On the stability of the quadratic equation on groups

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Abstract

In this paper the stability of the quadratic equation is considered on arbitrary groups. Since the quadratic equation is stable on Abelian groups, this paper examines the stability of the quadratic equation on noncommutative groups. It is shown that the quadratic equation is stable on n-Abelian groups when n is a positive integer. The stability of the quadratic equation is also established on the noncommutative group $T(2, K)$, where K is an arbitrary commutative field. It is proved that every group can be embedded into a group in which the quadratic equation is stable.

1 Introduction

In 1940 to the audience of the Mathematics Club of the University of Wisconsin S. M. Ulam presented a list of unsolved problems [20]. One of these problems can be considered as the starting point of a new line of investigations: the stability problem. The problem was posed as follows. If we replace a given functional equation by a functional inequality, then under what conditions we can say that the solutions of the inequality are close to the solutions of the equation. For example, given a group G_1, a metric group (G_2, d) and a positive number ε, the Ulam question is: Does there exist a $\delta > 0$ such that if the map $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $T : G_1 \to G_2$ exists with $d(f(x), T(x)) < \varepsilon$ for all $x, y \in G_1$? In the case of a positive answer to this problem, we say that Cauchy

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functional equation $f(xy) = f(x)f(y)$ is *stable* for the pair (G_1, G_2). The interested reader should refer to [20] and [12] for an account on Ulam's problem.

Hyers [11] proved the following result to give an affirmative answer to Ulam's problem. Let X, Y be Banach spaces and let $f : X \to Y$ be a mapping satisfying

$$||f(x + y) - f(x) - f(y)|| \leq \varepsilon$$

for all x, y in X. Then there exists a unique additive mapping $A : X \to Y$satisfying

$$||f(x) - A(x)|| \leq \varepsilon$$

for all x in X. This pioneer result of Hyers can be expressed in the following way: Cauchy's functional equation is stable for any pair of Banach spaces.

The quadratic functional equation

$$f(xy) + f(xy^{-1}) = 2f(x) + 2f(y) \tag{1.1}$$

where f is defined on a group G and takes its values from a vector space E, is an important equation in the theory of functional equations and it plays an important role in the characterization of inner product spaces [7]. The stability of the quadratic functional equation (1.1) was first proved by Skof [19] for functions from a normed space into a Banach space. Cholewa [2] demonstrated that Skof's theorem is also valid if the relevant domain is replaced by an Abelian group. Later, Fenyő [8] improved the bound obtained and Cholewa from $\frac{\varepsilon}{2}$ to $\frac{\varepsilon + \|f(0)\|}{3}$ (cf. [3]).

Theorem 1.1. *Let G be an Abelian group and let E be a Banach space. If a function $f : G \to E$ satisfies the inequality*

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \varepsilon$$

for some $\varepsilon \geq 0$ and for all $x, y \in G$, then there exists a unique quadratic function $q : G \to E$such that

$$\|f(x) - q(x)\| \leq \frac{1}{3}(\varepsilon + \|f(0)\|)$$

for all $x \in G$.

The above theorem can be expressed in the following way: The quadratic functional equation is stable for the pair (G, E), where G is an Abelian group and E is a Banach space [7].

Various works on stability of the quadratic functional equation can be found in Skof [19], Cholewa [2], Fenyő [8], Ger [10], Czerwik [3], [4], [5], [6], Jung [13], [14], Jung and Sahoo [15], and Rassias [18]. In all these works, the stability of the quadratic equation or a more general quadratic equation was treated for the pair (G, E) when G is an Abelian group.

In the present paper, we consider the stability of the functional equation (1.1) for the pair (G, E) when G is an arbitrary group and E is a real Banach space. We prove that if G is an n-Abelian group with $n \in \mathbb{N}$, then the functional equation (1.1) is stable. The Skof's result [19] is a particular case of this result. Stability of the quadratic equation is established on the group $T(2, G)$. We also show that any group can be embedded into a group G such that the functional equation (1.1) is stable on G.

2 Preliminary results

Definition 2.1. Let G be an arbitrary group and E a Banach space. We say that a mapping $f : G \to E$ is a *quasiquadratic mapping* if there exists a nonnegative number δ such that

$$\|f(xy) + f(xy^{-1}) - 2f(x) - 2f(y)\| \leq \delta \tag{2.1}$$

for all $x, y \in G$.

Definition 2.2. Let G be an arbitrary group and E a Banach space. We say that $f : G \to E$ is a *quadratic mapping* if it satisfies the quadratic equation

$$f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) = 0 \tag{2.2}$$

for all $x, y \in G$.

It is clear that the set of all quasiquadratic mappings from G to E is a real linear space relative to the usual operations. Let us denote it by $KQ(G; E)$. The subspace of $KQ(G, E)$ consisting of all quadratic functions will be denoted by $Q(G; E)$.

In this sequel, we will write the arbitrary group G in multiplicative notation so that 1 will denote the identity element of G.

Substituting 1 for x in (2.1), we get

$$\|f(y) + f(y^{-1}) - 2f(1) - 2f(y)\| \leq \delta.$$

Hence

$$\|f(y^{-1}) - f(y)\| \leq c_1 \quad \text{for all} \quad y \in G, \tag{2.3}$$

where $c_1 = 2\|f(1)\| + \delta$.

