Extrinsic spheres in a real space form

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Abstract

Let M be an n-dimensional orientable compact hypersurface in an $(n+1)$-dimensional real space form $\overline{M}(c)$, $n \geq 2$. If the lengths $\|R\|$, $\|A\|$ and $\|\nabla\alpha\|$ of the curvature tensor field R, the shape operator A, the gradient $\nabla\alpha$ of the mean curvature α and the scalar curvature S of the hypersurface M satisfy the inequality

$$\frac{1}{2}\|R\|^2 \leq cS + \delta\|A\|^2 - n(n-1)\|\nabla\alpha\|^2$$

where $\delta = \min Ric = \min_{p\in M} \min_{v\in T_pM} \min_{\|v\|=1} Ric_p(v)$, Ric is Ricci curvature of the hypersurface, then it is shown that M is an extrinsic sphere in $\overline{M}(c)$. In particular we deduce that the condition $\frac{1}{2}\|R\|^2 \leq \delta\|A\|^2 - n(n-1)\|\nabla\alpha\|^2$ characterizes spheres in the Euclidean space R^{n+1} among the compact orientable hypersurfaces whose Ricci curvatures are bounded below by a constant $\delta > 0$.

1 Introduction:

A totally umbilical hypersurface in a real space form $\overline{M}(c)$ (Riemannian manifold of constant curvature c) is called an extrinsic sphere. The class of compact hypersurfaces in a real space form $\overline{M}(c)$ is quite large and therefore it is an interesting question in Geometry to obtain conditions which characterize extrinsic spheres in this class. This question has been of considerable interest to many geometers and had been approached using various invariants of the hypersurfaces. Most natural invariants of a hypersurface are the mean curvature, Ricci curvature and scalar curvature. Nomizu and Smyth have studied non-negatively curved hypersurfaces

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with constant mean curvature in real space form and in particular they have shown that such compact hypersurfaces in a Euclidean space are spheres (cf. [10],[11]). Hypersurfaces with constant mean curvature and higher order mean curvatures in a real space form have also been studied by Chen, Montiel-Ros, Ripoll, Ros and obtained different characterizations for extrinsic spheres (cf. [2], [3], [4], [9], [12], [14]). Similarly Ros[13] has studied compact embedded hypersurfaces with constant scalar curvature in an Euclidean space and proved that they are essentially spheres. We denote by R the curvature tensor field of the hypersurface M of the real space form $\overline{M}(c)$. For a local orthonormal frame $\{e_1, .., e_n\}$ on M the length $\|R\|$ of the curvature tensor field is given by $\|R\|^2 = \sum_{ijk} \|R(e_i, e_j)e_k\|^2$. In this paper we use the invariant $\|R\|$ of the hypersurface to characterize the extrinsic spheres. Let A be the shape operator of the hypersurface, α its mean curvature and S be its scalar curvature. An interesting question would be, using the invariants $\alpha, S, \|A\|, \|R\|$ of the hypersurface, how to characterize extrinsic spheres in the real space form $\overline{M}(c)$? The motivation of this question comes from the following: A sphere $S^n(c)$ in R^{n+1}, satisfies the equality

$$\frac{1}{2}\|R\|^2 = \delta \|A\|^2 - n(n-1)\|\nabla\alpha\|^2$$

$\delta = \min Ric = \min_{p\in M} \min_{v\in T_pM} \min_{\|v\|=1} Ric_p(v)$, Ric being the Ricci curvatures and $\nabla\alpha$ being the gradient of the mean curvature α (This follows from the Gauss equation for the hypersurface in a Euclidean space and the fact that α is constant for the sphere $S^n(c)$) . This raises a question, does a compact hypersurface of R^{n+1} satisfying above equality necessarily a sphere? In this paper we show that the answer to this question is in affirmative, and indeed we prove the following general result which gives a characterization for extrinsic spheres in a real space form $\overline{M}(c)$ and as a particular case we get the characterization of spheres in R^{n+1}.

