n(x)\rangle \end{aligned}$$

From the orthogonality of $\{S_n\}_{n\geq 0}$, we obtain

$$\begin{aligned} \langle u, x^{k+1}Z_{n+1}(x)\rangle &= 0, \;\; 0 \leq k \leq n-1, \; n \geq 1, \\ \langle u, x^{n+1}Z_{n+1}(x)\rangle &= \lambda a_n\langle v, S_n^2\rangle \neq 0, \; n \geq 0. \end{aligned}$$

Consequently, the precedent relation and (1.11) prove that $\{Z_n\}_{n\geq 0}$ is orthogonal with respect to u. This proves the Lemma.

Based on (1.7) and (1.11), we get

$$\begin{aligned}
0 &= S_{n+3}(0) + c_{n+2}S_{n+2}(0) + b_{n+1}S_{n+1}(0) + a_nS_n(0),\ n \geq 0,\\
0 &= S'_{n+3}(0) + c_{n+2}S'_{n+2}(0) + b_{n+1}S'_{n+1}(0) + a_nS'_n(0),\ n \geq 0,\\
0 &= \langle u, Z_{n+1}\rangle\\
&= \langle u, \theta_0^2 S_{n+3}\rangle + c_{n+2}\langle u, \theta_0^2 S_{n+2}\rangle + b_{n+1}\langle u, \theta_0^2 S_{n+1}\rangle + a_n\langle u, \theta_0^2 S_n\rangle,\ n \geq 0, \qquad (1.12)
\end{aligned}$$

with the following initial conditions:

$$\begin{aligned}
0 &= S_2(0) + c_1S_1(0) + b_0S_0(0),\\
0 &= S'_2(0) + c_1S'_1(0) + b_0S'_0(0).
\end{aligned} \qquad (1.13)$$

If we denote

$$\Delta_n := \begin{vmatrix} S_{n+2}(0) & S_{n+1}(0) & S_n(0)\\ S'_{n+2}(0) & S'_{n+1}(0) & S'_n(0)\\ \langle u, \theta_0^2 S_{n+2}\rangle & \langle u, \theta_0^2 S_{n+1}\rangle & \langle u, \theta_0^2 S_n\rangle \end{vmatrix},\ n \geq 0. \qquad (1.14)$$

From the Cramer rule, we get

$$\Delta_n a_n = -\Delta_{n+1},\ n \geq 0. \qquad (1.15)$$

$$\Delta_n b_{n+1} = \begin{vmatrix} S_{n+2}(0) & -S_{n+3}(0) & S_n(0)\\ S'_{n+2}(0) & -S'_{n+3}(0) & S'_n(0)\\ \langle u, \theta_0^2 S_{n+2}\rangle & -\langle u, \theta_0^2 S_{n+3}\rangle & \langle u, \theta_0^2 S_n\rangle \end{vmatrix},\ n \geq 0. \qquad (1.16)$$

$$\Delta_n c_{n+2} = \begin{vmatrix} -S_{n+3}(0) & S_{n+1}(0) & S_n(0)\\ -S'_{n+3}(0) & S'_{n+1}(0) & S'_n(0)\\ -\langle u, \theta_0^2 S_{n+3}\rangle & \langle u, \theta_0^2 S_{n+1}\rangle & \langle u, \theta_0^2 S_n\rangle \end{vmatrix},\ n \geq 0. \qquad (1.17)$$

Proposition 1.2. *The form u is regular if and only if $\Delta_n \neq 0$, $n \geq 0$. In this case the coefficients of the second-order recurrence relation of $\{Z_n\}_{n\geq 0}$ are given by the following formulas:*

$$\gamma_1 = -\lambda\frac{\Delta_1}{\Delta_0}. \qquad (1.18)$$

$$\gamma_{n+2} = \frac{\Delta_n\Delta_{n+2}}{\Delta_{n+1}^2}\sigma_{n+1},\ n \geq 0. \qquad (1.19)$$

$$\beta_0 = \lambda b_0. \qquad (1.20)$$

$$\beta_{n+1} = -b_{n+1}\frac{\Delta_n}{\Delta_{n+1}}\sigma_{n+1} + c_{n+2} - \xi_{n+2} - \xi_{n+1},\ n \geq 0. \qquad (1.21)$$

Proof.
Necessity.
Through (1.14), we have

$$\Delta_0 = -S'_1(0)\langle u, \theta_0^2 S_2\rangle = -1. \qquad (1.22)$$

$\{Z_n\}_{n\geq 0}$ is orthogonal with respect to u, hence it is strictly quasi-orthogonal of order one with respect to x^2v, which satisfies (1.7) with $a_n \neq 0$, $n \geq 0$. This implies $\Delta_n \neq 0$, $n \geq 0$. Assuming the contrary, there exists an $n_0 \geq 1$ such that $\Delta_{n_0} = 0$. Then from (1.15), $\Delta_0 = 0$ becomes a contradiction.
Sufficiency.
Let

$$c_1 = -S_2'(0). \tag{1.23}$$
$$b_0 = -c_1 S_1(0) - S_2(0). \tag{1.24}$$

Then the initial conditions (1.13) are satisfied.
Furthermore, the system (1.12) is a Cramer system whose solution is given by (1.15), (1.16) and (1.17). The numbers a_n, b_n and c_n ($n \geq 0$) define a sequence of polynomials $\{Z_n\}_{n\geq 0}$ by (1.7). Therefore, it follows from (1.12) and Lemma 1.1 that u is regular ($\{Z_n\}_{n\geq 0}$ is the corresponding monic orthogonal polynomial sequence).
Moreover, we have

$$\langle u, Z_{n+1}^2\rangle = \langle u, x^{n+1}Z_{n+1}(x)\rangle = \lambda\langle v, x^{n+2}Z_{n+1}(x)\rangle,\ n \geq 0,$$

by (1.7) and the orthogonality of $\{S_n\}_{n\geq 0}$. We get

$$\langle u, Z_{n+1}^2\rangle = \lambda a_n\langle v, S_n^2\rangle,\ n \geq 0.$$

Taking the relation (1.15) into account, we obtain

$$\langle u, Z_{n+1}^2\rangle = -\lambda\frac{\Delta_{n+1}}{\Delta_n}\langle v, S_n^2\rangle,\ n \geq 0. \tag{1.25}$$

Making $n = 0$ in the latter equation, we get (1.18).
On the other hand, we have

$$\gamma_{n+2} = \frac{\langle u, Z_{n+2}^2\rangle}{\langle u, Z_{n+1}^2\rangle},\ n \geq 0.$$

Based on the relation (1.25), we can deduce (1.19).
We have $\beta_0 = \langle u, x\rangle = \lambda\langle v, x^2 Z_0(x)\rangle$ and by (1.7) and the orthogonality of $\{S_n\}_{n\geq 0}$ we obtain (1.20).
From (1.9) and the orthogonality of $\{Z_n\}_{n\geq 0}$, we obtain

$$\langle u, xZ_{n+1}^2(x)\rangle = \langle u, Z_{n+1}\theta_0 S_{n+3}\rangle + c_{n+2}\langle u, Z_{n+1}^2\rangle,\ n \geq 0. \tag{1.26}$$

Using (1.4), we have

$$\theta_0 S_{n+3} = S_{n+2} - \xi_{n+2}\theta_0 S_{n+2} - \sigma_{n+2}\theta_0 S_{n+1},\ n \geq 0.$$

Through the latter relation and the orthogonality of $\{Z_n\}_{n\geq 0}$, we get

$$\langle u, Z_{n+1}\theta_0 S_{n+3}\rangle = \langle u, Z_{n+1}S_{n+2}\rangle - \xi_{n+2}\langle u, Z_{n+1}^2\rangle,\ n \geq 0.$$

However, we have

$$\begin{aligned}\langle u, Z_{n+1}S_{n+2}\rangle &= \langle xu, Z_{n+1}S_{n+1}\rangle - \xi_{n+1}\langle u, Z_{n+1}^2\rangle \text{ (by(1.4))}\\ &= \lambda\langle v, x^2 Z_{n+1}(x)S_{n+1}(x)\rangle - \xi_{n+1}\langle u, Z_{n+1}^2\rangle,\ n \geq 0, \text{(by (1.1))}.\end{aligned}$$

On account of (1.7) and the orthogonality of $\{S_n\}_{n\geq 0}$, we get

$$\langle u, Z_{n+1}S_{n+2}\rangle = \lambda b_{n+1}\langle v, S_{n+1}^2\rangle - \xi_{n+1}\langle u, Z_{n+1}^2\rangle,\ n\geq 0,$$

then the latter becomes

$$\langle u, Z_{n+1}\theta_0 S_{n+3}\rangle = \lambda b_{n+1}\langle v, S_{n+1}^2\rangle - (\xi_{n+1}+\xi_{n+2})\langle u, Z_{n+1}^2\rangle,\ n\geq 0.$$

Therefore, (1.26) can be written as the following

$$\langle u, xZ_{n+1}^2(x)\rangle = \lambda b_{n+1}\langle v, S_{n+1}^2\rangle + (c_{n+2}-\xi_{n+1}-\xi_{n+2})\langle u, Z_{n+1}^2\rangle,\ n\geq 0.$$

As a matter of fact, we get

$$\beta_{n+1} = \frac{\langle u, xZ_{n+1}^2(x)\rangle}{\langle u, Z_{n+1}^2\rangle} = \lambda b_{n+1}\frac{\langle v, S_{n+1}^2\rangle}{\langle u, Z_{n+1}^2\rangle} + c_{n+2}-\xi_{n+1}-\xi_{n+2},\ \ n\geq 0.$$

By virtue of (1.25), we can deduce (1.21).

