Elliptic patching of harmonic functions

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Abstract

Given two harmonic functions $u_+(x,y)$, $u_-(x,y)$ defined on opposite sides of the y-axis in $\mathbb{R}^2$ and periodic in y, we consider the problem of constructing a *family of gluing elliptic functions*, i.e. a family of functions $u_\epsilon(x,y)$ of class $\mathcal{C}^{1,1}$ that coincide with u_+ and u_- outside neighborhoods of the y-axis of width less than ϵ and are solutions to linear, uniformly elliptic equations without zero order terms. We first show that not always there is such a family and we give a necessary condition for its existence. Then we give a sufficient condition for the existence of a family of gluing elliptic functions and a way for its construction.

1 Introduction

Let $u_+(x,y)$ and $u_-(x,y)$ be two different harmonic functions defined on a neighbourhood Ω of the y-axis in $\mathbb{R}^2$, periodic of period 2π in y and such that $u_+ \equiv u_-$ on $\{x=0\}$. For given $\epsilon > 0$, by the unique continuation property of analytic functions (see e.g. [6]), it is well known that there exists no harmonic function u_ϵ with the property that $u_\epsilon \equiv u_-$ in $\Omega \cap \{x \leq -\epsilon\}$ and $u_\epsilon \equiv u_+$ in $\Omega \cap \{x \geq \epsilon\}$.

Nevertheless, one could look for functions u_ϵ, which satisfy the above matching conditions and are solutions to linear, uniformly elliptic equations $L_\epsilon u_\epsilon = 0$ without zero order terms. Such functions would satisfy several useful properties, like e.g. the maximum principle and Harnack inequalities ([5]).

In this paper, we study the existence of a family of gluing elliptic functions u_ϵ of class $\mathcal{C}^{1,1}(\Omega)$ for all ϵ is some interval $]0,\epsilon_o[$. When a pair (u_-,u_+) admits such a

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class of gluing elliptic functions, we will say that it is a *gluable pair* (see §2, for the exact definitions).

In [9], M. V. Safonov constructed a smooth "patching" between two functions on the unit sphere $S^2 \subset \mathbb{R}^3$, satisfying an elliptic equation. In [1], the authors make some modifications of Safonov's construction and they obtain the existence of gluing elliptic functions for pairs (u_-, u_+) of some special kind. In particular they prove two "gluing theorems": the first one allows to patch together u_+ and u_- if the derivative with respect to x of the difference $u_+ - u_-$ on the y-axis has constant sign; the second one gives the existence of a family of gluing elliptic function under weaker conditions on $u_{+,x} - u_{-,x}$, but assumes other hypotheses on the values of u_+ and u_- on $\{x = 0\}$. A similar result for pairs of harmonic functions of three variables can be found in [4].

In this paper we determine a condition that must be satisfied by any gluable pair (Th. 3.1). It states that if the trivial non-smooth patching of u_- and u_+ (see formula (2.2)) has a strict local maximum or minimum, then (u_-, u_+) is not gluable. By means of an example, we show that this claim is no longer true if the trivial non-smooth patching has a weak local maximum (or minimum) (Remark 3.3). Finally, using some ideas of [1], we determine a new sufficient condition for a given pair (u_-, u_+) to be gluable (Th. 5.1).

Notice that, via the transformation $x = \ln r$, $y = \theta$, the functions $u_+(x, y)$, $u_-(x, y)$ can be always seen as harmonic functions defined in an annulus $\{1 - \delta < r < 1 + \delta\}$ and coinciding on the circumference S^1. In particular, the results of this paper can be reformulated as patching conditions for pairs of harmonic functions defined on a neighbourhood of S^1.

We conclude pointing out that the question on the existence of gluing elliptic functions is naturally related with the following inverse problem (see e.g. [10, 2, 3]): given a pair of functions f, g on the boundary of a domain D, find an elliptic equation $Lu = 0$ in D and a solution u_0 of such equation, which has f and g as Dirichlet and Neumann boundary values, respectively. In [2], the authors construct a non trivial solution of such problem in a sufficiently large disc with $f = g = 0$ and $u_0 \in W^{2,p}$, with $1 < p < 2$. In [3], another construction is given and provides a solution of the same problem in the unit disc and with f and g real analytic. In both papers, a key ingredient is represented by the gluing theorems for harmonic functions of [1] and it is expected that general results on patching of harmonic functions will be useful in finding new constructions and solutions to the above problem for boundary data f, g of weaker regularity.

2 The definition of gluing elliptic function

Throughout the paper, $T = \mathbb{R}/(2\pi\mathbb{Z}) \sim (-\pi, \pi]$ will denote the 1-dimensional torus. Any real function u on the cylinder $\mathbb{R} \times T$ is naturally identified with a real function of two real variables x and y, which is periodic in y of period 2π. We will also indicate by Ω the region $\Omega := (-\alpha, \alpha) \times T \subset \mathbb{R} \times T$, $\alpha > 0$, and by Γ the periodic axis $\Gamma := \{0\} \times T \subset \Omega$.

Finally, we will always denote by u_- and u_+ two harmonic functions defined in Ω and such that

$$u_+ \equiv u_- \quad \text{on } \Gamma; \qquad u_+ \not\equiv u_- \quad \text{in } \Omega. \tag{2.1}$$

We give the following definition.

