Ultrametric C^n-Spaces of Countable Type

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Abstract

Let K be a non-trivially non-archimedean valued field that is complete with respect to the valuation $|\ |: K \longrightarrow [0, \infty)$, let X be a non-empty subset of K without isolated points. For $n \in \{0, 1, \dots\}$ the K-Banach space $BC^n(X)$, consisting of all C^n-functions $X \longrightarrow K$ whose difference quotients up to order n are bounded, is defined in a natural way. It is proved that $BC^n(X)$ is of countable type if and only if X is compact. In addition we will show that $BC^\infty(X) := \bigcap_n BC^n(X)$, which is a Fréchet space with its usual projective topology, is of countable type if and only if X is precompact.

1 Preliminaries

For a non-empty subset V of $\mathbb{R}$ its supremum is denoted by $\sup V$, where by convention $\sup V = \infty$ if V is not bounded above. The closure of a subset Y of a topological space is denoted by $\overline{Y}$.

Throughout this paper $K = (K, |\ |)$ is a non-archimedean valued field as in the Abstract. For basic notions and facts on normed and locally convex spaces over K we refer to [2] and [5] respectively.

For a finite subset S of K containing at least two elements we denote its diameter $\max\{|x - y| : x, y \in S\}$ by d_S. Then $0 < d_S < \infty$.

For a non-empty topological space Z we denote by $C(Z)$ the collection of all continuous functions $Z \longrightarrow K$. It is a K-vector space under pointwise operations. For $f \in C(Z)$ we put $\|f\| := \sup\{|f(z)| : z \in Z\}$. We set $BC(Z) := \{f \in C(Z) :$

2000 *Mathematics Subject Classification :* Primary 46S10, Secondary 26E30.

Key words and phrases : non-archimedean Banach spaces, differentiable functions, spaces of countable type.

$\|f\| < \infty\}$. It is easily seen that $(BC(Z), \|\ \|)$ is a Banach space over K. If Z is compact then $BC(Z) = C(Z)$.

Recall ([2], p. 66) that a normed space over K is said to be **of countable type** if it contains a countable set whose linear hull is dense. The natural extension to locally convex spaces reads as follows. A locally convex space E over K is called **of countable type** if for every continuous seminorm p, (1) below holds.

The space $E_p := E/\mathrm{Ker}p$, equipped with the norm $\overline{p}$ defined by the formula $\overline{p}(x + \mathrm{Ker}p) = p(x)$, is of countable type. (1)

It is easily seen that E is of countable type as soon as (1) holds for each $p \in \mathcal{P}$, where $\mathcal{P}$ is a collection of seminorms on E generating the topology.

Proposition 1.1 ([4] 4.12) *Subspaces, continuous linear images, and products of locally convex spaces of countable type are of countable type.*

The Banach space $BC(I)$, where I carries the discrete topology is usually called $\ell^\infty(I)$, and $\ell^\infty := BC(\mathbb{N})$. The following fact is well-known, but it seems hard to find a direct reference, so we provide a proof.

Proposition 1.2 *If I is infinite then $\ell^\infty(I)$ is not of countable type.*

Proof. If $\ell^\infty(I)$ $(= BC(I)$, I with the discrete topology) were of countable type then, by 1.1, the subspace $RC(I)$ of all functions $I \longrightarrow K$ with precompact image would be also of countable type, which implies that I is compact ([1], Theorem 14), a contradiction.

2 An inequality for an arbitrary function $X \longrightarrow K$

Throughout this section we fix an infinite subset X of K, an $n \in \{0, 1, \ldots\}$ and a function $f : X \longrightarrow K$. In the spirit of [3] 29.1 we put

$$\bigtriangledown^{n+1} X := \{(x_1, \ldots, x_{n+1}) \in X^{n+1} : \text{if } i \neq j \text{ then } x_i \neq x_j\ \}$$

(notice that $\nabla^1 X = X$) and define the nth order difference quotient $\Phi_n f : \bigtriangledown^{n+1} X \longrightarrow K$ inductively by $\Phi_0 f := f$ and, for $n \geq 1$,

$$\Phi_n f(x_1, \ldots, x_{n+1}) := \frac{\Phi_{n-1} f(x_1, x_3, \ldots, x_{n+1}) - \Phi_{n-1} f(x_2, x_3, \ldots, x_{n+1})}{x_1 - x_2}.$$

We set

$$\begin{aligned} \|\Phi_n f\| &:= \sup\{|\Phi_n f(v)| : v \in \bigtriangledown^{n+1} X\} \text{ and} \\ \|f\|_n &:= \max\{\|\Phi_i f\| : 0 \leq i \leq n\} \end{aligned}$$

(allowing $\|f\|_n$ to be ∞ if $\|\Phi_i f\| = \infty$ for some i).