1.2 The computation of Δ_n

As we have seen in the proposition 1.2, it is very important to have an explicit expression of Δ_n.
First, we need the following lemma:
Lemma 1.3. *The following formulas hold*

$$\langle u, \theta_0 S_n\rangle = \lambda\langle v, S_n\rangle - \lambda S_n(0) + (1-\lambda(v)_1)S_n'(0),\ n\geq 0. \tag{1.27}$$

$$\langle u, \theta_0^2 S_n\rangle = \frac{1}{2}S_n''(0) + \lambda(S_{n-1}^{(1)}(0) - S_n'(0) - \frac{1}{2}(v)_1 S_n''(0)),\ n\geq 0, \tag{1.28}$$

$$\langle v, S_n^2\rangle = S_n(0)S_n^{(1)}(0) - S_{n+1}(0)S_{n-1}^{(1)}(0),\ n\geq 0, \tag{1.29}$$

with $S_n^{(1)}(x) = \langle v, \frac{S_{n+1}(x)-S_{n+1}(\xi)}{x-\xi}\rangle$, $n\geq 0$ *and* $S_{-1}^{(1)}(x)=0$.
Proof.
Both formulas (1.27) and (1.28) can be deduced from (1.2).
The formula (1.29) is proved in [23].
By (1.4), we successively obtain the following relations:

$$S_{n+2}(0) = -\xi_{n+1}S_{n+1}(0) - \sigma_{n+1}S_n(0),\ n\geq 0. \tag{1.30}$$

$$S_{n+2}'(0) = S_{n+1}(0) - \xi_{n+1}S_{n+1}'(0) - \sigma_{n+1}S_n'(0),\ n\geq 0. \tag{1.31}$$

$$(\theta_0 S_{n+2})(x) = S_{n+1}(x) - \xi_{n+1}(\theta_0 S_{n+1})(x) - \sigma_{n+1}(\theta_0 S_n)(x),\ n\geq 0. \tag{1.32}$$

$$(\theta_0^2 S_{n+2})(x) = (\theta_0 S_{n+1})(x) - \xi_{n+1}(\theta_0^2 S_{n+1})(x) - \sigma_{n+1}(\theta_0^2 S_n)(x),\ n\geq 0. \tag{1.33}$$

Using (1.33), we get

$$\langle u, \theta_0^2 S_{n+2}\rangle = \langle u, \theta_0 S_{n+1}\rangle - \xi_{n+1}\langle u, \theta_0^2 S_{n+1}\rangle - \sigma_{n+1}\langle u, \theta_0^2 S_n\rangle,\ n\geq 0. \tag{1.34}$$

Taking the relations (1.30), (1.31) and (1.34) into account, we get (1.14) written as the following:

$$\Delta_n = \begin{vmatrix} 0 & S_{n+1}(0) & S_n(0) \\ S_{n+1}(0) & S_{n+1}'(0) & S_n'(0) \\ \langle u, \theta_0 S_{n+1}\rangle & \langle u, \theta_0^2 S_{n+1}\rangle & \langle u, \theta_0^2 S_n\rangle \end{vmatrix},\ n\geq 0,$$

that is

$$\Delta_n = -S_{n+1}(0)\Big\{S_{n+1}(0)\langle u, \theta_0^2 S_n\rangle - S_n(0)\langle u, \theta_0^2 S_{n+1}\rangle\Big\} \\ + \langle u, \theta_0 S_{n+1}\rangle\Big\{S_{n+1}(0)S_n'(0) - S_n(0)S_{n+1}'(0)\Big\},\ n \geq 0.$$

From the relations (1.27), (1.28) and (1.29), we get

$$\Delta_n = \lambda\Big\{S_{n+1}(0)\langle v, S_n^2\rangle - (v)_1\Big(\frac{1}{2}S_{n+1}(0)\chi_n'(0) - S_{n+1}'(0)\chi_n(0)\Big)\Big\} \\ + \frac{1}{2}S_{n+1}(0)\chi_n'(0) - S_{n+1}'(0)\chi_n(0),\ n \geq 0, \quad (1.35)$$

with

$$\chi_n(x) = S_n(x)S_{n+1}'(x) - S_{n+1}(x)S_n'(x),\ n \geq 0. \quad (1.36)$$

If the form u is regular, for (1.15), (1.16) and (1.17) we obtain

$$a_n = -\frac{\Delta_{n+1}}{\Delta_n},\ n \geq 0. \quad (1.37)$$

$$b_{n+1} = \Delta_n^{-1}(\lambda E_n + F_n) + \sigma_{n+2},\ n \geq 0. \quad (1.38)$$

$$c_{n+2} = -\Delta_n^{-1}(\lambda G_n + H_n) + \xi_{n+2},\ n \geq 0, \quad (1.39)$$

where

$$E_n = S_{n+2}(0)\Big(\Theta_n(0) + \frac{1}{2}(v)_1\mu_n'(0)\Big) - (v)_1 S_{n+2}'(0)\mu_n(0),\ n \geq 0. \quad (1.40)$$

$$F_n = -\frac{1}{2}S_{n+2}(0)\mu_n'(0) + S_{n+2}'(0)\mu_n(0),\ n \geq 0. \quad (1.41)$$

$$G_n = S_{n+2}(0)\Big(\langle v, S_n^2\rangle - \frac{1}{2}(v)_1\chi_n'(0)\Big) + (v)_1\chi_n(0)S_{n+2}'(0),\ n \geq 0. \quad (1.42)$$

$$H_n = -S_{n+2}'(0)\chi_n(0) + \frac{1}{2}S_{n+2}(0)\chi_n'(0),\ n \geq 0, \quad (1.43)$$

with

$$\mu_n(x) = S_{n+2}(x)S_n'(x) - S_{n+2}'(x)S_n(x),\ n \geq 0. \quad (1.44)$$

$$\Theta_n(x) = S_n(x)S_{n+1}^{(1)}(x) - S_{n+2}(x)S_{n-1}^{(1)}(x),\ n \geq 0. \quad (1.45)$$

1.3 The case where v is a symmetric form

In the following sequel we will assume that v is a symmetric regular form.
We need the following result:
Lemma 1.4. [23] *When $\{S_n\}_{\geq 0}$ is a symmetric sequence, we have*

$$S_{2n}(0) = \frac{(-1)^n}{\sigma_{2n+1}}\prod_{\mu=0}^{n}\sigma_{2\mu+1}\ ,\ n \geq 0\ ,\quad S_{2n+1}(0) = 0\ ,\ n \geq 0.$$

$$S_{2n+1}^{(1)}(0) = 0\ ,\ n \geq 0\ ,\quad S_{2n}'(0) = 0\ ,\ n \geq 0.$$

$$S_{2n+1}'(0) = (-1)^n\Lambda_n\prod_{\mu=0}^{n}\sigma_{2\mu}\ ,\ n \geq 0\ ,\quad S_{2n+1}''(0) = 0\ ,\ n \geq 0,$$

where

$$\Lambda_n = \sum_{\nu=0}^{n} \frac{1}{\sigma_{2\nu+1}} \prod_{\mu=0}^{\nu} \frac{\sigma_{2\mu+1}}{\sigma_{2\mu}},\ n \geq 0\,, \tag{1.46}$$

with $\sigma_0 = (u)_0 = 1$.