Definition 2.1. We say that (u_-, u_+) is a *gluable pair* if there exists a positive constant ϵ_0 such that for any $0 < \epsilon < \epsilon_0$ there exists a function u_ϵ of class $C^{1,1}(\Omega)$ such that

i) u_ϵ solves a linear, uniformly elliptic equation $L_\epsilon u_\epsilon = 0$, with bounded measurable coefficients in Ω and without zero order term;

ii) $u_\epsilon \equiv u_-$ in $\Omega \cap \{x \leq -\epsilon\}$ and $u_\epsilon \equiv u_+$ in $\Omega \cap \{x \geq \epsilon\}$;

iii) there exists a positive constant M such that $|u_{\epsilon,x}| < M$ in $[-\epsilon, \epsilon] \times T$ for all $\epsilon < \epsilon_0$.

A family $\{u_\epsilon : \ \epsilon \in (0, \epsilon_0)\}$ which satisfies i), ii), iii) is called a *family of gluing elliptic functions for* (u_-, u_+).

Notice that condition iii) in Definition 2.1 implies that any sequence $\{u_{\epsilon_n}\}$, with $\epsilon_n \to 0$ for $n \to \infty$, converges uniformly in Ω to the trivial nonsmooth patching of u_- and u_+

$$u_0 = \begin{cases} u_- & \text{in } \Omega \cap \{x \leq 0\} \\ u_+ & \text{in } \Omega \cap \{x > 0\}. \end{cases} \tag{2.2}$$

3 A necessary condition for a gluable pair

Theorem 3.1. *If the function u_0 defined in (2.2) has a point of strict local minimum (maximum) on Γ, then the pair (u_-, u_+) is not gluable.*

Proof. Let $z_0 = (0, y_0) \in \Gamma$ be a point of strict local minimum for u_0. By the definition of u_0, it means that there exists a square $Q_r(z_0) = \{|x| < r, |y - y_0| < r\}$ such that

$$u_-(z) \geq u_-(z_0) = u_0(z_0) \quad \forall z \in \bar{Q}_r(z_0) \cap \{x \leq 0\}$$
$$u_+(z) \geq u_+(z_0) = u_0(z_0) \quad \forall z \in \bar{Q}_r(z_0) \cap \{x \geq 0\}$$

and in both inequalities we have the equal sign only for $z = z_0$. Now, let us take $m = \min_{\partial Q_r(z_0)} u_0$ and $\eta = m - u_0(z_0) > 0$. Since u_0 is a continuous function, there exists $0 < \delta < r$ such that $|u_0(z) - u_0(z_0)| < \eta/2$ for all $z \in \bar{Q}_\delta(z_0)$. It follows that the minimum of u_0 on $D = \bar{Q}_r(z_0) \cap \{|x| \geq \delta\}$ is achieved on $x = -\delta$ or $x = \delta$. In fact, we have that

$$\min_{Q_r(z_0) \cap \{|x| = \delta\}} u_0 \leq u_0(z_0) + \eta/2 = m - \eta/2 \ ,$$

while, on the other parts of the boundary, u_0 is bigger or equal than m. Let z_1 be a point on $x = -\delta$ or $x = \delta$ such that $u_0(z_1) = \min_D u_0$. Now, let us assume that the pair (u_-, u_+) is gluable and let u_δ be a gluing elliptic function of a family which satisfies conditions i), ii) and iii) in Definition 2.1. Since u_δ satisfies a uniformly elliptic equation $Lu_\delta = 0$, without zero order term in Ω, the minimum of u_δ in $H = \bar{Q}_r(z_0) \cap \{|x| \le \delta\}$ is achieved on ∂H. Let us prove that it is assumed on $x = -\delta$ or $x = \delta$. In fact, by conditions ii) and iii) in Definition 2.1, we have that

$$\min_{\partial H \cap \partial Q_r(z_0)} u_\delta \ge m - M\delta \ .$$

Thus, we can choose δ in such a way that

$$\min_{\partial H \cap \partial Q_r(z_0)} u_\delta \ge m - \frac{\eta}{4} \ .$$

On the other hand, u_δ coincides with u_0 on $x = -\delta$ and $x = \delta$. Then we have

$$\min_{\partial H \cap \{|x| = \delta\}} u_\delta < m - \frac{\eta}{2} \ .$$

It follows that $u_\delta(z_1) = \min_{\bar{H}} u_\delta$. But, since $u_\delta \equiv u_0$ for $|x| \ge \delta/2$, we have also

$$u_\delta(z_1) = \min_{Q_r} u_\delta$$

contradicting Alexandrov-Pucci maximum principle. ■

Remark 3.2. Assume that u_0 has a local minimum at $z_0 = (0, y_0) \in \Gamma$. From the fact that u_0 is harmonic outside Γ and $u_0|_\Gamma$ is analytic, it follows that either u_0 has a strict local minimum at z_0 or u_0 is constant on Γ.

Remark 3.3. By minor modifications of the proof of Theorem 3.1, one can see that if $u_0|_\Gamma = const.$ and such value is a global minimum for u_0, then the pair (u_-, u_+) is not gluable. For instance, the pair $u_-(x, y) = -x$, $u_+(x, y) = x$ is not gluable.