Replacing y by x in (2.1), we obtain

$$\|f(x^2) + f(1) - 4f(x)\| \leq \delta.$$

Therefore

$$\|f(x^2) - 4f(x)\| \leq \|f(1)\| + \delta \quad \text{for all} \quad x \in G. \tag{2.4}$$

Again, substituting x^2 for y in (2.1), we see that

$$\|f(x^3) + f(x^{-1}) - 2f(x) - 2f(x^2)\| \leq \delta.$$

Using (2.3), we have

$$\|f(x^3) - f(x) - 2f(x^2)\| \leq c_1 + \delta.$$

From the last inequality and (2.4) it follows that

$$\|f(x^3) - 9f(x)\| \leq c_1 + 2\|f(1)\| + 3\delta. \tag{2.5}$$

Lemma 2.3. *Let $f \in KQ(G, E)$. Then for any integer $m \geq 1$ there is a $\delta_m > 0$ such that for each $x \in G$*

$$\|f(x^m) - m^2 f(x)\| \leq \delta_m. \tag{2.6}$$

Proof. If we put $\delta_1 = \delta$, $\delta_2 = \|f(1)\| + \delta$ and $\delta_3 = 4(\|f(1)\| + \delta)$, then (2.6) follows from (2.4) and (2.5) for $m = 1, 2, 3$. So for $m = 1, 2, 3$ the lemma is easily established. Next we prove the lemma for $m \geq 4$ by induction on m. Let $m \geq 4$ and suppose (2.6) has been already established for m, and let us check it for $m + 1$. From (2.1), we have

$$\|f(x^{m+1}) + f(x^{m-1}) - 2f(x^m) - 2f(x)\| \leq \delta. \tag{2.7}$$

Now from the induction hypothesis we have

$$\|f(x^{m-1}) - (m-1)^2 f(x)\| \leq \delta_{m-1},$$

and

$$\|f(x^m) - m^2 f(x)\| \leq \delta_m.$$

From (2.7) and (2.6) we obtain

$$\|f(x^{m+1}) + (m-1)^2 f(x) - 2m^2 f(x) - 2f(x)\| \leq \delta + \delta_{m-1} + 2\delta_m.$$

which is

$$\|f(x^{m+1}) - (m+1)^2 f(x)\| \leq \delta + \delta_{m-1} + 2\delta_m.$$

So letting $\delta_{m+1} = \delta + \delta_{m-1} + 2\delta_m$ we get (2.6). This completes the proof of the lemma. ∎

Lemma 2.4. *Suppose $f \in KQ(G, E)$. Then for any $k, m \in \mathbb{N}$ with $m \geq 2$ and any $x \in G$, the following relation*

$$\left\| \frac{1}{m^{2k}} f(x^{m^k}) - f(x) \right\| \leq 2b_m \tag{2.8}$$

holds, where $b_m = \frac{1}{m^2}\delta_m$.

Proof. The proof is by induction on k. If $k = 1$, then the assertion is clearly true by Lemma 2.3. Let $k > 1$. From Lemma 2.3, we have

$$\left\| \frac{1}{m^2} f(x^m) - f(x) \right\| \leq b_m. \tag{2.9}$$

Replacing x by x^m in (2.9), we get

$$\left\| \frac{1}{m^2} f(x^{m^2}) - f(x^m) \right\| \leq b_m. \tag{2.10}$$

Hence, as above, we get

$$\left\| \frac{1}{m^2}\frac{1}{m^2} f(x^{m^2}) - \frac{1}{m^2} f(x^m) \right\| \leq b_m \frac{1}{m^2}. \tag{2.11}$$

Now from the last inequality and (2.9), we see that

$$\left\| \tfrac{1}{m^{2\cdot 2}} f(x^{m^2}) - f(x) \right\| \leq b_m[1 + \tfrac{1}{m^2}].$$

Substituting x^m for x in the last inequality, we obtain

$$\left\| \tfrac{1}{m^{2\cdot 2}} f(x^{m^3}) - f(x^m) \right\| \le b_m[1 + \tfrac{1}{m^2}].$$

Hence

$$\left\| \tfrac{1}{m^{2\cdot 3}} f(x^{m^3}) - f(x) \right\| \le b_m[1 + \tfrac{1}{m^2} + \tfrac{1}{m^{2\cdot 2}}].$$

Continuing in this manner, we obtain the formula

$$\left\| \tfrac{1}{m^{2\cdot k}} f(x^{m^k}) - f(x) \right\| \le b_m \left[1 + \tfrac{1}{m^2} + \tfrac{1}{m^{2\cdot 2}} + \cdots + \tfrac{1}{m^{2\cdot (k-1)}}\right] \le 2b_m,$$

and this completes the proof of the lemma. ∎

From (2.8) it follows that for any $x \in G$ and any $m \in \mathbb{N}$ the set

$$\left\{ \frac{1}{m^{2k}} f(x^{m^k}) \;:\; k \in \mathbb{N} \right\}$$

is bounded. Let us verify that the sequence $\left\{ \frac{1}{m^{2k}} f(x^{m^k}) \right\}_{k=1}^{\infty}$ has a limit. From (2.8) it follows that for any $x \in G$ and any $m, n \in \mathbb{N}$

$$\left\| \frac{1}{m^{2k}} f((x^{m^n})^{m^k}) - f(x^{m^n}) \right\| \le 2b_m,$$

that is

$$\left\| \frac{1}{m^{2(n+k)}} f(x^{m^{n+k}}) - \frac{1}{m^{2n}} f(x^{m^n}) \right\| \le 2\,\frac{b_m}{m^{2n}}.$$