Proposition 1.5. *We have the following formulas:*

$$\begin{cases} \Delta_{2n} = \dfrac{(-1)^{n+1}}{\sigma_{2n+1}} \Big(\prod_{\mu=0}^{n} \sigma_{2\mu}\Big)^2 \Big(\prod_{\mu=0}^{n} \sigma_{2\mu+1}\Big) \Lambda_n^2,\ n \geq 0. \\ \Delta_{2n+1} = \lambda(-1)^{n+1} \Big(\prod_{\mu=0}^{n} \sigma_{2\mu}\Big) \Big(\prod_{\mu=0}^{n} \sigma_{2\mu+1}\Big)^2,\ n \geq 0, \end{cases} \tag{1.47}$$

Proof.
By virtue of lemma 1.4, for (1.36) we get

$$\begin{aligned} &\chi_{2n}(0) = \frac{\Lambda_n}{\sigma_{2n+1}} \prod_{\mu=0}^{2n+1} \sigma_\mu,\ n \geq 0\ ;\ \ \chi_{2n+1}(0) = \Lambda_n \prod_{\mu=0}^{2n+1} \sigma_\mu,\ n \geq 0. \\ &\chi_n'(0) = 0,\ n \geq 0 \end{aligned} \tag{1.48}$$

When v is a symmetric form, we have $(v)_1 = 0$, then (1.35) becomes

$$\Delta_n = \lambda S_{n+1}(0)\langle v, S_n^2\rangle + \frac{1}{2} S_{n+1}(0)\chi_n'(0) - S_{n+1}'(0)\chi_n(0),\ \ n \geq 0,$$

by (1.48), we get (1.47).

Theorem 1.6. *The form* u *is regular if and only if* $\Lambda_n \neq 0$, $n \geq 0$.
Proof.
We get the desired result from the proposition 1.5.

Corollary 1.7. *When* v *is a positive definite form* u *is a regular form.*
Proof.
If v is a positive definite then $\sigma_n > 0$. Therefore, we obtain $\Lambda_n > 0$, $n \geq 0$, thus the desired result.

Proposition 1.8. *When* u *is a regular form, we have*

$$\begin{aligned} a_{2n} &= -\lambda \sigma_{2n+1} \Lambda_n^{-2} \prod_{\mu=0}^{n} \frac{\sigma_{2\mu+1}}{\sigma_{2\mu}},\ n \geq 0, \\ a_{2n+1} &= \lambda^{-1} \sigma_{2n+2}^2 \Lambda_{n+1}^2 \prod_{\mu=0}^{n} \frac{\sigma_{2\mu}}{\sigma_{2\mu+1}},\ n \geq 0. \end{aligned} \tag{1.49}$$

$$\begin{aligned} b_{2n} &= \sigma_{2n+1},\ n \geq 0, \\ b_{2n+1} &= \sigma_{2n+2} + \Lambda_n^{-1} \prod_{\mu=0}^{n} \frac{\sigma_{2\mu+1}}{\sigma_{2\mu}},\ n \geq 0. \end{aligned} \tag{1.50}$$

$$\begin{aligned} c_1 &= 0, \\ c_{2n+2} &= -\lambda \Lambda_n^{-2} \prod_{\mu=0}^{n} \frac{\sigma_{2\mu+1}}{\sigma_{2\mu}},\ n \geq 0, \\ c_{2n+3} &= \lambda^{-1} \Lambda_n \Lambda_{n+1} \sigma_{2n+2} \prod_{\mu=0}^{n} \frac{\sigma_{2\mu}}{\sigma_{2\mu+1}},\ n \geq 0. \end{aligned} \tag{1.51}$$

Proof.
On account of (1.47) and (1.37), we get (1.49).
By (1.13), it follows that

$$b_0 = \sigma_1 \quad , \quad c_1 = 0. \tag{1.52}$$

For (1.44) and (1.45) we have

$$\mu_n(0) = 0,\ n \geq 0 \ \ ; \ \ \Theta_n(0) = 0,\ n \geq 0,$$
$$\mu'_{2n}(0) = -2\frac{\Lambda_n}{\sigma_{2n+1}}\Big(\prod_{\mu=0}^{n} \sigma_{2\mu}\Big)\Big(\prod_{\mu=0}^{n} \sigma_{2\mu+1}\Big),\ n \geq 0 \ \ , \ \ \mu'_{2n+1}(0) = 0,\ n \geq 0,$$

by the preceding relations and (1.48), for (1.40)-(1.43) we obtain

$$E_n = 0,\ n \geq 0 \ \ ; \ \ F_{2n} = (-1)^{n+1}\frac{\Lambda_n}{\sigma_{2n+1}}\Big(\prod_{\mu=0}^{n} \sigma_{2\mu}\Big)\Big(\prod_{\mu=0}^{n} \sigma_{2\mu+1}\Big)^2,\ n \geq 0,$$
$$F_{2n+1} = 0,\ n \geq 0 \ \ ; \ \ G_{2n} = \frac{(-1)^{n+1}}{\sigma_{2n+1}}\Big(\prod_{\mu=0}^{n} \sigma_{2\mu}\Big)\Big(\prod_{\mu=0}^{n} \sigma_{2\mu+1}\Big)^2,\ n \geq 0,$$
$$G_{2n+1} = 0,\ n \geq 0 \ \ ; \ \ H_{2n} = 0,\ n \geq 0,$$
$$H_{2n+1} = (-1)^n \sigma_{2n+2}\Lambda_n\Lambda_{n+1}\Big(\prod_{\mu=0}^{n} \sigma_{2\mu}\Big)^2 \prod_{\mu=0}^{n} \sigma_{2\mu+1},\ n \geq 0.$$

Taking the previous relations and (1.52) into account, the relations (1.38) and (1.39) give (1.50) and (1.51).

2 Some results on the semi-classical case

Let us recall that a form u is called semi-classical if it is regular and there exists two polynomials ϕ and ψ such that

$$(\phi u)' + \psi u = 0,$$

where the distributional derivative w' of a form w is defined by $\langle w', p\rangle = -\langle w, p'\rangle$, $p \in \mathcal{P}$.
The class of the semi-classical form u is $s = \max(\deg\phi - 2, \deg\psi - 1)$ if and only if the following condition is satisfied:

$$\prod_c \Big(\mid \psi(c) + \phi'(c) \mid + \mid \langle u, \theta_c\psi + \theta_c^2\phi\rangle \mid \Big) > 0, \tag{2.1}$$

where $c \in \{x : \phi(x) = 0\}$ [16].
In the following sequel, the form v is taken to be semi-classical of class s satisfying $(\phi v)' + \psi v = 0$.
From (1.1) when the form u is regular, it is also semi-classical and it satisfies

$$(\tilde{\phi} u)' + \tilde{\psi} u = 0,$$

with

$$\tilde{\phi}(x) = x^2\phi(x) \qquad \text{and} \qquad \tilde{\psi}(x) = x^2\psi(x) - 3x\phi(x). \tag{2.2}$$

Lemma 2.1.
(a) *We have the following formulas:*

$$(\theta_c(fg))(x) = f(x)(\theta_c g)(x) + g(c)(\theta_c f)(x)\ ,\ f, g \in \mathcal{P}. \tag{2.3}$$

$$\langle xw, \theta_c f\rangle = \langle w, f\rangle + c\langle w, \theta_c f\rangle - (w)_0 f(c)\ ,\ f \in \mathcal{P}\ ,\ w \in \mathcal{P}'. \tag{2.4}$$

(b) *Let $f, g \in \mathcal{P}$, $w \in \mathcal{P}'$, if we have $(fw)' + gw = 0$ then $\langle w, g\rangle = 0$.*

Proposition 2.2. *The class of u depends only on the zero $x = 0$.*

We use the following lemma to prove it:

Lemma 2.3. *For all zero c of ϕ, we have*

$$\langle u, \theta_c\tilde{\psi} + \theta_c^2\tilde{\phi}\rangle = \lambda c^3\langle v, \theta_c\psi + \theta_c^2\phi\rangle + (\psi(c) + \phi'(c))\Big\{c + (u)_1 - \lambda(c^2 + c(v)_1 + (v)_2)\Big\}, \tag{2.5}$$

and

$$\tilde{\psi}(c) + \tilde{\phi}'(c) = c^2\Big(\psi(c) + \phi'(c)\Big). \tag{2.6}$$

Proof.
Let c be a zero of ϕ, we can write the following equation:

$$\tilde{\phi}(x) = x^2(x - c)(\theta_c\phi)(x). \tag{2.7}$$

On account of (2.3), we successively obtain

$$(\theta_c^2\tilde{\phi})(x) = x^2(\theta_c^2\phi)(x) + \phi'(c)(\theta_c(t^2))(x). \tag{2.8}$$

$$(\theta_c\tilde{\psi})(x) = x^2(\theta_c\psi)(x) + \psi(c)(\theta_c(t^2))(x) - 3x(\theta_c\phi)(x). \tag{2.9}$$