Theorem: Let M be an n-dimensional orientable compact hypersurface of the simply connected real space form $\overline{M}(c)$, $n \geq 2$. If the scalar curvature S, the shape operator A, the mean curvature α and the curvature tensor field R of M satisfy

$$\frac{1}{2}\|R\|^2 \leq cS + \delta \|A\|^2 - n(n-1)\|\nabla\alpha\|^2,$$

where $\delta = \min Ric = \min_{p\in M} \min_{v\in T_pM} \min_{\|v\|=1} Ric_p(v)$, then M is an extrinsic sphere.

As a particular case of above theorem we have

Corollary: Let M be an n-dimensional orientable compact hypersurface of the Euclidean space R^{n+1}, $n \geq 2$. If the shape operator A, the mean curvature α and the curvature tensor field R of M satisfy

$$\frac{1}{2}\|R\|^2 \leq \delta \|A\|^2 - n(n-1)\|\nabla\alpha\|^2,$$

where $\delta = \min Ric = \min_{p\in M} \min_{v\in T_pM} \min_{\|v\|=1} Ric_p(v)$, then M is a sphere.

2 Preliminaries

Let M be an orientable hypersurface of the real space form $\overline{M}(c)$. We denote the induced metric on M by g. Let $\overline{\nabla}$ be the Riemannian connection on the real space form $\overline{M}(c)$ and ∇ be the Riemannian connection on M with respect to the induced metric g. Let N be the unit normal vector field and A be the shape operator of M. Then the Gauss and Weingarten formulas for the hypersurface are (cf. [1])

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \overline{\nabla}_X N = -AX, \qquad X, Y \in \mathfrak{X}(M) \tag{2.1}$$

where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M. We also have the following Gauss and Codazzi equations

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\} + g(AY, Z)AX - g(AX, Z)AY \tag{2.2}$$

$$(\nabla A)(X, Y) = (\nabla A)(Y, X), \qquad X, Y \in \mathfrak{X}(M) \tag{2.3}$$

where R is the curvature tensor field of the hypersurface and $(\nabla A)(X, Y) = \nabla_X AY - A\nabla_X Y$. The mean curvature α of the hypersurface is given by $n\alpha = \sum_i g(Ae_i, e_i)$, where $\{e_1, .., e_n\}$ is a local orthonormal frame on M. If $A = \lambda I$ holds for a constant λ, then the hypersurface is said to be totally umbilical and a totally umbilical hypersurface is called and an extrinsic sphere. The square of the length of the shape operator A is given by

$$\|A\|^2 = \sum_{ij} g(Ae_i, e_j)^2 = tr.A^2$$

From equation (2.2) we get the following expression for the Ricci tensor field

$$Ric(X, Y) = (n-1)cg(X, Y) + n\alpha g(AX, Y) - g(AX, AY) \tag{2.4}$$

The scalar curvature S of the hypersurface is given by

$$S = n(n-1)c + n^2\alpha^2 - \|A\|^2 \tag{2.5}$$

3 Some Lemmas

Let M be an orientable hypersurface of the real space form $\overline{M}(c)$ and $\nabla\alpha$ be the gradient of the mean curvature function α. Then we have

Lemma 3.1 Let M be an n-dimensional orientable hypersurface of the real space form $\overline{M}(c)$ and $\{e_1, .., e_n\}$ be a local orthonormal frame on the hypersurface M. Then

$$\sum_i (\nabla A)(e_i, e_i) = n\nabla\alpha.$$

The proof is straightforward and follows from the symmetry of A and the equation (2.3).

Lemma 3.2 Let M be an orientable hypersurface of the real space form $\overline{M}(c)$. Then the Ricci curvature tensor field of the hypersurface M satisfies

$$Ric(AX, Y) = Ric(X, AY),\ X, Y \in \mathfrak{X}(M).$$

The proof follows immediately from equation (2.4) and the symmetry of A.

Lemma 3.3 Let M be an n-dimensional orientable hypersurface of the real space form $\overline{M}(c)$. Then for a local orthonormal frame $\{e_1, .., e_n\}$ on M the following holds

$$\sum_{ij} g(R(e_j, e_i; Ae_j, Ae_i) = cS - \frac{1}{2}\|R\|^2,$$

where $R(X, Y; Z, W) = g(R(X, Y)Z, W)$, $X, Y, Z, W \in \mathfrak{X}(M)$.