From the last inequality it follows that if $n \to \infty$, then

$$\left\| \frac{1}{m^{2(n+k)}} f(x^{m^{n+k}}) - \frac{1}{m^{2n}} f(x^{m^n}) \right\| \to 0.$$

So, the sequence $\left\{ \frac{1}{m^{2k}} f(x^{m^k}) \right\}_{k=1}^{\infty}$ is a Cauchy sequence and has a limit, say $\varphi_m(x)$. It is clear that for any $x \in G$ we have

$$\|\varphi_m(x) - f(x)\| \le 2b_m. \tag{2.12}$$

Obviously, for any natural number m, the function φ_m belongs to the space $KQ(G; E)$. Now let us verify that for any $x \in G$, the following relation

$$\varphi_m(x^{m^n}) = m^{2n}\varphi_m(x) \tag{2.13}$$

holds. Indeed

$$\begin{aligned}\varphi_m(x^{m^n}) &= \lim_{k\to\infty} \frac{1}{m^{2k}}\, f((x^{m^n})^{m^k}) = \lim_{k\to\infty} \frac{m^{2n}}{m^{2(n+k)}}\, f(x^{m^{n+k}}) \\ &= m^{2n} \lim_{k\to\infty} \frac{1}{m^{2k}}\, f(x^{m^k}) = m^{2n}\varphi_m(x).\end{aligned}$$

Lemma 2.5. *Let $f \in KQ(G,E)$ and $\varphi_m(x) = \frac{1}{m^{2k}} f(x^{m^k})$. Then for any positive integer $m \geq 2$, the relation $\varphi_2 = \varphi_m$ holds.*

Proof. The functions φ_2, φ_m belong to the space $KQ(G;E)$. Hence the mapping

$$g(x) = \lim_{k\to\infty} \frac{1}{m^{2k}} \varphi_2(x^{m^k})$$

is well defined and belongs to the space $KQ(G;E)$. It is clear that

$$g(x^{m^k}) = m^{2k} g(x) \quad \text{and} \quad g(x^{2^k}) = 2^{2k} g(x) \tag{2.14}$$

for any $x \in G$ and any $k \in \mathbb{N}$. From (2.12) it follows that there are exist positive numbers d_1, d_2 such that for any $x \in G$

$$\|\varphi_2(x) - g(x)\| \leq d_1 \text{ and } \|\varphi_m(x) - g(x)\| \leq d_2. \tag{2.15}$$

Replacing x by x^{2^k} in (2.15), we get

$$\|\varphi_2(x^{2^k}) - g(x^{2^k})\| \leq d_1.$$

Now using (2.13) and (2.14), we have

$$2^k \|\varphi_2(x) - g(x)\| \leq d_1,$$

which is

$$\|\varphi_2(x) - g(x)\| \leq \frac{d_1}{2^k}.$$

Hence $\varphi_2(x) = g(x)$. Similarly, we obtain $\varphi_m(x) = g(x)$, and $\varphi_2 \equiv \varphi_m$ follows. This completes the proof of the theorem. ■

Let

$$\widehat{f}(x) = \lim_{k\to\infty} \frac{1}{4^k} f(x^{2^k}). \tag{2.16}$$

By Lemma 2.5, we have

$$\widehat{f}(x^p) = \varphi_2(x^p) = \varphi_p(x^p) = p^2 \varphi_p(x) = p^2 \varphi_2(x) = p^2 \widehat{f}(x).$$

Thus

$$\widehat{f}(x^p) = p^2 f(x) \tag{2.17}$$

for any $x \in G$ and for any $p \in \mathbb{N}$.

Definition 2.6. By a *pseudoquadratic mapping*, defined on a group G, we mean a quasiquadratic mapping f such that $f(x^n) = n^2 f(x)$ for any $x \in G$ and any $n \in \mathbb{N}$.

The set of all pseudoquadratic mappings will be denoted by $PQ(G;E)$. We will say that a pseudoquadratic mapping f is *nontrivial* if $f \notin Q(G;E)$. The space of all bounded mappings $f : G \to E$ will be denote by $B(G;E)$.

Theorem 2.7. *For any group G we have the following decomposition*

$$KQ(G;E) = PQ(G;E) \oplus B(G;E).$$

Proof. It is clear that $B(G;E)$ is a subspace of $KQ(G;E)$ and $PQ(G;E) \cap B(G;E) = \{0\}$. Hence a subspace of $KQ(G;E)$ generated by $PQ(G;E)$ and $B(G;E)$ is their direct sum. Let us verify that $KQ(G;E) \subseteq PQ(G;E) \oplus B(G;E)$. Indeed, if $f \in KQ(G;E)$, then we have $\widehat{f} \in PQ(G;E)$ and $f - \widehat{f} \in B(G;E)$. ■

3 Stability

Suppose that G is a group and E is a real Banach space.