Now let X_n be the collection of all subsets of X containing precisely $n + 1$ elements. Since $\Phi_n f$ is a symmetric function of its $n + 1$ variables ([3] 29.2) it induces naturally a function $\tilde{\Phi}_n f : X_n \longrightarrow K$ via the formula

$$\tilde{\Phi}_n f(S) = \Phi_n f(x_1, \ldots, x_{n+1}) \quad (S := \{x_1, \ldots, x_{n+1}\} \in X_n).$$

Setting $\|\tilde{\Phi}_n f\| := \sup\{|\Phi_n f(S)| : S \in X_n\}$ we have obviously $\|\Phi_n f\| = \|\tilde{\Phi}_n f\|$.

Lemma 2.1 *Let $n \geq 2$, $S_n \in X_n$. Then there exists an $S_{n-1} \in X_{n-1}$ such that $S_{n-1} \subset S_n$ and*

$$\left|\tilde{\Phi}_n f(S_n)\right| \leq d_{S_n}^{-1} \left|\tilde{\Phi}_{n-1} f(S_{n-1})\right|.$$

Proof. Let $x, y \in S_n$ be such that $|x-y| = d_{S_n}$. Then writing $S_n = \{x, y, x_1, \ldots, x_{n-1}\}$ we obtain

$$\begin{aligned} \left|\tilde{\Phi}_n f(S_n)\right| &= |\Phi_n f(x, y, x_1, \ldots, x_{n-1})| \\ &= |x-y|^{-1} |\Phi_{n-1} f(x, x_1, \ldots, x_{n-1}) - \Phi_{n-1} f(y, x_1, \ldots, x_{n-1})| \\ &\leq d_{S_n}^{-1} \max\left(|\Phi_{n-1} f(x, x_1, \ldots, x_{n-1})|, |\Phi_{n-1} f(y, x_1, \ldots, x_{n-1})|\right) \\ &= d_{S_n}^{-1} \left|\tilde{\Phi}_{n-1} f(S_{n-1})\right|, \end{aligned}$$

where $S_{n-1} \in X_{n-1}$ is either $\{x, x_1, \ldots, x_{n-1}\}$ (if $|\Phi_{n-1} f(x, x_1, \ldots, x_{n-1})| \geq |\Phi_{n-1} f(y, x_1, \ldots, x_{n-1})|$) or $\{y, x_1, \ldots, x_{n-1}\}$ (otherwise).

For convenience we introduce yet another quantity. We put

$${}_n\|f\| := \sup\left\{\frac{|f(x)-f(y)|}{|x-y|^n} : x, y \in X,\ x \neq y\right\}.$$

Notice that ${}_0\|f\| \leq \|f\|$ and ${}_1\|f\| = \|\Phi_1 f\|$.

Proposition 2.2 *Let $n \geq 1$. Then $\|\Phi_n f\| \leq {}_n\|f\|$.*

Proof. For $n = 1$ the formula (even with an equality sign) holds trivially, so let $n \geq 2$ and $S_n \in X_n$. By using 2.1 repeatedly we arrive at sets $S_n \supset S_{n-1} \supset \ldots \supset S_1$, where $S_i \in X_i$ for $i \in \{1, \ldots, n\}$, for which

$$\left|\tilde{\Phi}_n f(S_n)\right| \leq d_{S_n}^{-1} \ldots d_{S_2}^{-1} \left|\tilde{\Phi}_1 f(S_1)\right|. \tag{2}$$

Now let $S_1 := \{x, y\}$. From $|x-y| = d_{S_1} \leq d_{S_2} \leq \ldots \leq d_{S_n}$ we obtain $d_{S_i}^{-1} \leq |x-y|^{-1}$ for $i \in \{2, \ldots, n\}$ so that (2) yields the further estimate

$$\left|\tilde{\Phi}_n f(S_n)\right| \leq |x-y|^{1-n} |\Phi_1 f(x, y)| = |x-y|^{-n} |f(x) - f(y)| \leq {}_n\|f\|,$$

and $\|\Phi_n f\| = \|\tilde{\Phi}_n f\| \leq {}_n\|f\|$.

Remark 2.3 The inequality does not hold for $n = 0$ as is easily seen by taking for f a non-zero constant function.

Proposition 2.4 *For all $n \geq 0$*

$$\|f\|_n \leq \max(\|f\|, {}_n\|f\|).$$

Proof. We have to prove that $\|\Phi_i f\| \leq \max(\|f\|, {}_n\|f\|)$ for all $i \in \{0, 1, \ldots, n\}$. To this end we may assume that $i \geq 1$. Let $x, y \in X$, $x \neq y$. If $|x-y| \geq 1$ then $|x-y|^{-i} |f(x) - f(y)| \leq |f(x) - f(y)| \leq \|f\|$, whereas, if $|x-y| < 1$, we have $|x-y|^{-i} |f(x) - f(y)| \leq |x-y|^{-n} |f(x) - f(y)| \leq {}_n\|f\|$. We see that ${}_i\|f\| \leq \max(\|f\|, {}_n\|f\|)$, and by 2.2 we find $\|\Phi_i f\| \leq \max(\|f\|, {}_n\|f\|)$.