On the other hand, it may happen that u_0 has a point of (non strict) local minimum (maximum) on Γ and that the pair (u_-, u_+) is gluable. For instance, let us consider $u_-(x, y) = -\sinh x \cos y$ and $u_+(x, y) = \sinh x \cos y$. Then $u_0(x, y) = |\sinh x| \cos y$ is identically zero on Γ and any point $(0, y) \in \Gamma$ with $y \in (-\pi/2, \pi/2)$ is of local minimum for u_0. Now, let us show that the pair (u_-, u_+) is gluable. For any $\epsilon > 0$, let us define

$$u_\epsilon(x, y) = \begin{cases} a_\epsilon \cosh(b_\epsilon x) \cos y & \text{for } |x| < \epsilon \\ u_0 & \text{for } |x| \ge \epsilon, \end{cases}$$

where b_ϵ is the unique positive solution of the equation $b_\epsilon \tanh(\epsilon b_\epsilon) = \tanh^{-1}(\epsilon)$ and $a_\epsilon = \sinh \epsilon / \cosh(\epsilon b_\epsilon)$ in such a way that u_ϵ is of class $C^{1,1}(\mathbb{R}^2)$. Then, we have $|u_{\epsilon,x}| \le a_\epsilon b_\epsilon |\sinh(\epsilon b_\epsilon)| \le \cosh \epsilon \le \cosh 1$ in $|x| \le \epsilon$ for any $\epsilon < 1$, so that condition iii) in Definition 2.1 is satisfied. Moreover, we have that u_ϵ solves the equation $u_{xx} + b_\epsilon^2 u_{yy} = 0$ in $|x| \le \epsilon$ and hence it is a gluing elliptic function for (u_-, u_+).

4 An interpolating function for (u_-, u_+)

In this section, it is given the definition of interpolating function u_ϵ for a pair (u_-, u_+) of harmonic functions. A sufficient condition for u_ϵ to be a gluing elliptic function will be proved in the next section.

In what follows, let us denote by U the difference $U = u_+ - u_-$. Notice that U is harmonic in Ω, it satisfies

$$U \equiv U_y \equiv 0 \quad \text{on } \Gamma \tag{4.1}$$

and it is odd in the x variable.

We start with the following lemma.

Lemma 4.1. *The derivative with respect to x of the function U has at most a finite number of zeros on Γ.*

Proof. The restriction $U_x|_\Gamma$ of U_x to the compact curve Γ is real analytic and thus it is either identically zero or with a finite number of zeros. Let us prove that it cannot be $U_x|_\Gamma \equiv 0$. In fact, if $U_x|_\Gamma \equiv 0$, by (4.1), it follows that the function $f = U_x - iU_y$, which is holomorphic in Ω, is identically zero on Γ and hence in all Ω. Then U is constant and, in particular, $U \equiv 0$ in Ω, in contradiction with (2.1). ∎

In the next proposition we summarize some facts on the zeros of harmonic functions in two variables that one can obtain by considering them as real parts of holomorphic functions (see e.g. [11]).

Proposition 4.2. *Let v be a non identically vanishing harmonic function on a simply connected domain $D \subset \mathbb{R}^2 = \mathbb{C}$. Let also $z_0 \in D$ be such that $v(z_0) = 0$. Then there exist a neighbourhood $\mathcal{U}(z_0)$ of z_0 and a biholomorphism $h : B_r \to \mathcal{U}(z_0)$ from $B_r = \{|z| < r\}$ to $U(z_0)$ such that $h(0) = z_0$ and $v(h(\zeta)) = \operatorname{Re} \zeta^n$ for an integer $n \geq 1$, which is uniquely determined by u and z_0.*

Moreover, the zeros set $\{v = 0\} \cap \mathcal{U}(z_0)$ is of the form

$$\{v = 0\} \cap \mathcal{U}(z_0) = \cup_{k=1}^{n} \gamma_k(I) \ ,$$

where $\gamma_k : I = (-1, 1) \to \mathcal{U}(z_0)$ are n analytic curves, such that $\gamma_k(0) = z_0$ and the angles between γ_k and γ_{k+1} at z_0 are equal to π/n for any k (we set $\gamma_{n+1} = \gamma_1$).

Notice that the number n coincides with the order of the zero at z_0 of the unique holomorphic function f in D, with $\operatorname{Re} f = u$ and such that $f(z_0) = 0$. In the sequel, we will call such integer the *order* of the zero z_0.

An a consequence of Proposition 4.2 we have the following lemma.

Lemma 4.3. *Let $z_1 = (0, y_1)$ and $z_2 = (0, y_2)$ be two consecutive zeros of U_x on Γ, with $y_1 < y_2$. Then, for any $y \in (y_1, y_2)$, there exists a neighbourhood $\mathcal{U}(z)$ of the point $z = (0, y)$ in $\mathbb{R}^2$ such that the only zeros of U in $\mathcal{U}(z)$ are on Γ.*

Proof. Let $y_0 \in (y_1, y_2)$ and $z_0 = (0, y_0) \in \Gamma$. Since $U_x(z_0) \neq 0$, from Proposition 4.2 it follows that the order of z_0 is equal to 1 and hence there exists a neighbourhood $\mathcal{U}(z_0)$ in which Γ is the unique curve of zeros of U. ∎

The situation is different at the points z_1 and z_2, where the gradient of U vanishes and hence the orders n_1 and n_2 of z_1 and z_2, respectively, are larger than 1. By Proposition 4.2, there exist neighbourhoods $\mathcal{U}(z_1)$ and $\mathcal{U}(z_2)$ such that the set of zeros of U in $\mathcal{U}(z_1)$ is given by n_1 analytic curves throughout z_1 and the set of zeros of U in $\mathcal{U}(z_2)$ is given by n_2 analytic curves throughout z_2. In both cases, one of these curves coincides with Γ and the others form with Γ positive angles.