Definition 3.1. The quadratic equation (2.2) is said to be *stable* for the pair $(G; E)$ if for any $f : G \to E$ satisfying functional inequality

$$\| f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) \| \leq a \quad \forall x, y \in G$$

(for some $a \geq 0$), there is a solution q of functional equation (2.2) such that the function $f(x) - q(x)$ belongs to the space $B(G; E)$.

Theorem 3.2. *The functional equation* (2.2) *is stable for the pair* $(G; E)$ *if and only if* $PQ(G; E) = Q(G; E)$.

Proof. It is clear that $Q(G; E) \subseteq PQ(G; E)$.

Now suppose that there is $f \in PQ(G; E) \setminus Q(G; E)$. Let us show that the equation (2.2) is not stable. Indeed, if there is $q \in Q(G; E)$ such that for some positive number a

$$\| f(x) - q(x) \| \leq a,$$

then,

$$\| f(x) - q(x) \| = \tfrac{1}{4^n} \| f(x^{2^n}) - q(x^{2^n}) \| \leq \tfrac{a}{4^n}$$

and we see that $f(x) = q(x)$. Thus we come to a contradiction with the assumption about f. So if equation(2.2) is stable, then $PQ(G; E) = Q(G; E)$.

Now suppose that $PQ(G; E) = Q(G; E)$. Let us show that the equation (2.2) is stable. By Theorem 2.7 for any group G we have the decomposition

$$KQ(G; E) = PQ(G) \oplus B(G; E).$$

Hence, in our case we get

$$KQ(G; E) = Q(G; E) \oplus B(G; E).$$

It follows that for any f satisfying the functional inequality

$$\| f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) \| \leq a \quad \forall x, y \in G$$

(for some $a \geq 0$), there is a solution q of functional equation (2.2) such that the function $f(x) - q(x)$ belongs to the space $B(G; E)$. So, the equation (2.2) is stable. This completes the proof of the theorem. ■

Theorem 3.3. *Let E_1, E_2 be a Banach spaces over reals. Then the equation* (2.2) *is stable for the pair $(G; E_1)$ if and only if it is stable for the pair $(G; E_2)$.*

Proof. Let E be a Banach space and $\mathbb{R}$ be the set of reals. Let the equation (2.2) is stable for the pair $(G; E)$. Suppose that (2.2) is not stable for the pair $(G, \mathbb{R})$, then there is a nontrivial pseudoquadratic function f on G. So, for some $a \geq 0$ we have

$$\| f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) \| \leq a \quad \forall x, y \in G.$$

Now let $e \in E$ and $\|e\| = 1$. Consider the function $\varphi : G \to E$ given by the formula $\varphi(x) = f(x) \cdot e$. It is clear that φ is a nontrivial pseudoquadratic E–valued function, and we obtain a contradiction.

Now suppose that the equation (2.2) is stable for the pair $(G, \mathbb{R})$, that is, $PQ(G; \mathbb{R}) = Q(G, \mathbb{R})$. Denote by E^* the space of linear bounded functionals on E endowed by functional norm topology. Let us verify that for any $\varphi \in PQ(G; E)$ and any $\lambda \in E^*$ the function $\psi = \lambda \circ \varphi$ belongs to the space $PQ(G, \mathbb{R})$. Indeed, if a a nonnegative number such that for any $x, y \in G$ we have inequality $\|\varphi(xy) + \varphi(xy^{-1}) - 2\varphi(x) - 2\varphi(y)\| \leq a$, then

$$\begin{aligned}
&|\psi(xy) + \psi(xy^{-1}) - 2\psi(x) - 2\psi(y)| \\
&\qquad = |\lambda(\varphi(xy)) + \lambda(\varphi(xy^{-1})) - 2\lambda(\varphi(x)) - 2\lambda(\varphi(y))| \\
&\qquad = |\lambda(\varphi(xy)) + \lambda(\varphi(xy^{-1})) - \lambda(2\varphi(x)) - \lambda(2\varphi(y))| \\
&\qquad = |\lambda(\varphi(xy) + \varphi(xy^{-1}) - 2\varphi(x) - 2\varphi(y))| \\
&\qquad \leq \|\lambda\|\, a.
\end{aligned}$$

Obviously $\lambda(\varphi(x^n)) = n^2\lambda(\varphi(x))$ for any $x \in G$ and for any $n \in \mathbb{N}$. Hence the function $\lambda \circ \varphi$ belongs to the space $PQ(G, \mathbb{R})$. Let $f : G \to E$ be a nontrivial pseudoquadratic mapping. Then there are $x, y \in G$ such that $f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) \neq 0$. Hahn–Banach Theorem implies that there is a $\ell \in E^*$ such that $\ell(f(xy) + f(xy^{-1}) - 2f(x) - 2f(y)) \neq 0$, and we see that $\ell \circ f$ is a nontrivial pseudoquadratic real–valued function on G. This contradiction establishes the theorem. ■

Due to the last theorem we may simply say that the equation (2.2) is stable or not stable on a group G. In what follows, the spaces $PQ(G, \mathbb{R})$ and $Q(G, \mathbb{R})$ will be denoted by $PQ(G)$ and $Q(G)$, respectively.

Let n be an integer. A group G is said to be an n-Abelian group if $(xy)^n = x^n y^n$ for every x and y in G (see Levi [16], Baer [1], Li [17] and Gallian and Reid [9]).