Then

$$\langle u, \theta_c\tilde{\psi} + \theta_c^2\tilde{\phi}\rangle = \langle x^2u, \theta_c\psi + \theta_c^2\phi\rangle - 3\langle xu, \theta_c\phi\rangle + (\psi(c) + \phi'(c))\langle u, \theta_c(t^2)(x)\rangle,$$

by (1.1), we have $xu = \lambda x^2v$ and $x^2u = \lambda x^3v$ therefore, it follows that

$$\langle u, \theta_c\tilde{\psi} + \theta_c^2\tilde{\phi}\rangle = \lambda\langle x^3v, \theta_c\psi + \theta_c^2\phi\rangle - 3\lambda\langle x^2v, \theta_c\phi\rangle + (\psi(c) + \phi'(c))\langle u, \theta_c(t^2)(x)\rangle. \tag{2.10}$$

Using (2.4), we get successively

$$\langle x^3v, \theta_c\psi + \theta_c^2\phi\rangle = \langle v, x^2\psi\rangle + c\langle v, x\psi\rangle + c^2\langle v, \psi\rangle + \langle v, x\phi\rangle + 2c\langle v, \phi\rangle + 3c^2\langle v, \theta_c\phi\rangle + c^3\langle v, \theta_c\psi + \theta_c^2\phi\rangle - (\psi(c) + \phi'(c))\Big((v)_2 + c(v)_1 + c^2\Big),$$

$$\langle x^2v, \theta_c\phi\rangle = \langle v, x\phi\rangle + c\langle v, \phi\rangle + c^2\langle v, \theta_c\phi\rangle.$$

Consequently (2.10) can be written

$$\langle u, \theta_c\tilde{\psi} + \theta_c^2\tilde{\phi}\rangle = \lambda\langle v, x^2\psi - 2x\phi\rangle + \lambda c\langle v, x\psi - \phi\rangle + \lambda c^2\langle v, \psi\rangle + \lambda c^3\langle v, \theta_c\psi + \theta_c^2\phi\rangle + \{\langle u, \theta_c(t^2)(x)\rangle - \lambda(c^2 + c(v)_1 + (v)_2)\}(\psi(c) + \phi'(c)).$$

But $(\phi v)' + \psi v = 0$. Then $(x\phi v)' + (x\psi - \phi)v = 0$ and $(x^2\phi v)' + (x^2\psi - 2x\phi)v = 0$, by the lemma 2.1, we obtain

$$\langle v, \psi\rangle = 0, \quad \langle v, x\psi - \phi\rangle = 0, \quad \langle v, x^2\psi - 2x\phi\rangle = 0.$$

Therefore,

$$\langle u, \theta_c\tilde{\psi} + \theta_c^2\tilde{\phi}\rangle = \lambda c^3\langle v, \theta_c\psi + \theta_c^2\phi\rangle + \{\langle u, \theta_c(t^2)(x)\rangle - \lambda(c^2 + c(v)_1 + (v)_2)\}(\psi(c) + \phi'(c)).$$

On the other hand, $\langle u, \theta_c(t^2)(x)\rangle = \langle u, x + c\rangle = (u)_1 + c$, thus (2.5).
From (2.2), we can deduce (2.6).

Proof of the proposition 2.2.
Let c be a zero of ϕ such that $c \neq 0$.
If $\psi(c) + \phi'(c) = 0$, using (2.5), $\langle u, \theta_c\tilde{\psi} + \theta_c^2\tilde{\phi}\rangle = \lambda c^3\langle v, \theta_c\psi + \theta_c^2\phi\rangle \neq 0$ since v is semi-classical of class s and so satisfies (2.1).
If $\psi(c) + \phi'(c) \neq 0$, then $\tilde{\psi}(c) + \tilde{\phi}'(c) \neq 0$, from (2.6).
In all cases, we cannot simplify (2.2) by $x - c$.

Proposition 2.4. *Let v be a semi-classical form of class s satisfying*

$$(\phi v)' + \psi v = 0,$$

and introduce

$$\vartheta_1 := (1 - \lambda(v)_1)\phi(0), \tag{2.11}$$
$$\vartheta_2 := (1 - \lambda(v)_1)(\psi(0) - \phi'(0)), \tag{2.12}$$
$$\vartheta_3 := (1 - \lambda(v)_1)\psi'(0). \tag{2.13}$$

The form u given by (1.1) *is also a semi-classical of class $\tilde{s}$ satisfying*

$$(\tilde{\phi}u)' + \tilde{\psi}u = 0.$$

Moreover,
(1) *if $\vartheta_1 \neq 0$, then $\tilde{s} = s + 2$ and $\tilde{\phi}(x) = x^2\phi(x)$, $\tilde{\psi}(x) = x^2\psi(x) - 3x\phi(x)$;*
(2) *if $\vartheta_1 = 0$ and $\vartheta_2 \neq 0$ or $\phi(0) \neq 0$, then $\tilde{s} = s + 1$ and $\tilde{\phi}(x) = x\phi(x)$, $\tilde{\psi}(x) = x\psi(x) - 2\phi(x)$;*
(3) *if $\vartheta_1 = 0$, $\vartheta_2 = 0$, $\phi(0) = 0$ and $\vartheta_3 \neq 0$ or $\psi(0) \neq 0$, then $\tilde{s} = s$ and $\tilde{\phi}(x) = \phi(x)$, $\tilde{\psi}(x) = \psi(x) - (\theta_0\phi)(x)$.*

Proof.
(1) From (2.2), we have

$$\tilde{\psi}(0) + \tilde{\phi}'(0) = 0,$$

and

$$\langle u, \theta_0\tilde{\psi} + \theta_0^2\tilde{\phi}\rangle = \langle u, x\psi(x) - 2\phi(x)\rangle = \langle xu, \psi\rangle - 2\langle u, \phi\rangle.$$

Taking into account the relation (1.2), we obtain

$$\langle u, \theta_0\tilde{\psi} + \theta_0^2\tilde{\phi}\rangle = \lambda\langle v, x^2\psi(x) - 2x\phi(x)\rangle - 2(1 - \lambda(v)_1)\phi(0).$$

But $(\phi v)' + \psi v = 0$, then $(x^2\phi(x)v)' + (x^2\psi(x) - 2x\phi(x))v = 0$. By virtue of the lemma 2.1, we have $\langle v, x^2\psi(x) - 2x\phi(x)\rangle = 0$ so, the latter becomes

$$\langle u, \theta_0\tilde{\psi} + \theta_0^2\tilde{\phi}\rangle = -2(1 - \lambda(v)_1)\phi(0) = -2\vartheta_1. \tag{2.14}$$

Therefore, if $\vartheta_1 \neq 0$, it is not possible to simplify from (2.1), which means that the class of u is $\tilde{s} = s + 2$ and u satisfies

$$(\tilde{\phi}u)' + \tilde{\psi}u = 0, \tag{2.15}$$

with

$$\tilde{\phi}(x) = x^2\phi(x), \quad \tilde{\psi}(x) = x^2\psi(x) - 3x\phi(x).$$

(2) If $\vartheta_1 = 0$, by (2.14) and (2.15) u satisfies

$$(\tilde{\phi}_0 u)' + \tilde{\psi}_0 u = 0, \tag{2.16}$$

with

$$\tilde{\phi}_0(x) = x\phi(x), \quad \tilde{\psi}_0(x) = x\psi(x) - 2\phi(x).$$

Then

$$\tilde{\psi}_0(0) + \tilde{\phi}_0'(0) = -\phi(0), \tag{2.17}$$

and

$$\begin{aligned}\langle u, \theta_0\tilde{\psi}_0 + \theta_0^2\tilde{\phi}_0\rangle &= \langle u, \psi - \theta_0\phi\rangle \\ &= \lambda\langle v, x\psi(x) - x(\theta_0\phi)(x)\rangle + (1 - \lambda(v)_1)(\psi(0) - \phi'(0)) \\ &= \lambda\langle v, x\psi(x) - \phi(x)\rangle + \lambda\phi(0) + (1 - \lambda(v)_1)(\psi(0) - \phi'(0)).\end{aligned}$$

But $(\phi v)' + \psi v = 0$, then $(x\phi(x)v)' + (x\psi(x) - \phi(x))v = 0$. By lemma 2.1 we obtain $\langle v, x\psi(x) - \phi(x)\rangle = 0$. As result, we get

$$\langle u, \theta_0\tilde{\psi}_0 + \theta_0^2\tilde{\phi}_0\rangle = \lambda\phi(0) + \vartheta_2. \tag{2.18}$$

On account of (2.17), (2.18) and (2.1), we can deduce that when $\phi(0) \neq 0$ or $\vartheta_2 \neq 0$, it impossible to simplify equation (2.16), which means that the class of u is $\tilde{s} = s+1$.
(3) When $\vartheta_1 = 0$, $\vartheta_2 = 0$ and $\phi(0) = 0$, by (2.16) and (2.18) u satisfies

$$(\tilde{\phi}_1 u)' + \tilde{\psi}_1 u = 0, \tag{2.19}$$

with

$$\tilde{\phi}_1(x) = \phi(x), \quad \tilde{\psi}_1(x) = \psi(x) - (\theta_0\phi)(x). \tag{2.20}$$

Then

$$\tilde{\psi}_1(0) + \tilde{\phi}_1'(0) = \psi(0), \tag{2.21}$$

and

$$\langle u, \theta_0\tilde{\psi}_1 + \theta_0^2\tilde{\phi}_1\rangle = \langle u, \theta_0\psi\rangle = \lambda\langle v, x(\theta_0\psi)(x)\rangle + (1 - \lambda(v)_1)\psi'(0).$$

Consequently, it follows that

$$\langle u, \theta_0\tilde{\psi}_1 + \theta_0^2\tilde{\phi}_1\rangle = -\lambda\psi(0) + \vartheta_3. \tag{2.22}$$

From (2.21) and (2.22), we can deduce that if $\psi(0) \neq 0$ or $\vartheta_3 \neq 0$ which means it is impossible to simplify (2.19) and $\tilde{s} = s$.