This brings to the following lemma.

Lemma 4.4. *Let $0 < \epsilon < \alpha$ and let $z_1 = (0, y_1)$ and $z_2 = (0, y_2)$ be two consecutive zeros of U_x on Γ, with $y_1 < y_2$. Assume also that $U_x(0, y) > 0$ for all $y \in (y_1, y_2)$. Then there exist two regular curves γ_-^ϵ and γ_+^ϵ contained in $\Omega \cap \{-\epsilon \le x \le 0\}$ and $\Omega \cap \{0 \le x \le \epsilon\}$ respectively, such that the following holds:*

i) γ_-^ϵ is symmetric to γ_+^ϵ with respect to Γ;

ii) γ_+^ϵ has z_1 and z_2 as endpoints, forms positive angles with Γ at those points and it admits a parameterization of the form $\gamma_+^\epsilon(y) = (\gamma_+^{\epsilon,x}(y), y)$, $y_1 \le y \le y_2$ with $\gamma_+^{\epsilon,x}(y_1) = \gamma_+^{\epsilon,x}(y_2) = 0$ and $0 < \gamma_+^{\epsilon,x}(y) \le \epsilon$ for $y_1 < y < y_2$;

iii) the domain D^ϵ bounded by γ_-^ϵ and γ_+^ϵ is so that $U_x > 0$ on D^ϵ and hence $U > 0$ on $\bar{D}^\epsilon \cap \{x > 0\}$ and $U < 0$ on $\bar{D}^\epsilon \cap \{x < 0\}$.

Proof. Consider the function U_x which is harmonic and equal to zero in z_1 and z_2. By Proposition 4.2, in a neighbourhood $\mathcal{U}(z_1)$ of z_1 the zeros of U_x are located on a finite number of curves throughout z_1 which form positive angles with Γ. The same holds in a suitable neighbourhood $\mathcal{U}(z_2)$ of z_2. Moreover, for each point $z = (0, y)$, $y \in (y_1, y_2)$, there exists a neighbourhood $\mathcal{U}(z)$ such that $U_x > 0$ in $\mathcal{U}(z)$. Therefore we may consider a finite open subcover $\{\mathcal{U}(z_1), \mathcal{U}(z_2), \mathcal{U}(z_3), \ldots, \mathcal{U}(z_N))\}$ of $\{(0, y) : y \in [y_1, y_2]\}$ out of the collection $\{\mathcal{U}(z_1), \mathcal{U}(z_2)\} \cup \{\mathcal{U}(z), z = (0, y), y \in (y_1, y_2)\}$.

Thus, we can take two symmetric curves γ_+^ϵ and γ_-^ϵ, which are graphs of two functions of y, so that they are contained in $\cup_{i=1}^N \mathcal{U}(z^k)$ and they form sufficiently small angles with Γ in z_1 and z_2 so that they intersect the curves of zeros of U_x only at the endpoints z_1 and z_2. With such choice of γ_+^ϵ and γ_-^ϵ, we have $U_x > 0$ in D^ϵ. ∎

If D^ϵ is the region in Lemma 4.4, the restriction of U on each horizontal segment in D^ϵ is an increasing function. In particular, notice that

$$U(-\gamma_+^{\epsilon,x}(y), y) \le U(x, y) \le U(\gamma_+^{\epsilon,x}(y), y) \quad \forall (x, y) \in D^\epsilon \ . \tag{4.2}$$

On the other hand, in case $U_x(0, y) < 0$ at any $y \in (y_1, y_2)$, it is possible to choose γ_+^ϵ and γ_-^ϵ so that $U > 0$ in $D^\epsilon \cap \{x < 0\}$, $U < 0$ in $D^\epsilon \cap \{x > 0\}$ and $U_x < 0$ in D^ϵ. In this case, the restriction of U to each horizontal segment in D^ϵ is a decreasing function and the inequalities in (4.2) are reversed.

Now, with the help of Lemma 4.4, for any $\epsilon \in (0, \alpha)$ we define two special curves $\Gamma_+^\epsilon(y) := (\Gamma^{\epsilon,x}(y), y)$ and $\Gamma_-^\epsilon(y) := (-\Gamma^{\epsilon,x}(y), y)$, for any $y \in T$, which are symmetric with respect to Γ and which bound a region D^ϵ on which we are going to modify the function u_0. The modified function will be called "interpolating" and will be a candidate for being a gluing elliptic function.

If $U_x|_\Gamma$ never vanishes, we set $\Gamma_+^{\epsilon,x}(y)$ equal to the constant function $\Gamma^{\epsilon,x}(y) = \epsilon$, so that Γ_+^ϵ and Γ_-^ϵ bound a strip around Γ. Otherwise, let us first denote by z_1, z_2, ..., z_n the zeros of $U_x|_\Gamma$ (which we know are in finite number by Lemma 4.1) and assume that they are ordered so that their coordinates are $z_i = (0, y_i)$ with $y_1 < y_2 < \cdots < y_n$. Let us also denote by $z_{n+1} = (0, y_{n+1})$ the point of Γ with