Theorem 3.4. *Let $n \in \mathbb{N}$ and G be an n-Abelian group. The equation*

$$f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) = 0$$

is stable on group G.

Proof. Let G be an n-Abelian group. Let $f \in PQ(G)$ and $\delta > 0$ be such that for any $x, y \in G$ the inequality

$$|f(xy) + f(xy) - 2f(x) - 2f(y)| \leq \delta \tag{3.1}$$

holds. Let u, v be arbitrary elements of G and $n \in \mathbb{N}$ such that $(uv)^n = u^n v^n$. From the latter relation, we get

$$(uv)^{n^k} = u^{n^k} v^{n^k} \tag{3.2}$$

for any $k \in \mathbb{N}$. Let us proof this by induction on k. If $k = 1$ the relation (3.2) is true. Suppose that (3.2) is true for k. Then we have $(uv)^{n^{k+1}} = ((uv)^{n^k})^n = (u^{n^k} v^{n^k})^n = u^{n^{k+1}} v^{n^{k+1}}$. Thus, for any $k \in \mathbb{N}$, we have

$$\begin{aligned}
&n^{2k}|f(uv) + f(uv^{-1}) - 2f(u) - 2f(v)| \\
&\qquad = |f((uv)^{n^k}) + f((uv^{-1})^{n^k}) - 2f(u^{n^k}) - 2f(v^{n^k})| \\
&\qquad = |f((u^{n^k} v^{n^k}) + f((u^{n^k} (v^{-1})^{n^k}) - 2f(u^{n^k}) - 2f(v^{n^k})| \leq \delta.
\end{aligned}$$

Hence

$$|f(uv) + f(uv^{-1}) - 2f(u) - 2f(v)| \leq \frac{1}{n^{2k}}\delta.$$

Therefore, it follows that $f(uv) + f(uv^{-1}) - 2f(u) - 2f(v) = 0$ and the proof is now complete. ■

It is well known that if $n = 2$, then an n-Abelian group is an Abelian group. Thus we get the result obtained by Skof in [19] as a corollary.

Corollary 3.5. *The quadratic functional equation*

$$f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) = 0$$

is stable on any Abelian group.

Let K be an arbitrary commutative field. Let K^* be the set nonzero elements of K with operation of multiplication. Denote by G the group $T(2, K)$ consisting of matrices of the form

$$\begin{bmatrix} \alpha & t \\ 0 & \beta \end{bmatrix} \quad ; \quad \alpha, \beta \in K^*; \quad t \in K.$$

Denote by T, E, D subgroups of $G = T(2, K)$ consisting of matrices

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}; \qquad \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}; \qquad \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix};$$

where $a, b \in K^*, t \in K$, respectively. It is clear that $T \triangleleft G$ and we have the following semidirect products, $G = D \cdot T$. Subgroup C of G generated by T and E is a semidirect product $C = E \cdot T$. In the remaining of this section, we investigate the stability of the quadratic equation on the group $T(2, K)$.

Let $f \in PQ(G)$ and $f\Big|_D \equiv 0$. Then for some positive number Δ and any $x, y \in G$ we have

$$|f(xy) + f(xy^{-1}) - 2f(x) - 2f(y)| \leq \Delta. \tag{3.3}$$

Let

$$u = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad v = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad w = \begin{bmatrix} 1 & \frac{b}{c}t \\ 0 & 1 \end{bmatrix}.$$

From the equality

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} 1 & \frac{b}{a}t \\ 0 & 1 \end{bmatrix} \tag{3.4}$$

we get

$$uv = vw, \quad vw^{-1} = u^{-1}v. \tag{3.5}$$

From (3.3), we have

$$|f(uv) + f(uv^{-1}) - 2f(u) - 2f(v)| \leq \Delta,$$
$$|f(vw) + f(vw^{-1}) - 2f(v) - 2f(w)| \leq \Delta.$$

Taking into account (3.5) and (2.3) it follows from the last two relations that there is a positive number Δ_1 such that

$$|f(u) - f(w)| \leq \Delta_1.$$

That is

$$\left| f\left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right) - f\left(\begin{bmatrix} 1 & \frac{b}{a}t \\ 0 & 1 \end{bmatrix}\right)\right| \leq \Delta_1$$

for any $t \in K$ and $a, b \in K^*$. Therefore f is bounded on T. But for any $n \in \mathbb{N}$ and $x \in T$ we have $f(x^n) = n^2 f(x)$. It follows $f\Big|_T \equiv 0$. Therefore $f\Big|_{D\cup T} \equiv 0$.