3 Some examples

3.1. Let us describe the case $v := \mathcal{H}(\tau)$, where $\mathcal{H}(\tau)$ is the generalized Hermite form. Here is [5]

$$\xi_n = 0,\ n \geq 0, \qquad \sigma_{n+1} = \frac{n+1+\tau(1+(-1)^n)}{2},\ n \geq 0. \tag{3.1}$$

Then

$$\prod_{\mu=0}^{n} \sigma_{2\mu+1} = \frac{\Gamma(n+\tau+3/2)}{\Gamma(\tau+1/2)},\ n \geq 0, \qquad \prod_{\mu=0}^{n} \sigma_{2\mu} = \Gamma(n+1),\ n \geq 0. \tag{3.2}$$

We want

$$\Lambda_n = \sum_{\nu=0}^{n} \frac{1}{\sigma_{2\nu+1}} \prod_{\mu=0}^{\nu} \frac{\sigma_{2\mu+1}}{\sigma_{2\mu}},\ n \geq 0.$$

From (3.1) and (3.2), we have

$$\frac{1}{\sigma_{2\nu+1}} \prod_{\mu=0}^{\nu} \frac{\sigma_{2\mu+1}}{\sigma_{2\mu}} = \frac{\Gamma(\nu+\tau+3/2)}{(\nu+\tau+1/2)\Gamma(\nu+1)\Gamma(\tau+1/2)} = \frac{1}{\Gamma(\tau+1/2)} h_\nu,$$

where

$$h_\nu = \frac{\Gamma(\nu+\tau+1/2)}{\Gamma(\nu+1)},\ \nu \geq 0,$$

fulfilling

$$(\nu+1)h_{\nu+1} - \nu h_\nu = (\tau+1/2)h_\nu,$$

and so

$$\Lambda_n = \frac{1}{\Gamma(\tau+1/2)} \sum_{\nu=0}^{n} h_\nu = \frac{1}{(\tau+1/2)\Gamma(\tau+1/2)} \sum_{\nu=0}^{n} \Big\{(\nu+1)h_{\nu+1} - \nu h_\nu\Big\}.$$

We can deduce that

$$\Lambda_n = \frac{(n+1)h_{n+1}}{\Gamma(\tau+3/2)} = \frac{\Gamma(n+\tau+3/2)}{\Gamma(\tau+3/2)\Gamma(n+1)},\ n \geq 0. \tag{3.3}$$

Therefore we have the following table:
Table 1

Δ_n	$\Delta_{2n} = (-1)^{n+1}\frac{\tau+1/2}{\Gamma^3(\tau+3/2)}\frac{\Gamma^3(n+\tau+3/2)}{n+\tau+1/2},\ n\geq 0,$ $\Delta_{2n+1} = (-1)^{n+1}\frac{\lambda}{\Gamma^2(\tau+1/2)}\Gamma(n+1)\Gamma^2(n+\tau+3/2),\ n\geq 0.$
a_n	$a_{2n} = -\lambda(\tau+1/2)\Gamma(\tau+3/2)\frac{\Gamma(n+1)}{\Gamma(n+\tau+1/2)},\ n\geq 0,$ $a_{2n+1} = \frac{1}{\lambda(\tau+1/2)\Gamma(\tau+3/2)}\frac{(n+\tau+3/2)\Gamma(n+\tau+5/2)}{\Gamma(n+1)},\ n\geq 0.$
b_n	$b_{2n} = n+\tau+1/2,\ n\geq 0$, $b_{2n+1} = n+\tau+3/2,\ n\geq 0.$
c_n	$c_{2n+2} = -\lambda(\tau+1/2)\Gamma(\tau+3/2)\frac{\Gamma(n+1)}{\Gamma(n+\tau+3/2)},\ n\geq 0,$ $c_1=0$, $c_{2n+3} = \frac{1}{\lambda(\tau+1/2)\Gamma(\tau+3/2)}\frac{\Gamma(n+\tau+5/2)}{\Gamma(n+1)},\ n\geq 0.$
γ_{n+1}	$\gamma_1 = -\lambda^2(\tau+1/2)^2,$ $\gamma_{2n+3} = -\lambda^2(\tau+1/2)^2\Gamma^2(\tau+3/2)\frac{\Gamma^2(n+2)}{\Gamma^2(n+\tau+5/2)},\ n\geq 0,$ $\gamma_{2n+2} = -\frac{1}{\lambda^2(\tau+1/2)^2\Gamma^2(\tau+3/2)}\frac{\Gamma^2(n+\tau+5/2)}{\Gamma^2(n+1)},\ n\geq 0.$
β_n	$\beta_0 = \lambda(\tau+1/2),$ $\beta_{2n+2} = \lambda(\tau+1/2)\Gamma(\tau+3/2)\frac{\Gamma(n+2)}{\Gamma(n+\tau+5/2)} + \frac{1}{\lambda(\tau+1/2)\Gamma(\tau+3/2)}\frac{\Gamma(n+\tau+5/2)}{\Gamma(n+1)},\ n\geq 0,$ $\beta_{2n+1} = -\frac{1}{\lambda(\tau+1/2)\Gamma(\tau+3/2)}\frac{\Gamma(n+\tau+5/2)}{\Gamma(n+1)} - \lambda(\tau+1/2)\Gamma(\tau+3/2)\frac{\Gamma(n+1)}{\Gamma(n+\tau+3/2)},\ n\geq 0.$

Proposition 3.1. *If* $v=\mathcal{H}(\tau)$ *is the generalized Hermite form, then the form* u *given by* (1.1) *possesses the following integral representation:*

$$\langle u,f\rangle = \frac{\lambda}{\Gamma(\tau+1/2)}\int_{-\infty}^{+\infty} x\mid x\mid^{2\tau} e^{-x^2}f(x)dx + f(0),\ \forall f\in\mathcal{P},\ \Re\tau>-1/2. \tag{3.4}$$

It is a quasi-antisymmetric and semi-classical form of class s *satisfying the following functional equation*

$$(x^2u)' + (2x^3-(2\tau+3)x)u = 0,\quad \tau\neq -1,\quad s=2. \tag{3.5}$$

$$(xu)' + 2x^2u = 0,\quad \tau=-1,\quad s=1. \tag{3.6}$$

Proof.
It is well known that the generalized Hermite form possesses the following integral representation [5]

$$\langle v, f\rangle = \int_{-\infty}^{+\infty} V(x)f(x)dx,\ \forall f \in \mathcal{P},$$

with $V(x) = \dfrac{1}{\Gamma(\tau+1/2)} \mid x \mid^{2\tau}$, $x \in \mathbb{R}$, $\Re\tau > -1/2$. Following from (1.3), we easily obtain (3.4).
Also, the form u is quasi-antisymmetric because it satisfies

$$\langle u, x^{2n+2}\rangle = \lambda\langle v, x^{2n+3}\rangle = 0,\ n \geq 0.$$

When $\tau = 0$, v is the classical Hermite form. The latter satisfies [17]

$$(\phi_0 v)' + \psi_0 v = 0,$$

with $\phi_0(x) = 1$, $\psi_0(x) = 2x$. Therefore, (2.15) becomes $\vartheta_1 = 1 \neq 0$. By virtue of the proposition 2.4, we get

$$(\tilde{\phi}_0 u)' + \tilde{\psi}_0 u = 0, \tag{3.7}$$

where $\tilde{\phi}_0(x) = x^2$, $\tilde{\psi}_0(x) = 2x^3 - 3x$, with u a semi-classical form of class $s = 2$.
When $\tau \neq 0$, the generalized Hermite form is a semi-classical of class one and satisfies [1]

$$(\phi_1 v)' + \psi_1 v = 0,$$

with $\phi_1(x) = x$, $\psi_1(x) = 2x^2 - 2\tau - 1$. In this case, for (2.15) and (2.16) we have

$$\vartheta_1 = 0,\quad \vartheta_2 = -2(\tau+1).$$

If $\tau \neq -1$, by virtue of the proposition 2.4, we get

$$(\tilde{\phi}_1 u)' + \tilde{\psi}_1 u = 0, \tag{3.8}$$

with $\tilde{\phi}_1(x) = x^2$, $\tilde{\psi}_1(x) = 2x^3 - (2\tau+3)x$ and u a semi-classical form of class $s = 2$. Then, (3.8) gives (3.5).
When $\tau = -1$, we have $\psi_1(0) = 1 \neq 0$, by virtue of the proposition 2.4, we can deduce (3.6).