$y_{n+1} = y_1 + 2\pi$, so that $z_{n+1} = z_1$ on Γ. Then, for any $i = 1, \dots, n$, we fix a pair of symmetric curves

$$\gamma^{\epsilon}_{+,[z_i,z_{i+1}]} = (\gamma_i^{\epsilon,x}(y), y) \ , \qquad \gamma^{\epsilon}_{-,[z_i,z_{i+1}]} = (-\gamma_i^{\epsilon,x}(y), y)$$

satisfying the conditions in Lemma 4.4 with endpoints z_i and z_{i+1}, and we indicate with $D^{\epsilon}_{[z_i,z_{i+1}]}$ the domain bounded by these two curves. Finally, we define as $\Gamma^{\epsilon}_{+} = (\Gamma^{\epsilon,x}(y), y)$ and $\Gamma^{\epsilon}_{-} = (-\Gamma^{\epsilon,x}(y), y)$ the two piecewise regular curves given by the union of the curves $\gamma^{\epsilon}_{+,[z_i,z_{i+1}]}$ and $\gamma^{\epsilon}_{-,[z_i,z_{i+1}]}$, respectively, and we call D^{ϵ} the set bounded by these two curves, namely $D^{\epsilon} := \cup_{i=1}^{n} D^{\epsilon}_{[z_i,z_{i+1}]}$.

Now, we are able to give the definition of interpolating function.

Definition 4.5. For any set $D^{\epsilon} \subset \Omega$, bounded by two symmetric curves Γ^{ϵ}_{+} and Γ^{ϵ}_{-} defined as above, we call *interpolating function of the pair* (u_-, u_+) *on* D^{ϵ} the function u_{ϵ} defined as follows:

$$u_{\epsilon}(x,y) = u_-(x,y) \qquad \text{if } (x,y) \in (\Omega \setminus D^{\epsilon}) \cap \{x \le 0\};$$

$$u_{\epsilon}(x,y) = u_+(x,y) \qquad \text{if } (x,y) \in (\Omega \setminus D^{\epsilon}) \cap \{x > 0\};$$

$$u_{\epsilon}(x,y) = \frac{u_+(x,y) + u_-(x,y)}{2} + \frac{U^2(x,y)}{4U_{\epsilon}(y)} + \frac{U_{\epsilon}(y)}{4} \ \text{ if } (x,y) \in D^{\epsilon} \ ,$$

where $U_{\epsilon}(y) := U(\Gamma^{\epsilon,x}(y), y)$.

The regularity properties of the interpolating functions are given in the next lemma.

Lemma 4.6. *For any* $0 < \epsilon < \alpha$, *let* D^{ϵ} *be as in Definition 4.5 and* u_{ϵ} *be the corresponding interpolating function. Then the family of the functions* u_{ϵ} *satisfies conditions ii) and iii) of Definition 2.1 and each* u_{ϵ} *is of class* $C^{1,1}(\Omega)$. *In particular, the second derivatives of* u_{ϵ} *are discontinuous only on* Γ^{ϵ}_{-} *and* Γ^{ϵ}_{+}.

Proof. Notice that from definitions and (4.2)

$$\frac{|U(x,y)|}{U_{\epsilon}(y)} \le 1 \qquad \text{for any } (x,y) \in D^{\epsilon}.$$

It follows that u_{ϵ} is continuous in $\bar{D}^{\epsilon}$. Moreover, we have that $u_{\epsilon} = u_-$ on Γ^{ϵ}_{-} and $u_{\epsilon} = u_+$ on Γ^{ϵ}_{+} and this implies that u_{ϵ} is continuous in Ω. Since $\Gamma^{\epsilon}_{+}, \Gamma^{\epsilon}_{-} \subset \{|x| \le \epsilon\}$, we also have that condition ii) in Definition 2.1 is satisfied.

Moreover, on D^{ϵ} we have

$$u_{\epsilon,x} = \frac{u_{+,x} + u_{-,x}}{2} + \frac{UU_x}{2U_{\epsilon}}$$

$$u_{\epsilon,y} = \frac{u_{+,y} + u_{-,y}}{2} + \frac{UU_y}{2U_{\epsilon}} + \frac{U'_{\epsilon}}{4}\left(1 - \frac{U^2}{U_{\epsilon}^2}\right) \ .$$

Observe that $u_{\epsilon,x} = u_{-,x}$ and $u_{\epsilon,y} = u_{-,y}$ on Γ^{ϵ}_{-}, $u_{\epsilon,x} = u_{+,x}$ and $u_{\epsilon,y} = u_{+,y}$ on Γ^{ϵ}_{+}, and that $u_{-,x}(z_i) = u_{+,x}(z_i)$ and $u_{-,y}(z_i) = u_{+,y}(z_i)$ at any point z_i. It follows that u_ϵ is $C^1(\Omega)$. Moreover, we have $|u_{\epsilon,x}| \le |u_{+,x}| + |u_{-,x}|$ and so also condition iii) of Definition 2.1 holds.

Now let us consider the second derivatives of u_ϵ in D^ϵ:

$$
\begin{aligned}
u_{\epsilon,xx} &= u_{+,xx}\left(\frac{1}{2}+\frac{U}{2U_\epsilon}\right)+u_{-,xx}\left(\frac{1}{2}-\frac{U}{2U_\epsilon}\right)+\frac{U_x^2}{2U_\epsilon}\\
u_{\epsilon,xy} &= u_{+,xy}\left(\frac{1}{2}+\frac{U}{2U_\epsilon}\right)+u_{-,xy}\left(\frac{1}{2}-\frac{U}{2U_\epsilon}\right)+\frac{U_xU_y}{2U_\epsilon}-\frac{U_xUU_\epsilon'}{2U_\epsilon^2}\\
u_{\epsilon,yy} &= u_{+,yy}\left(\frac{1}{2}+\frac{U}{2U_\epsilon}\right)+u_{-,yy}\left(\frac{1}{2}-\frac{U}{2U_\epsilon}\right)+\frac{U_y^2}{2U_\epsilon}-\frac{U_yUU_\epsilon'}{U_\epsilon^2}+\\
&\quad +\frac{U^2(U_\epsilon')^2}{2U_\epsilon^3}+\frac{U_\epsilon''}{4}\left(1-\frac{U^2}{U_\epsilon^2}\right)\ .
\end{aligned}
$$