Let $x = a^{-1}u,\ y = av,\ a \in D,\ u, v \in T$. Then from (3.3), we have

$$|f(a^{-1}uav) + f(a^{-1}uv^{-1}a^{-1}) - 2f(a^{-1}u) - 2f(av)| \leq \Delta.$$

Denoting $a^{-1}ua$ by u^a, we obtain

$$|f(u^a v) + f(a^{-1}uv^{-1}a^{-1}) - 2f(a^{-1}u) - 2f(av)| \leq \Delta.$$

Letting $v = 1$ and simplifying we see that

$$|f(u^a) + f(a^{-2}u^{a^{-1}}) - 2f(a^{-1}u) - 2f(a)| \leq \Delta.$$

Since $f\Big|_T \equiv 0$, $f(a) = 0$ and from last inequality, we have

$$|f(u^a) + f(a^{-2}u^{a^{-1}}) - 2f(a^{-1}u)| \leq \Delta.$$

Further, since $u^a \in D \cup T$, $f(u^a) = 0$. Hence we obtain

$$|f(a^{-2}u^{a^{-1}}) - 2f(a^{-1}u)| \leq \Delta$$

for all $a \in D$ and $u \in T$. Replacing a^{-1} by a, we have

$$|f(a^2u^a) - 2f(au)| \leq \Delta \tag{3.6}$$

for all $a \in D$ and $u \in T$. Next letting $x = a^2u^a$ and $y = u$ in (3.3), we have

$$|f(a^2u^au) + f(a^2u^au^{-1}) - 2f(a^2u^a) - 2f(u)| \leq \Delta.$$

Since $(au)^2 = a^2u^au$, the last inequality yields

$$|f((au)^2) + f(a^2u^au^{-1}) - 2f(a^2u^a) - 2f(u)| \leq \Delta.$$

Since $f \in PQ(G)$, $f(x^2) = 4f(x)$ and from the last inequality, we have

$$|4f(au) + f(a^2u^au^{-1}) - 2f(a^2u^a) - 2f(u)| \leq \Delta. \tag{3.7}$$

From (3.6) and (3.7) we get

$$|4f(au) + f(a^2u^au^{-1}) - 4f(au) - 2f(u)| \leq 3\Delta.$$

Since $u \in T$ and $f\Big|_T \equiv 0$, $f(u) = 0$. Therefore the last inequality yields

$$|f(a^2u^au^{-1})| \leq 3\Delta. \tag{3.8}$$

Lemma 3.6. *Let $f \in PQ(G)$ and $f\Big|_D \equiv 0$. Suppose for some positive number Δ the inequality* (3.3) *holds for all $x, y \in G$. Then f is bounded on the set*

$$M_1 = \left\{ \begin{bmatrix} a^2 & t \\ 0 & b^2 \end{bmatrix} \;\middle|\; a, b \in K^*,\ t \in K \quad \textit{and} \quad a^2 \neq b^2 \right\}.$$

Proof. Let us show that any element g from the set M_1 is representable in the form $g = c^2 u^c u^{-1}$ for some $c \in D$ and $u \in T$.

Let $g = \begin{bmatrix} a^2 & t \\ 0 & b^2 \end{bmatrix}$ be an arbitrary element from M_1, and let $c = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, and $u = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}$. We have $u^c u^{-1} = \begin{bmatrix} 1 & \frac{b}{a}\tau \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -\tau \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & (\frac{b}{a} - 1)\tau \\ 0 & 1 \end{bmatrix}$ So $c^2 u^c u^{-1} = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} \cdot \begin{bmatrix} 1 & (\frac{b}{a} - 1)\tau \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a^2 & a(b-a)\tau \\ 0 & b^2 \end{bmatrix}$. Hence we see that if $\tau = \frac{t}{a(b-a)}$, then $g = c^2 u^c u^{-1}$. Now from (3.8) it follows that for any $g \in M_1$ we have

$$|f(g)| \leq 3\Delta \tag{3.9}$$

and the proof of the lemma is now complete. ■

Lemma 3.7. *Let $f \in PQ(G)$ and $f\Big|_D \equiv 0$. Suppose for some positive number Δ the inequality* (3.3) *holds for all $x, y \in G$. Let*

$$M_2 = \left\{ \begin{bmatrix} \alpha^2 & t \\ 0 & \alpha^2 \end{bmatrix} \;\middle|\; \alpha \in K^*, t \in K \right\}.$$

Then $f\Big|_{M_2} \equiv 0$.

Proof. Let us show that f is bounded on the set M_2. Any element g from M_2 is representable in the form $g = au$, where $a \in D$ and $u \in T$. Let $b = \begin{bmatrix} \beta^2 & \tau \\ 0 & 1 \end{bmatrix}$, where $\beta^2 \neq 1$. Then for any $v \in G$ we have

$$|f(gv) + f(gv^{-1}) - 2f(g) - 2f(v)| \leq \Delta,$$
$$|f(aub) + f(aub^{-1}) - 2f(au) - 2f(b)| \leq \Delta,$$

$$|f(abu^b) + f(ab^{-1}u^{b^{-1}}) - 2f(au) - 2f(b)| \leq \Delta, \tag{3.10}$$

The diagonal elements of the matrices $b, abu^b, ab^{-1}u^{b^{-1}}$ are different. Hence from (3.9), we have

$$|f(b)| \leq 3\Delta, \quad |f(abu^b)| \leq 3\Delta, \quad |f(ab^{-1}u^{b^{-1}})| \leq 3\Delta.$$

From (3.10) we get

$$|f(au)| \leq 5\Delta, \tag{3.11}$$

and we see that f is bounded on M_2. It is clear that M_2 is a subgroup of G, hence from (3.11) it follows that $f\Big|_{M_2} \equiv 0$. ■

Theorem 3.8. *Quadratic equation is stable on $T(2,K)$, where K is an arbitrary commutative field.*