Proposition 3.2. *When* $\tau = -1$, *the form* u *satisfying the equation* (3.6) *has the following integral representation:*

$$\langle u, f\rangle = -\frac{\lambda}{2\Gamma(1/2)} P\int_{-\infty}^{+\infty} \frac{e^{-x^2}}{x} f(x)dx + f(0),\ \forall f \in \mathcal{P}, \tag{3.9}$$

where [7]

$$P\int_{-\infty}^{+\infty} \frac{V(x)}{x}dx = \lim_{\epsilon\to 0}\Big(\int_{-\infty}^{-\epsilon} \frac{V(x)}{x}dx + \int_{\epsilon}^{+\infty} \frac{V(x)}{x}dx\Big).$$

Proof.
By virtue of the previous proposition, the form u is quasi antisymmetric

$$(u)_{2n+2} = 0,\ n \geq 0. \tag{3.10}$$

On account of (1.1), we get $\langle xu, 1\rangle = \lambda\langle x^2 v, 1\rangle$ and we have

$$(u)_1 = \lambda(v)_2 = \lambda\sigma_1.$$

By (3.1), we obtain

$$(u)_1 = -\frac{\lambda}{2}. \tag{3.11}$$

From the functional equation (3.6), we get

$$\langle (xu)' + 2x^2 u, x^{2n+1}\rangle = 0,\ \ n \geq 0,$$

which is equivalent to

$$(u)_{2n+3} = (n + 1/2)(u)_{2n+1},\ \ n \geq 0,$$

consequently

$$(u)_{2n+3} = \frac{\Gamma(n+3/2)}{\Gamma(1/2)}(u)_1,\ \ n \geq 0.$$

By (3.11), we can deduce that

$$(u)_{2n+1} = -\frac{\lambda}{2\Gamma(1/2)}\Gamma(n+1/2),\ \ n \geq 0. \tag{3.12}$$

From the definition of the gamma function, we get

$$\begin{aligned}\langle u, x^{2n+1}\rangle &= -\frac{\lambda}{2\Gamma(1/2)}\int_0^{+\infty} x^{n-1/2}e^{-x}dx = -\frac{\lambda}{\Gamma(1/2)}\int_0^{+\infty} x^{2n}e^{-x^2}dx \\ &= -\frac{\lambda}{2\Gamma(1/2)}\int_{-\infty}^{+\infty} x^{2n}e^{-x^2}dx,\ n \geq 0.\end{aligned}$$

Then, we can deduce

$$\langle u, x^{2n+1}\rangle = -\frac{\lambda}{2\Gamma(1/2)}\lim_{\varepsilon\to 0}\Big(\int_{-\infty}^{-\epsilon}\frac{e^{-x^2}}{x}x^{2n+1}dx + \int_{\epsilon}^{+\infty}\frac{e^{-x^2}}{x}x^{2n+1}dx\Big),\ n \geq 0.$$

On account of (3.10), we can write

$$\langle u, x^n\rangle = -\frac{\lambda}{2\Gamma(1/2)}\lim_{\varepsilon\to 0}\Big(\int_{-\infty}^{-\epsilon}\frac{e^{-x^2}}{x}x^n dx + \int_{\epsilon}^{+\infty}\frac{e^{-x^2}}{x}x^n dx\Big),\ n \geq 1,$$

taking (3.11) into account, we get

$$\langle u, x^n\rangle = -\frac{\lambda}{2\Gamma(1/2)}\lim_{\varepsilon\to 0}\Big(\int_{-\infty}^{-\epsilon}\frac{e^{-x^2}}{x}x^n dx + \int_{\epsilon}^{+\infty}\frac{e^{-x^2}}{x}x^n dx\Big) - \frac{\lambda}{2}\langle\delta, x^n\rangle,\ n \geq 0.$$

Hence (3.9).

Remark. The integral representation given in (3.9) does not exist in the list given in [4].

3.2. Let us describe the case $v := \mathcal{J}_{(1/2,1/2)}$. It is the second kind Chebyshev functional, which is a particular case of the Jacobi form $\mathcal{J}_{(\alpha,\beta)}$ for $\alpha = \beta = 1/2$. Here is [5]:

$$\xi_n = 0,\ n \geq 0 \quad , \quad \sigma_{n+1} = \frac{1}{4},\ n \geq 0. \tag{3.13}$$

Then,

$$\prod_{\mu=0}^{n} \sigma_{2\mu+1} = \frac{1}{4^{n+1}},\ n \geq 0 \quad , \quad \prod_{\mu=0}^{n} \sigma_{2\mu} = \frac{1}{4^{n}},\ n \geq 0. \tag{3.14}$$

So, for (1.46) we get

$$\Lambda_n = n+1,\ n \geq 0. \tag{3.15}$$

Therefore, we obtain the table below:
Table 2

Δ_n	$\Delta_{2n} = (-1)^{n+1}\frac{(n+1)^2}{4^{3n}},\ n \geq 0 \quad , \quad \Delta_{2n+1} = \lambda\frac{(-1)^{n+1}}{4^{3n+2}},\ n \geq 0.$
a_n	$a_{2n} = -\frac{\lambda}{4^2(n+1)^2},\ n \geq 0 \quad , \quad a_{2n+1} = \frac{(n+2)^2}{4\lambda},\ n \geq 0.$
b_n	$b_{2n} = \frac{1}{4},\ n \geq 0 \quad , \quad b_{2n+1} = \frac{n+2}{4(n+1)},\ n \geq 0.$
c_n	$c_1 = 0 \quad , \quad c_{2n+3} = \frac{(n+1)(n+2)}{\lambda},\ n \geq 0 \quad , \quad c_{2n+2} = -\frac{\lambda}{4(n+1)^2},\ n \geq 0.$
γ_{n+1}	$\gamma_{2n+2} = -\lambda^{-2}(n+1)^2(n+2)^2,\ n \geq 0 \quad , \quad \gamma_{2n+1} = -\frac{\lambda^2}{4^2(n+1)^2},\ n \geq 0.$
β_n	$\beta_0 = \frac{\lambda}{4} \quad , \quad \beta_{2n+2} = \frac{\lambda}{4(n+2)^2} + (n+1)(n+2)\lambda^{-1},\ n \geq 0$ $\beta_{2n+1} = -\frac{\lambda}{4(n+1)^2} - (n+1)(n+2)\lambda^{-1},\ n \geq 0$

Proposition 3.3. *If $v = \mathcal{J}_{(1/2,1/2)}$, the second kind Chebyshev form, then the form u given by* (1.1) *possesses the following integral representation:*

$$\langle u, f\rangle = f(0) + \lambda\sqrt{\frac{2}{\pi}}\int_{-1}^{1} x\sqrt{1-x^2}f(x)dx,\ \ f \in \mathcal{P}. \tag{3.16}$$

The form u is a quasi-antisymmetric and semi-classical of class $s = 2$ satisfying the following functional equation:

$$\left(x^2(x^2-1)u\right)' - 3x(2x^2-1)u = 0. \tag{3.17}$$

Proof.
It is well known that the second kind Chebyshev form possesses the following integral representation [5]:

$$\langle v, f\rangle = \int_{-1}^{1} V(x)dx, \forall f \in \mathcal{P},$$

with $V(x)=\sqrt{\frac{2}{\pi}}\sqrt{1-x^2}, x\in]-1,1[$. Following from (1.3), we get (3.16).
Also, u is quasi-antisymmetric because it satisfies

$$\langle u, x^{2n+2}\rangle = \lambda\langle v, x^{2n+3}\rangle = 0,\ n\geq 0.$$

The form v is classical and it satisfies [17]

$$\big((x^2-1)v\big)' - 3xv = 0.$$

Then, $\vartheta_1=-1\neq 0$, by virtue of the proposition 2.4, we get (3.17).