Proof. Using equations (2.2) and (2.5) we compute

$$\begin{aligned}\|R\|^2 &= \sum_{ijk} g(R(e_i, e_j)e_k, R(e_i, e_j)e_k) \\ &= \sum_{ijk} R(e_i, e_j; e_k, R(e_i, e_j)e_k) \\ &= 2cS + 2\sum_{ij} g(R(e_i, e_j, Ae_j, Ae_i)\end{aligned}$$

and this proves the Lemma.

Lemma 3.4 Let M be an n-dimensional orientable hypersurface of the real space form $\overline{M}(c)$, $n \geq 2$. Then

$$\|\nabla A\|^2 \geq n\|\nabla\alpha\|^2,$$

where $\|\nabla A\|^2 = \sum_{ij} \|(\nabla A)(e_i, e_j)\|^2$ for a local orthonormal frame $\{e_1, .., e_n\}$ on M, moreover the equality holds if and only if α is constant and A is parallel.

Proof. Define an operator $B : \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $B = A - \alpha I$. Then we have

$$(\nabla B)(X, Y) = (\nabla A)(X, Y) - (X\alpha)Y,$$

which gives

$$\begin{aligned}\|\nabla B\|^2 &= \|\nabla A\|^2 + n\|\nabla\alpha\|^2 - 2\sum_{ij} g\left((\nabla A)(e_i, e_j), e_j\right) g(\nabla\alpha, e_i) \\ &= \|\nabla A\|^2 + n\|\nabla\alpha\|^2 - 2\sum_{j} g\left(\nabla\alpha, (\nabla A)(e_j, e_j)\right) \\ &= \|\nabla A\|^2 - n\|\nabla\alpha\|^2\end{aligned}$$

This proves that $\|\nabla A\|^2 \geq n\|\nabla\alpha\|^2$. The equality holds if and only if $\nabla B = 0$ that is, $(\nabla A)(X, Y) = X(\alpha)Y$. Using Codazzi equation (2.3) and $n \geq 2$ we get that α is a constant and that $\nabla A = 0$.

Lemma 3.5 Let M be an n-dimensional orientable compact hypersurface of the real space form $\overline{M}(c)$. Then

$$\int_M \left(\sum_i g(\nabla_{e_i}(\nabla\alpha), Ae_i)\right) dV = -n\int_M \|\nabla\alpha\|^2\, dV$$

where $\{e_1, .., e_n\}$ is a local orthonormal frame on M.

Proof. Choosing a point wise covariant constant local orthonormal frame $\{e_1, ..., e_n\}$ on M, we compute

$$\begin{aligned} div\,(A(\nabla\alpha)) = \sum_i e_i g(\nabla\alpha, Ae_i) &= \sum_i g(\nabla_{e_i}(\nabla\alpha), Ae_i) + \sum_i g(\nabla\alpha, (\nabla A)(e_i, e_i)) \\ &= \sum_i g(\nabla_{e_i}(\nabla\alpha), Ae_i) + n\,\|\nabla\alpha\|^2 \end{aligned}$$

Integrating this equation we get the Lemma.

We define the second covariant derivative $(\nabla^2 A)(X, Y, Z)$ as

$$(\nabla^2 A)(X, Y, Z) = \nabla_X(\nabla A)(Y, Z) - A(\nabla_X Y, Z) - A(Y, \nabla_X Z),$$

then using the Ricci identity we get

$$(\nabla^2 A)(X, Y, Z) - (\nabla^2 A)(Y, X, Z) = R(X, Y)AZ - AR(X, Y)Z \tag{3.1}$$

4 Proof of the Theorem

Let M be an n-dimensional orientable compact hypersurface of the real space form $\overline{M}(c)$. Define a function $f : M \to R$ by $f = \frac{1}{2}\|A\|^2$. Then by a straightforward computation we get the Laplacian Δf of the smooth function f as

$$\Delta f = \|\nabla A\|^2 + \sum_{ij} g\left((\nabla^2 A)(e_j, e_j, e_i), Ae_i\right) \tag{4.1}$$

where $\{e_1, ..., e_n\}$ is local orthonormal frame on M.