Proof. Let $\varphi \in PQ(G)$ and $\psi = \varphi\Big|_D$. The subgroup D is an Abelian group. By Theorem 3.4 we have $\psi \in Q(D)$. Hence a function $\widehat{\psi}$ defined by the rule $\widehat{\psi}(x) = \psi(\pi(x))$, where $\pi : G \to D$ an epimorphism such that $\pi : g = \begin{bmatrix} a & t \\ 0 & b \end{bmatrix} \to \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, is a quadratic function on G such that $\widehat{\psi}\Big|_D = \varphi\Big|_D$. Now consider function $f(x) = \varphi(x) - \widehat{\psi}(x)$. It is clear that $f\Big|_D \equiv 0$. By Lemmas 3.6 and 3.7 we get that there exists a positive δ such that for any g belonging to the set

$$M = \left\{ \begin{bmatrix} a^2 & t \\ 0 & b^2 \end{bmatrix} \;\middle|\; a, b \in K^*,\, t \in K \right\}$$

we have the following estimation $|f(g)| \le \delta$. Now if x an arbitrary element from G, then $x^2 \in M$ and we have $|f(x)| = \frac{1}{4}|f(x^2)| \le \frac{1}{4}\delta$. Therefore the function f is bounded on G. Hence, $f \equiv 0$. and we see that $\varphi = \widehat{\psi} \in Q(G)$. ∎

4 Embedding

Let G be an arbitrary group, $f \in PQ(G)$, and for any $x, y \in G$ the following relation

$$|f(xy) + f(xy^{-1}) - 2f(x) - 2f(y)| \le \delta \tag{4.1}$$

holds. Let b, c, u, v be elements of G, and let $x = bu$ and $y = cv$. Below we will use notation a^b for element $b^{-1}ab$; $\forall a, b \in G$. Then from (4.1), we get

$$\begin{aligned} &|f(bucv) + f(buv^{-1}c^{-1}) - 2f(bu) - 2f(cv)| \\ &\qquad = |f(bcu^cv) + f(bc^{-1}(uv^{-1})^{c^{-1}}) - 2f(bu) - 2f(cv)| \le \delta. \end{aligned}$$

Thus

$$|f(bcu^cv) + f(bc^{-1}(uv^{-1})^{c^{-1}}) - 2f(bu) - 2f(cv)| \le \delta \tag{4.2}$$

and if $b = c$, we get

$$|f(c^2u^cv) + f((uv^{-1})^{c^{-1}}) - 2f(cu) - 2f(cv)| \le \delta. \tag{4.3}$$

If $b = c$ and $u = v$, then from (4.2) it follows that

$$|f(c^2u^cu) + f((uu^{-1})^{c^{-1}}) - 2f(cu) - 2f(cu)| \le \delta. \tag{4.4}$$

Hence

$$|f(c^2u^cu) - 4f(cu)| \le \delta.$$

If $c^2 = 1$, then the inequality implies that

$$|f(u^cu) - 4f(cu)| \le \delta. \tag{4.5}$$

If we put $b = c,\ c^2 = 1$ and $u = 1$, then from (4.2), we get

$$|f(v) + f((v^{-1})^{c^{-1}}) - 2f(cv)| \le \delta \tag{4.6}$$

which is

$$|f(v) + f(v^c) - 2f(cv)| \le \delta. \tag{4.7}$$

From (4.5) and (4.7), we have

$$|f(u^c u) - 2f(u) - 2f(u^c)| \le 3\delta. \tag{4.8}$$

Substituting $c^2 = 1,\ v = 1$, in (4.3), we see that

$$|f(u^c) + f(u^c) - 2f(cu)| \le \delta$$

which is

$$|f(u^c) - f(cu)| \le \tfrac{1}{2}\delta. \tag{4.9}$$

From (4.9) and (4.7) it follows that

$$|f(u^c) - f(u)| \le 2\delta.$$

Since the last inequality holds for any $u \in G$, we get

$$|f((u^n)^c) - f(u^n)| \le 2\delta, \quad \forall n \in \mathbb{N}.$$

Hence

$$n^2|f(u^c) - f(u)| \le 2\delta, \quad \forall n \in \mathbb{N}.$$

and we see that the last relation is possible only if

$$f(u^c) = f(u). \tag{4.10}$$

From (4.9) and (4.10), we get

$$|f(cu) - f(u)| \le \tfrac{1}{2}\delta. \tag{4.11}$$

From (4.5), (4.7) and (4.10), we get

$$|f(u^c u) - 4f(u)| \le 3\delta. \tag{4.12}$$

Indeed

$$\begin{aligned}|f(u^c u) - 4f(u)| &= |f(u^c u) - 4f(cu) - 2f(u) - 2f(u^c) + 4f(cu)| \\ &\le |f(u^c u) - 4f(cu)| + 2|f(u) + f(u^c) - 2f(cu)| \\ &\le 3\delta.\end{aligned}$$

Lemma 4.1. *Let G be an arbitrary group, $f \in PQ(G)$ and $u, c \in G$. Suppose that $u^c u = u u^c$. Then*

$$f(u^c u) = 4f(u). \tag{4.13}$$

Proof. For any $n \in \mathbb{N}$, we have

$$n^2 |f(u^c u) - 4f(u)| = |f((u^c u)^n) - 4f(u^n)| = |f((u^n)^c u^n) - 4f(u^n)| \leq \delta.$$