3.3 Let us describe $v=\mathcal{J}_{(-1/2,1/2)}$, the third kind Chebyshev form. The latter is the co-recursive of the second kind Chebyshev form. We have [5]

$$\xi_0=-\frac{1}{2},\ \ \xi_{n+1}=0,\, n\geq 0\ \ ,\ \ \sigma_{n+1}=\frac{1}{4},\, n\geq 0. \tag{3.18}$$

We have the following results:
Lemma 3.4. [23] *The following formulas hold*

$$S_{2n}(0)=\frac{(-1)^n}{2^{2n}},\, n\geq 0\ \ ,\ \ S_{2n+1}(0)=\frac{(-1)^n}{2^{2n+1}},\, n\geq 0,$$
$$S^{(1)}_{2n}(0)=\frac{(-1)^n}{2^{2n}},\, n\geq 0\ \ ,\ \ S^{(1)}_{2n+1}(0)=0,\, n\geq 0,$$
$$S'_{2n}(0)=(-1)^{n+1}\frac{n}{2^{2n-1}},\, n\geq 0\ \ ,\ \ S'_{2n+1}(0)=(-1)^n\frac{n+1}{2^{2n}},\, n\geq 0,$$
$$S''_{2n}(0)=(-1)^{n+1}\frac{n(n+1)}{2^{2n-2}},\, n\geq 0\ ,\ S''_{2n+1}(0)=(-1)^{n+1}\frac{n(n+1)}{2^{2n-1}},\, n\geq 0.$$

Following the previous lemma, for (1.36), (1.44) and (1.45) we get

$$\chi_{2n}(0)=\frac{2n+1}{2^{4n}},\, n\geq 0\ \ ,\ \ \chi_{2n+1}(0)=\frac{n+1}{2^{4n+1}},\, n\geq 0,$$
$$\chi'_{2n}(0)=0,\, n\geq 0\ \ ,\ \ \chi'_{2n+1}(0)=\frac{n+1}{2^{4n}},\, n\geq 0,$$
$$\mu_{2n}(0)=\frac{-1}{2^{4n+1}},\, n\geq 0\ \ ,\ \ \mu_{2n+1}(0)=\frac{1}{2^{4n+3}},\, n\geq 0,$$
$$\mu'_{2n}(0)=-\frac{n+1}{2^{4n-1}},\, n\geq 0\ \ ,\ \ \mu'_{2n+1}(0)=-\frac{n+1}{2^{4n+1}},\ n\geq 0,$$
$$\Theta_n(0)=0,\, n\geq 0,\, \langle v, S_n^2\rangle=\frac{1}{4^n},\, n\geq 0\,,\, (v)_1=-\frac{1}{2}.$$

Then, we obtain

$$\begin{aligned}\Delta_{2n}&=\lambda\frac{(-1)^{n+1}}{2^{6n+1}}\big((1+2\lambda^{-1})(n+1)(2n+1)-1\big),\, n\geq 0,\\ \Delta_{2n+1}&=\lambda\frac{(-1)^{n+1}}{2^{6n+4}}\big((1+2\lambda^{-1})(n+1)(2n+3)+1\big),\, n\geq 0.\end{aligned} \tag{3.19}$$

On account of the proposition 1.2, the form u is regular if and only if

$$t(n+1)(2n+1)-1\neq 0, n\geq 0\ ,\ t(n+1)(2n+3)+1\neq 0, n\geq 0, \tag{3.20}$$

where $t=1+2\lambda^{-1}$.

We assume that the previous conditions are satisfied. Therefore, we get the table below:
Table 3

a_n	$a_{2n}=-\frac{1}{8}\frac{t(n+1)(2n+3)+1}{t(n+1)(2n+1)-1},\ n\geq 0\ ,\ a_{2n+1}=\frac{1}{8}\frac{t(n+2)(2n+3)-1}{t(n+1)(2n+3)+1},\ n\geq 0.$
b_n	$b_0=\frac{1}{2}\ ,\ b_{2n+2}=\frac{1}{4}\frac{t(n+2)(2n+3)+1}{t(n+1)(2n+3)+1},\ n\geq 0,\ \ b_{2n+1}=\frac{1}{4}\frac{t(n+1)(2n+3)-1}{t(n+1)(2n+1)-1},\ n\geq 0.$
c_n	$c_1=-\frac{1}{2}\ ,\ c_{2n+3}=\frac{1}{2}\frac{t(n+1)(2n+3)-1}{t(n+1)(2n+3)+1},\ n\geq 0\ ,\ c_{2n+2}=-\frac{1}{2}\frac{t(n+1)(2n+1)+1}{t(n+1)(2n+1)-1},\ n\geq 0.$
γ_{n+1}	$\gamma_1=-\frac{\lambda(2\lambda+3)}{8}\ ,\ \gamma_{2n+3}=-\frac{1}{4}\frac{\big(t(n+1)(2n+3)+1\big)\big(t(n+2)(2n+5)-1\big)}{\big(t(n+2)(2n+3)-1\big)^2},\ n\geq 0,$ $\gamma_{2n+2}=-\frac{1}{4}\frac{\big(t(n+1)(2n+1)-1\big)\big(t(n+2)(2n+3)-1\big)}{\big(t(n+1)(2n+3)+1\big)^2},\ n\geq 0.$
β_n	$\beta_0=\frac{\lambda}{2}\ ,\ \beta_{2n+2}=\frac{t^2(n+1)(n+2)(2n+3)^2+1}{\big(t(n+2)(2n+3)-1\big)\big(t(n+1)(2n+3)+1\big)},\ n\geq 0,$ $\beta_{2n+1}=-\frac{t^2(n+1)^2(2n+1)(2n+3)+1}{\big(t(n+1)(2n+3)+1\big)\big(t(n+1)(2n+1)-1\big)},\ n\geq 0.$

Proposition 3.5. *If $v=\mathcal{J}_{(-1/2,1/2)}$, the third kind Chebyshev form, then the form u given by* (1.1) *possesses the following integral representation:*

$$\langle u,f\rangle=(1+\frac{1}{2}\lambda)f(0)+\frac{\lambda}{\pi}\int_{-1}^{1}x\sqrt{\frac{1-x}{1+x}}f(x)dx,\ f\in\mathcal{P}. \tag{3.21}$$

The form u is a semi-classical form of class s satisfying the following functional equation:

$$\begin{aligned}\lambda\neq -2\ ,\ s=2,\ \ (x^2(x^2-1)u)'-x(5x^2+x-3)u=0,\\ \lambda=-2\ ,\ s=1,\ \ (x(x^2-1)u)'-(4x^2+x-2)u=0.\end{aligned} \tag{3.22}$$

Proof.
It is well known that $v=\mathcal{J}_{(-1/2,1/2)}$ possesses the following integral representation [5]:

$$\langle v,f\rangle=\int_{-1}^{1}V(x)f(x)dx,\ f\in\mathcal{P},$$

with $V(x)=\frac{1}{\pi}\sqrt{\frac{1-x}{1+x}},\ x\in]-1,1[$. Following from (1.3), we easily obtain (3.21).
The form v is classical and satisfies [17]

$$(\phi v)'+\psi v=0,$$

with $\phi(x)=x^2-1,\ \psi(x)=-2x-1$. Then, (2.15) and (2.16) become

$$\vartheta_1=-\frac{1}{2}(\lambda+2)\ ,\ \vartheta_2=-\frac{1}{2}(\lambda+2),$$

and $\phi(0)=-1\neq 0$.
The proposition 2.4 is enough to obtain (3.22).

3.4. Let us describe the case where v is the form given in [11,22]. We have

$$\xi_n = (-1)^n,\ n \geq 0 \quad , \quad \sigma_{n+1} = -\frac{1}{4},\ n \geq 0. \tag{3.23}$$

Lemma 3.6. *We have the following formulas:*

$$S_n(0) = (-1)^{\nu_n}\frac{n+1}{2^n},\ n \geq 0. \tag{3.24}$$

$$S_n^{(1)}(0) = (-1)^{n+\nu_n}\frac{n+1}{2^n},\ n \geq 0. \tag{3.25}$$

$$S_n'(0) = (-1)^{\nu_n}\left((-1)^n - 1\right)\frac{n+1}{2^{n+1}},\ n \geq 0. \tag{3.26}$$

$$S_n''(0) = \frac{(-1)^{1+\nu_n}}{3.2^{n+2}}(n+1)\left(2n-1+(-1)^n\right)\left(2n+5-(-1)^n\right),\ n \geq 0, \tag{3.27}$$

where

$$\nu_n = \frac{2n+1-(-1)^n}{4},\ n \geq 0. \tag{3.28}$$

Proof.
In this case, (1.4) becomes

$$\begin{aligned} &S_0(x) = 1 \quad , \quad S_1(x) = x - 1, \\ &S_{n+2}(x) = (x + (-1)^n)S_{n+1}(x) + \frac{1}{4}S_n(x),\ n \geq 0. \end{aligned} \tag{3.29}$$