It is clear that the second derivatives of u_ϵ are discontinuous only on Γ_-^ϵ and Γ_+^ϵ. We have to prove that they are bounded. It is sufficient to prove it in suitable neighbourhoods of the points z_i, $1 \le i \le n$. To check that $u_{\epsilon,xx}$ is bounded it is enough to show that the quotient $U_x^2(x,y)/U_\epsilon(y)$ is bounded. By Proposition 4.2 and the remarks after Lemma 4.3, there exist a neighbourhood $\mathcal{U}(z_i)$ of z_i and a biholomorphism $h : B_r \to \mathcal{U}(z_i)$ such that $h(0) = z_i$ and $U(h(\zeta)) = \operatorname{Re}\zeta^{n_i}$ for some integer $n_i \ge 2$. Then we have $U(z) = \operatorname{Re}((h^{-1}(z))^{n_i})$ and

$$
U_x = \operatorname{Re}\left(n_i(h^{-1}(z))^{n_i-1}(h^{-1}(z))'\right) = \operatorname{Re}\left(n_i\zeta^{n_i-1}\frac{1}{h'(\zeta)}\right)\ .
$$

Now, let $z = (x,y) \in \mathcal{U}(z_1)$, $\tilde z = (\gamma_+^{\epsilon,x}(y), y)$, $\zeta = h^{-1}(z)$ and $\tilde\zeta = h^{-1}(\tilde z)$. We have

$$
\frac{U_x^2(x,y)}{|U_\epsilon(y)|} = \frac{\left[\operatorname{Re}\left(n_i\zeta^{n_i-1}\frac{1}{h'(\zeta)}\right)\right]^2}{|\operatorname{Re}\tilde\zeta^{n_i}|} \le \frac{n_i^2}{|h'(\zeta)|^2}\frac{|\zeta|^{2n_i-2}}{|\operatorname{Re}\tilde\zeta^{n_i}|}\ .
$$

Now the claim is proved if we prove that the quotient $|\zeta|^{2n_i-2}/|\operatorname{Re}\tilde\zeta^{n_i}|$ is bounded. Let $\zeta = \rho e^{i\theta}$ and $\tilde\zeta = \tilde\rho e^{i\tilde\theta}$. Since the curve Γ_+^ϵ is not tangent to Γ in z_i, it follows that there exists a positive constant a_i such that $|\cos(n_i\tilde\theta)| \ge a_i$ and thus

$$
\frac{|\zeta|^{2n_i-2}}{|\operatorname{Re}\tilde\zeta^{n_i}|} = \frac{\rho^{2n_i-2}}{\tilde\rho^{n_i}|\cos(n_i\tilde\theta)|} \le \frac{\rho^{2n_i-2}}{a_i\tilde\rho^{n_i}}\ .
$$

Moreover, since h is a biholomorphism, there exist positive constants K_1 and K_2 such that

$$
\begin{aligned}
K_1 \operatorname{dist}(z, z_i) \le |\zeta| \le K_2 \operatorname{dist}(z, z_i)\\
K_1 \operatorname{dist}(\tilde z, z_i) \le |\tilde\zeta| \le K_2 \operatorname{dist}(\tilde z, z_i)\ .
\end{aligned}
$$

Since $\operatorname{dist}(z, z_i) \le \operatorname{dist}(\tilde z, z_i)$ we obtain $|\zeta| \le (K_2/K_1)|\tilde\zeta|$ so that

$$
\frac{|\zeta|^{2n_i-2}}{|\operatorname{Re}\tilde\zeta^{n_i}|} \le \operatorname{cost}\rho^{n_i-2}
$$

and it is bounded, since $n_i \ge 2$. This concludes the proof that $u_{\epsilon,xx}$ is bounded in Ω. Similar arguments show that also $u_{\epsilon,xy}$ and $u_{\epsilon,yy}$ are bounded in Ω. ∎

Remark 4.7. In Definition 2.1 we require that u_ϵ coincides with the function u_0 in (2.2) outside the strip $\{|x| \le \epsilon\}$. Notice that the function u_ϵ we have defined satisfies something more. In fact, it coincides with u_0 on a set which is strictly larger than $\{|x| > \epsilon\}$, namely, it contains also the points $z_1, \ldots, z_n$ on Γ, i.e. the points on Γ on which u_0 is C^1.

5 A sufficient condition for a pair to be gluable

Now, we give sufficient conditions on u_+ and u_- so that the interpolating function u_ϵ satisfies also condition i) of Definition 2.1. This means that (u_-, u_+) is gluable.

Theorem 5.1. *Assume that the restriction $u_+|_\Gamma = u_-|_\Gamma$ satisfies*

$$u_{+,x}(z)u_{-,x}(z) > 0 \quad \text{at any} \quad z \in \Gamma \text{ s.t. } u_{+,y}(z) = u_{-,y}(z) = 0 \ . \tag{5.1}$$

Then the pair (u_-, u_+) is gluable. In fact, for any ϵ sufficiently small, there exists an interpolating function u_ϵ that is a gluing elliptic function for such pair.