Therefore

$$|f(u^c u) - 4f(u)| \leq \tfrac{1}{n^2}\delta.$$

Hence

$$f(u^c u) = 4f(u).$$

This completes the proof of the lemma. ■

Let A and B be arbitrary groups. For each $b \in B$ denote by $A(b)$ a group that is isomorphic to A under isomorphism $a \to a(b)$. Denote by $H = A^{(B)} = \prod_{b \in B} A(b)$ the direct product of groups $A(b)$. It is clear that if $a_1(b_1)a_2(b_2)\cdots a_k(b_k)$ is an element of H, then for any $b \in B$, the mapping

$$b^* : a_1(b_1)a_2(b_2)\cdots a_k(b_k) \to a_1(b_1 b)a_2(b_2 b)\cdots a_k(b_k b)$$

is an automorphism of H and $b \to b^*$ is an embedding of B into $Aut\, H$. Thus, we can form a semidirect product $G = B \cdot H$. This group is called *the wreath product* of the groups A and B, and will be denoted by $G = A \wr B$. We will identify the group A with subgroup $A(1)$ of H, where $1 \in B$. Hence, we can assume that A is a subgroup of H.

Let us denote, by C, the group of order 2 with the generator c. Consider the group $A \wr C$.

Lemma 4.2. *Let A be an arbitrary group and C be a group of order 2 with the generator c. Further, let $H = A^{(C)}$. If for some $a_1, b_1 \in A$ we have*

$$|f(a_1 b_1) + f(a_1 b_1^{-1}) - 2f(a_1) - 2f(b_1)| = \delta > 0$$

then for some $x, y \in H$ we have

$$|f(xy) + f(xy^{-1}) - 2f(x) - 2f(y)| = 4\delta.$$

Proof. Let $u = a_1 b_1$. Then we have $u^c u = u u^c$. Using the relation (4.13) we get

$$\begin{aligned} f(a_1 a_1^c b_1 b_1^c) &+ f(a_1 a_1^c (b_1^{-1})^c b_1^{-1}) - 2f(a_1 a_1^c) - 2f(b_1 b_1^c) \\ &= f(a_1 b_1 a_1^c b_1^c) + f(a_1 b_1^{-1} a_1^c (b_1^{-1})^c) - 2f(a_1 a_1^c) - 2f(b_1 b_1^c) \\ &= 4f(a_1 b_1) + 4f(a_1 b_1^{-1}) - 4 \cdot 2f(a_1) - 4 \cdot 2f(b_1) \\ &= 4\delta \end{aligned}$$

and the proof is now complete. ■

Theorem 4.3. *Any group A can be embedded into a group G such that the equation* (2.2) *is stable on G.*

Proof. Let C_i, for $i \in \mathbb{N}$, be a group of order 2. Consider the chain of groups defined as follows:

$$A_1 = A,\ A_2 = A_1 \wr C_1,\ A_3 = A_2 \wr C_2,\ \ldots,\ A_{k+1} = A_k \wr C_k, \ldots$$

Define a chain of embeddings

$$A_1 = A \ \rightarrow\ A_2 = A_1 \wr C_1 \ \rightarrow\ A_3 = A_2 \wr C_2 \ \rightarrow\ \cdots \rightarrow\ A_{k+1} = A_k \wr C_k \rightarrow\ \ldots \quad (4.14)$$

by identifying A_k with $A_k(1)$ a subgroup of A_{k+1}. Let G be the direct limit of the chain (4.14). Then we have $G = \cup_{k\in\mathbb{N}} A_k$ and

$$A_1 \subset A_2 \subset \cdots \subset A_k \subset A_{k+1} \subset \ldots\ldots \subset G.$$

Let $f \in PQ(G)$, and let for $k \in \mathbb{N}$

$$\delta_k = \sup\Big\{\, f(uv) + f(uv^{-1}) - 2f(u) - 2f(v) \ :\ u, v \in A_k \Big\}.$$

Let us verify that $\delta_k = 0$ for any k. Suppose that $\delta_1 > 0$. Then for some $a_1, b_1 \in A_1$, we have

$$|f(a_1b_1) + f(a_1b_1^{-1}) - 2f(a_1) - 2f(b_1)| = \delta > 0.$$

Then Lemma 4.2 implies that, for some $a_2, b_2 \in A_2$, we have

$$|f(a_2b_2) + f(a_2b_2^{-1}) - 2f(a_2) - 2f(b_2)| = 4\delta.$$

Again by Lemma 4.2 we can find $a_3, b_3 \in A_3$ such that

$$|f(a_3b_3) + f(a_3b_3^{-1}) - 2f(a_3) - 2f(b_3)| = 4^2\,\delta.$$

Continuing this process we see that one can choose $a_k, b_k \in A_k$ such that

$$|f(a_kb_k) + f(a_kb_k^{-1}) - 2f(a_k) - 2f(b_k)| = 4^{k-1}\delta \rightarrow \infty \ \text{ as } \ k \ \rightarrow \infty.$$

This contradicts the assumption that $f \in PQ(G)$. So we see that $\delta_1 = 0$. Similarly we can verify that $\delta_2 = \delta_3 = \cdots = \delta_k = \cdots = 0$. Therefore we have $f \in Q(G)$ and the proof is complete. ∎

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