Using the equation (2.3), we arrive at

$$g(\left(\nabla^2 A\right)(e_j, e_j, e_i), Ae_i) = g(\left(\nabla^2 A\right)(e_j, e_i, e_j), Ae_i)$$

Now using the Ricci identity (3.1) in above equation we get

$$\begin{aligned} g((\nabla^2 A)\,(e_j, e_j, e_i), Ae_i) &= g((\nabla^2 A)\,(e_i, e_j, e_j), Ae_i) + \\ &\quad g(R(e_j, e_i)Ae_j, Ae_i) - g(R(e_j, e_i)e_j, A^2 e_i) \end{aligned}$$

Thus in light of this equation the equation (4.1) takes the form

$$\begin{aligned} \Delta f &= \|\nabla A\|^2 + \sum_{ij} g((\nabla^2 A)\,(e_i, e_j, e_j), Ae_i) \\ &\quad + \sum_{ij}\left[g(R(e_j, e_i)Ae_j, Ae_i) - g(R(e_j, e_i)e_j, A^2 e_i)\right] \end{aligned} \tag{4.2}$$

Using Lemma 3.1, we get

$$\sum_j \left(\nabla^2 A\right)(e_i, e_j, e_j) = n\nabla_{e_i}(\nabla\alpha). \tag{4.3}$$

Now we use Lemmas 3.2 and 3.3 to compute

$$\begin{aligned} \sum_{ij}\left[g(R(e_j, e_i)Ae_j, Ae_i) - g(R(e_j, e_i)e_j, A^2 e_i)\right] &= cS - \frac{1}{2}\|R\|^2 \\ &\quad + \sum_i Ric(Ae_i, Ae_i) \end{aligned}$$

Using this last equation together with (4.3) in (4.2), we arrive at

$$\Delta f = \|\nabla A\|^2 + n\sum_i g(\nabla_{e_i}(\nabla\alpha), Ae_i) + cS - \frac{1}{2}\|R\|^2 + \sum_i Ric(Ae_i, Ae_i)$$

Integrating this equation and using Lemma 3.5 we arrive at

$$\int_M \left\{ \begin{array}{c} \left[\|\nabla A\|^2 - n\|\nabla\alpha\|^2\right] + cS - \frac{1}{2}\|R\|^2 - n(n-1)\|\nabla\alpha\|^2 \\ +\sum_i Ric(Ae_i, Ae_i) \end{array} \right\} dV = 0 \qquad (4.4)$$

Since $Ric \geq \delta$, the above equation takes the form

$$\int_M \left\{ \left[\|\nabla A\|^2 - n\|\nabla\alpha\|^2\right] + cS - \frac{1}{2}\|R\|^2 - n(n-1)\|\nabla\alpha\|^2 + \delta\|A\|^2 \right\} dV \leq 0$$

The condition in the statement of the theorem together with Lemma 3.4 and above inequality yields

$$\|\nabla A\|^2 = n\|\nabla\alpha\|^2 \qquad (4.5)$$

$$cS - \frac{1}{2}\|R\|^2 - n(n-1)\|\nabla\alpha\|^2 + \delta\|A\|^2 = 0 \qquad (4.6)$$

Using (4.5) and (4.6) in equation (4.4) we conclude

$$Ric\,(Ae_i, Ae_i) = \delta g\,(Ae_i, Ae_i) \qquad (4.7)$$

for each i. Thus equation (4.5) together with Lemma 3.4 we get that α is a constant and the shape operator A is parallel.

If $c = 0$, then by Theorem 4 in [8] we see that M being compact is an extrinsic sphere. If $c = 1$, then by the same result in [8] together with equation (4.7) implies that M is an extrinsic sphere as the tori $S^k(r) \times S^{n-k}\left(\sqrt{1-r^2}\right)$, $1 \leq k < n$ does not satisfy the equation (4.7) for each i. Finally if $c = -1$, the result in [8] together with compactness of M implies that M is an extrinsic sphere.

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