Proof. By Lemma 4.6, it is sufficient to prove that u_ϵ satisfies condition i) in Definition 2.1. We may reduce to consider the closed strip S between the two horizontal lines which pass through two consecutive zeros z_i and z_{i+1} of $U_x|_\Gamma$. Without loss of generality, we may also assume that $U_x > 0$ on $\Gamma \cap S$. To simplify the notation, in what follows we will always write γ_-^ϵ, γ_+^ϵ and D^ϵ in place of $\gamma^\epsilon_{-,[z_i,z_{i+1}]}$, $\gamma^\epsilon_{+,[z_i,z_{i+1}]}$ and $D^\epsilon_{[z_i,z_{i+1}]}$ respectively.

Now, in order to prove that u_ϵ is a solution to a uniformly elliptic equation $Lu_\epsilon = 0$ with bounded measurable coefficients and without zero order term, we want to show that $|\mathrm{grad}\, u_\epsilon| \ge C > 0$ in D^ϵ. In fact, if this occurs, then for any given uniformly elliptic operator $\tilde{L} = \sum_{ij=1}^2 a_{ij}\partial_{ij}$, if we set $b = -\frac{\tilde{L}u_\epsilon}{|\mathrm{grad}_{u_\epsilon}|^2}\mathrm{grad} u_\epsilon$, then u_ϵ satisfies the equation $\tilde{L}u + b \cdot \mathrm{grad} u = 0$.

To obtain a lower bound for $|\mathrm{grad}\, u_\epsilon|$ we will show that, for any $z_0 \in \Gamma \cap S$, we can choose a neighbourhood $\mathcal{U}(z_0)$ so that, for any sufficiently small ϵ, $|\mathrm{grad}\, u_\epsilon| \ge C(z_0) > 0$ on $U(z_0) \cap D^\epsilon$, where $C(z_0)$ is a constant depending only on z_0. Taking a finite subcover $\{\mathcal{U}(z_k) : \ k = 1 \cdots n\}$ and the curves γ_-^ϵ and γ_+^ϵ so that they are contained in $\cup_{k=1}^n \mathcal{U}(z_k)$, it follows that $|\mathrm{grad}\, u_\epsilon| \ge C > 0$ in D^ϵ as we need.

Let $z_0 \in \Gamma$ and suppose that $u_{+,y}(z_0) = u_{-,y}(z_0) \neq 0$. For any choice of D^ϵ, we have that at all its points

$$|u_{\epsilon,y}| \ge \left|\frac{u_{+,y} + u_{-,y}}{2}\right| - \left|\frac{U}{2}\right| - \left|\frac{U'_\epsilon}{4}\left(1 - \frac{U^2}{U_\epsilon^2}\right)\right| \ .$$

If $z \to z_0$, we have that $(u_{+,y} + u_{-,y})/2 \to u_{+,y}(z_0) = u_{-,y}(z_0) \neq 0$, while $U \to 0$. Then, there exist a neighbourhood $\mathcal{U}(z_0)$ and $k > 0$, both of them independent of ϵ, such that

$$\left|\frac{u_{+,y} + u_{-,y}}{2}\right| \ge k \quad \text{and} \quad |U| \le \frac{k}{2} \quad \text{in } \mathcal{U}(z_0) \ .$$

Moreover, in D^ϵ

$$\left|\frac{U'_\epsilon}{4}\left(1 - \frac{U^2}{U_\epsilon^2}\right)\right| \le \left|\frac{U'_\epsilon}{4}\right| \ .$$

We claim that, for any ϵ sufficiently small, the definition of the curves γ_-^ϵ and γ_+^ϵ can be given so that $|U'_\epsilon(y)| \leq k$ for all $y \in (y_1, y_2)$. In fact,

$$U'_\epsilon(y) = U_x(\gamma_+^{\epsilon,x}(y), y)(\gamma_+^{\epsilon,x})'(y) + U_y(\gamma_+^{\epsilon,x}(y), y) \ .$$

Moreover, for any ϵ sufficiently small, $|U_y| < k/2$ in D_ϵ and we may assume that the curve $(\gamma_+^{\epsilon,x})$ satisfies $|(\gamma_+^{\epsilon,x})'(y)| < M$ for some constant M. Since $|U_x| < \frac{k}{2M}$ on two suitable neighbourhoods of z_1 and z_2, if we define the curve $(\gamma_+^{\epsilon,x})$ so that $(\gamma_+^{\epsilon,x})'(y)$ is identically zero outside such neighbourhoods, we have $|U'_\epsilon(y)| \leq k$ for all $y \in (y_1, y_2)$. Fixing the curves γ_-^ϵ and γ_+^ϵ in this way, for any ϵ so small that they intersect $\mathcal{U}(z_0)$, we have

$$|\mathrm{grad}\, u_\epsilon|^2 \geq |u_{\epsilon,y}|^2 \geq (\frac{k}{2})^2 > 0 \quad \text{in } D^\epsilon \cap \mathcal{U}(z_0) \ .$$