So, we get

$$S_0(0) = 1, \quad S_1(0) = -1, \quad S_2(0) = -\frac{3}{4}, \tag{3.30}$$

$$S_{n+2}(0) = (-1)^n S_{n+1}(0) + \frac{1}{4}S_n(0),\ n \geq 0. \tag{3.31}$$

From (3.31), we can deduce the following relations:

$$S_{2n+1}(0) = S_{2n+2}(0) - \frac{1}{4}S_{2n}(0),\ n \geq 0. \tag{3.32}$$

$$S_{2n+3}(0) = -S_{2n+2}(0) + \frac{1}{4}S_{2n+1}(0),\ n \geq 0. \tag{3.33}$$

On account of (3.32), the relation (3.33) becomes

$$S_{2n+4}(0) + \frac{1}{2}S_{2n+2}(0) + \frac{1}{16}S_{2n}(0) = 0,\ n \geq 0,$$

by (3.30), we can deduce that

$$S_{2n}(0) = (-1)^n\frac{2n+1}{2^{2n}},\ n \geq 0. \tag{3.34}$$

By virtue of the previous relation and (3.32), we obtain

$$S_{2n+1}(0) = (-1)^{n+1}\frac{n+1}{2^{2n}},\ n \geq 0. \tag{3.35}$$

The relations (3.34) and (3.35) produce (3.24).
The sequence $\{S_n^{(1)}\}_{n\geq 0}$ satisfies the following recurrence relation

$$\begin{aligned} &S_0^{(1)}(x) = 1 \quad , \quad S_1^{(1)}(x) = x + 1, \\ &S_{n+2}^{(1)}(x) = (x - (-1)^n)S_{n+1}^{(1)}(x) + \frac{1}{4}S_n^{(1)}(x),\ n \geq 0. \end{aligned} \tag{3.36}$$

The above analogous calculations give (3.25).
From (3.29), we obtain

$$S_0'(0) = 0 \quad , \quad S_2'(0) = 0, \tag{3.37}$$

$$S_{n+2}'(0) = (-1)^n S_{n+1}'(0) + \frac{1}{4} S_n'(0) + S_{n+1}(0), \; n \geq 0. \tag{3.38}$$

Following (3.38), we get

$$S_{2n+1}'(0) = S_{2n+2}'(0) - \frac{1}{4} S_{2n}'(0) - S_{2n+1}(0), \; n \geq 0. \tag{3.39}$$

$$S_{2n+2}'(0) = -S_{2n+3}'(0) + \frac{1}{4} S_{2n+1}'(0) + S_{2n+2}(0), \; n \geq 0. \tag{3.40}$$

On account of (3.39), equation (3.40) can be written as following:

$$S_{2n+4}'(0) + \frac{1}{2} S_{2n+2}'(0) + \frac{1}{16} S_{2n}'(0) = S_{2n+3}(0) - \frac{1}{4} S_{2n+1}(0) + S_{2n+2}(0), \; n \geq 0.$$

By (3.24) and (3.37), we can deduce that

$$S_{2n}'(0) = 0, \; n \geq 0. \tag{3.41}$$

By virtue of the preceding relation and (3.24), equation (3.39) becomes

$$S_{2n+1}'(0) = (-1)^n \frac{n+1}{2^{2n}}, \; n \geq 0. \tag{3.42}$$

Then, (3.41) and (3.42) give (3.26).
On account of (3.29), we obtain

$$S_0''(0) = 0 \; , \; S_1''(0) = 0 \; , \; S_2''(0) = 2. \tag{3.43}$$

$$S_{n+2}''(0) = (-1)^n S_{n+1}''(0) + \frac{1}{4} S_n''(0) + 2 S_{n+1}'(0), \; n \geq 0. \tag{3.44}$$

Therefore, by (3.44), it follows that

$$S_{2n+1}''(0) = S_{2n+2}''(0) - \frac{1}{4} S_{2n}''(0) - 2 S_{2n+1}'(0), \; n \geq 0. \tag{3.45}$$

$$S_{2n+3}''(0) = -S_{2n+2}''(0) + \frac{1}{4} S_{2n+1}''(0) + 2 S_{2n+2}'(0), \; n \geq 0. \tag{3.46}$$

By (3.45) and (3.26), equation (3.46) can be written as

$$S_{2n+4}''(0) + \frac{1}{2} S_{2n+2}''(0) + \frac{1}{16} S_{2n}''(0) = (-1)^{n+1} \frac{4n+6}{4^{n+1}}, \; n \geq 0.$$

Then, we get

$$S_{2n}''(0) = (-1)^{n+1} \frac{n(n+1)(2n+1)}{3.2^{2n-2}}, \; n \geq 0. \tag{3.47}$$

On account of (3.47), (3.26) and (3.45), we obtain

$$S_{2n+1}''(0) = (-1)^n \frac{n(n+1)(n+2)}{3.2^{2n-2}}, \; n \geq 0. \tag{3.48}$$

Then (3.47) and (3.48) give (3.27).
Following from lemma 3.6, for (1.36), (1.44) and (1.45) we get

$$\chi_n(0) = \frac{(n+1)(n+2)}{2^{2n+1}},\ n \geq 0,$$
$$\chi'_n(0) = (-1)^n \frac{(n+1)(n+2)}{3.2^{2n+1}}\Big(2n+3-3(-1)^n\Big),\ n \geq 0,$$
$$\mu_n(0) = 0,\ n \geq 0\ \ ,\ \ \mu'_n(0) = -\frac{(n+1)(n+2)(n+3)}{3.2^{2n}},\ n \geq 0,$$
$$\Theta_n(0) = \frac{1}{2^{2n}},\ n \geq 0.$$

Then, we get

$$\Delta_n = \frac{(-1)^{n+1+\nu_{n+1}}}{3.2^{3n+2}}(n+2)t_n,\ n \geq 0, \tag{3.49}$$

where

$$t_n = (n+1)(n+2)(n+3)(\lambda-1) - 6\lambda. \tag{3.50}$$

On account of the proposition 1.2, the form u is regular if and only if $t_n \neq 0$, $n \geq 0$.
We assume that the previous condition is satisfied. Therefore, we obtain the following table:
Table 4

a_n	$\frac{(-1)^n}{8}\frac{n+3}{n+2}\frac{t_{n+1}}{t_n},\ n \geq 0.$
b_n	$b_0 = \frac{3}{4}\ ,\ b_{n+1} = \frac{n+4}{4(n+2)},\ n \geq 0.$
c_n	$c_1 = 0\ ,\ c_{n+2} = \frac{(-1)^n}{2}\frac{n+1}{n+2}\frac{t_{n+1}}{t_n},\ n \geq 0.$
γ_{n+1}	$\gamma_1 = -\lambda\frac{t_1}{2^5}\ \ ,\ \ \gamma_{n+2} = \frac{(n+2)(n+4)}{4(n+3)^2}\frac{t_n t_{n+2}}{t_{n+1}^2},\ n \geq 0.$
β_n	$\beta_0 = \frac{3}{4}\lambda\ ,\ \beta_{n+1} = \frac{(-1)^n}{2}\Big\{\frac{n+1}{n+2}\frac{t_{n+1}}{t_n} - \frac{n+4}{n+3}\frac{t_n}{t_{n+1}}\Big\},\ n \geq 0.$

Proposition 3.7. *The form u given by* (1.1) *have the following integral representation:*

$$\langle u, f\rangle = \frac{2\lambda}{\pi}\int_{-1}^{1} x^2\sqrt{\frac{1-x}{1+x}}f(x)dx + (1-\lambda)f(0),\ f \in \mathcal{P}. \tag{3.51}$$

The form u is a semi-classical form of class s satisfying the following functional equation:

$$\lambda \neq 1,\ s = 2,\quad \big(x^2(x^2-1)u\big)' + \big(-6x^3 + x^2 + 4x\big)u = 0, \tag{3.52}$$
$$\lambda = 1,\ s = 1,\quad \big(x(x^2-1)u\big)' + \big(-5x^2 + x + 3\big)u = 0. \tag{3.53}$$

Proof.
The form v has the following integral representation [22]:

$$\langle v, f\rangle = \int_{-1}^{1} V(x)f(x)dx,\ \ f \in \mathcal{P},$$

with $V(x) = \frac{2}{\pi}x\sqrt{\frac{1-x}{1+x}}$, $x \in]-1,1[$ and $(v)_1 = 1$. Following from (1.3) we obtain (3.51). The form v is a semi-classical of class one and satisfies [22]

$$(\phi v)' + \psi v = 0,$$

where $\phi(x) = x(x^2-1)$, $\psi(x) = -4x^2+x+2$. Then $\vartheta_1 = 0$, $\vartheta_2 = 3(1-\lambda)$, $\vartheta_3 = 0$, $\phi(0) = 0$ and $\psi(0) = 2 \neq 0$.
By virtue of the proposition 2.4 we get (3.52) and (3.53).

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