It remains to consider the case of a point $z_0 \in \Gamma$ such that $u_{+,y}(z_0) = u_{-,y}(z_0) = 0$. By hypothesis, we have $u_{+,x}(z_0)u_{-,x}(z_0) > 0$ and so there exist $\tilde{k} > 0$ and a neighbourhood $\mathcal{U}(z_0)$ of z_0 such that $u_{+,x}(z)u_{-,x}(z) > \tilde{k}$ for $z \in \mathcal{U}(z_0)$. Since $u_{-,x} \leq u_{\epsilon,x} \leq u_{+,x}$ in D^ϵ, if ϵ is so small so that γ_-^ϵ and γ_+^ϵ intersect $\mathcal{U}(z_0)$, we have that

$$|\mathrm{grad}\, u_\epsilon|^2 \geq |u_{\epsilon,x}|^2 \geq \tilde{k} > 0 \quad \text{in } D^\epsilon \cap \mathcal{U}(z_0) \ .$$

This concludes the proof. ∎

Remark 5.2. In the previous proof, the fact that u_ϵ satisfies a uniformly elliptic equation without zero order term is obtained as a consequence of the inequality $|\mathrm{grad}\, u_\epsilon| \geq C > 0$. The hypothesis of the theorem have been selected just to obtain such inequality. On the other hand, other sufficient conditions for gluability (maybe not so simple as in Theorem 5.1) might be found using the criterion for a function to be solution of elliptic equations proved by Pucci in [7, 8]. For reader's convenience, we recall such theorem. It is stated in terms of the so called *Pucci's extremal operators* (see also [5], Ch. 17): for any given $\alpha \in (0, 1/2]$ and $\beta \geq 0$ such operators on functions of two variables are defined as

$$M_{\alpha,\beta}[u] = \alpha\mathcal{C}_1(u) + (1-\alpha)\mathcal{C}_2(u) + \beta\,|\mathrm{grad}\, u| \ ,$$

$$m_{\alpha,\beta}[u] = (1-\alpha)\mathcal{C}_1(u) + \alpha\mathcal{C}_2(u) - \beta\,|\mathrm{grad}\, u|,$$

where $\mathcal{C}_1(u) \leq \mathcal{C}_2(u)$ are the eigenvalues of the Hessian matrix of u. Pucci's criterion is the following: *Let D be a bounded domain in the plane and $u \in C^{1,1}(\Omega)$. Then $Lu = 0$ a.e. in D for some linear, uniformly elliptic operator L without zero order term and with bounded, measurable coefficients in D if and only if there exist $\alpha \in (0, 1/2]$ and $\beta \geq 0$ for which the inequalities*

$$m_{\alpha,\beta}[u] \leq 0 \leq M_{\alpha,\beta}[u]$$

hold a.e.in D.

Example 5.3. By Theorem 5.1, it is immediate to see that the pair $u_-(x, y) = x$ and $u_+(x, y) = 2x$ is gluable.

Remark 5.4. Theorem 2.1 of [1] gives a different sufficient condition for the existence of a family of gluing elliptic functions. Using our notation, it states that a pair (u_-, u_+) is gluable if $U_x|_\Gamma > 0$, $(u_{+,yy} + u_{+,x})|_\Gamma < 0$ and $(u_{-,yy} + u_{-,x})|_\Gamma < 0$. In particular, it may be used only in the cases in which grad$U$ never vanishes on Γ.

In Theorem 2.2 of the same paper, the authors consider pairs (u_-, u_+) with $u_-|_\Gamma = u_+|_\Gamma = \sin y$. Under some additional hypothesis, they prove the existence of a family of gluing elliptic functions all satisfying elliptic equations with only principal part. We point out that by the hypothesis in that theorem, one can obtain gluability also by our Theorem 5.1. On the other hand, the gluing elliptic functions we construct solve elliptic equations with non zero first order terms. It would be nice to determine general sufficient conditions that give the existence of gluing elliptic functions satisfying elliptic equations with zero first order term.

References

[1] P. BUONOCORE - P. MANSELLI, *Nonunique continuation for plane uniformly elliptic equations in Sobolev spaces*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **29** (4) (2000), 731–754.

[2] P. BUONOCORE - P. MANSELLI, *Solutions to two dimensional, uniformly elliptic equations, that lie in Sobolev spaces and have compact support*, Rend. Circ. Mat. Palermo (2) **51** (3) (2002), 476–484.

[3] R. CAVAZZONI, *On the Cauchy problem for elliptic equations in a disk*, Rend. Circ. Mat. Palermo (2) **52** (1) (2003) 131–140

[4] C. GIANNOTTI, *A compactly supported solution to a three-dimensional uniformly elliptic equation without zero-order term*, J. Diff. Eq. **201** (2) (2004), 234–249.

[5] D. GILBARG - N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag (1983).

[6] F. JOHN, *Partial Differential Equations*, Applied mathematical sciences Springer-Verlag 1 (1982).

[7] C. PUCCI, *Un problema variazionale per i coefficienti di equazioni differenziali di tipo ellittico*, Ann. Sc. Norm. Pisa, ser.III **16** (1962), 159–171.

[8] C. PUCCI, *Operatori ellittici estremanti*, Ann. Mat. Pura Appl. (IV) **LXXII** (1966), 141–170.

[9] M. V. SAFONOV, *Unimprovability of estimates of Hölder constants for solutions of linear elliptic equations with measurable coefficients*, Math. USSR Sbornik **60** (1988), 269–281.

[10] T. H. WOLFF, *Some constructions with solutions of variable coefficient elliptic equations*, J. Geom. Anal. **3** (5) (1993), 423–511.

[11] Z.-Y. WEN - L.-M. WU - Y.-P. ZHANG, *Set of zeros of harmonic functions of two variables*, Lecture Notes in Math. 1494, Springer, Berlin, 1991 **CLXIII** (1993), 161–180.

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