3 The opposite inequality for a locally constant function $X \longrightarrow K$

From now on let X be a non-empty subset of K without isolated points. (Then X is infinite).

Definition 3.1 ([3] 29.1) Let $n \in \{0, 1, \ldots\}$, $f : X \longrightarrow K$. We say that $f \in C^n(X)$ if $\Phi_n f$ can (uniquely) be extended to a continuous function $\overline{\Phi}_n f : X^{n+1} \longrightarrow K$. For $x \in X$ we set $D_n f(x) := \overline{\Phi}_n f(x, x, \ldots, x)$. Also, $C^\infty(X) := \bigcap\{C^n(X) : n \geq 0\}$.

We recall some facts from basic theory of C^n-functions. Notice that $C^0(X) = C(X)$.

Proposition 3.2

(i) ([3] 29.3) $C^0(X) \supset C^1(X) \supset \ldots \supset C^\infty(X)$.

(ii) ([3] 29.3) $C^n(X)$ $(n \in \{0, 1, \ldots\})$ *and* $C^\infty(X)$ *are* K*-vector spaces under pointwise operations.*

(iii) ([3] 29.4, Taylor Formula) *Let* $f \in C^n(X)$ $(n \geq 1)$. *Then for all* $x, y \in X$

$$f(x) = \sum_{j=0}^{n-1} (x-y)^j D_j f(y) + (x-y)^n \overline{\Phi}_n f(x, y, y, \ldots, y).$$

(iv) ([3] 29.5) *Let* $f \in C^n(X)$ $(n \geq 1)$. *Then* f *is* n *times differentiable and* $j! D_j f = f^{(j)}$ *for* $1 \leq j \leq n$.

(v) ([3] 29.10) *A locally constant function* $f : X \longrightarrow K$ *is in* $C^\infty(X)$ *and* $D_j f = 0$ *for all* $j \in \{1, 2, \ldots\}$.

For $f \in C^n(X)$ $(n \in \{0, 1, \ldots\})$ we define $\|\overline{\Phi}_n f\| := \sup\{\left|\overline{\Phi}_n f(v)\right| : v \in X^{n+1}\}$. Then by continuity and density of $\bigtriangledown^{n+1} X$ in X^{n+1} we have $\|\overline{\Phi}_n f\| = \|\Phi_n f\|$.

We now arrive at our goal of this Section.

Theorem 3.3 *Let* $f : X \longrightarrow K$ *be locally constant. Then for* $n \in \{0, 1, \ldots\}$

$$\|f\|_n = \max(\|f\|, {}_n\|f\|).$$

Proof. By 2.4 and $\|f\| \leq \|f\|_n$ we only have to prove ${}_n\|f\| \leq \|f\|_n$. We may assume $n \geq 1$. By the Taylor Formula 3.2(iii) and the fact that $D_j f = 0$ for all $j \geq 1$ (3.2(v)) we get

$$f(x) = f(y) + (x-y)^n \, \overline{\Phi}_n f(x, y, \ldots, y) \quad (x, y \in X),$$

so that for $x \neq y$ and by continuity of $\overline{\Phi}_n f$,

$$\left| \frac{f(x) - f(y)}{(x-y)^n} \right| = \left| \overline{\Phi}_n f(x, y, \ldots, y) \right| \leq \|\Phi_n f\| \leq \|f\|_n$$

and the theorem is proved.

4 The Main Theorem

Definition 4.1 Let $n \in \{0, 1, \dots\}$. We set

$$BC^n(X) := \{f \in C^n(X) : \|f\|_n < \infty\}.$$

It is straightforward to check that $BC^n(X)$ is a subspace of $C^n(X)$ and that $\| \ \|_n$ is a norm on $BC^n(X)$ making it into a Banach space. Also notice that $BC^0(X) = BC(X)$. If X is compact then $BC^n(X) = C^n(X)$.

Lemma 4.2 *If X is compact then $BC^n(X)$ is of countable type.*

Proof. The map

$$f \longmapsto (f, \overline{\Phi}_1 f, \dots, \overline{\Phi}_n f)$$

is a linear isometry of $BC^n(X)$ into $\prod_{k=1}^{n+1} C(X^k)$. Now each X^k is ultrametrizable so, by [2] 3.T, $C(X^k)$ is of countable type, hence so are $\prod_{k=1}^{n+1} C(X^k)$ and its subspace $BC^n(X)$ (by 1.1).

Lemma 4.3 *Let $BC^n(X)$ be of countable type for some $n \in \{0, 1, \dots\}$. Then X is precompact.*

Proof. Suppose X is not precompact. Then, for some $r > 0$, the balls in X of radius r form an infinite covering of X, say $(B_i)_{i \in I}$. Choose $a_i \in B_i$ for each $i \in I$. Let D be the space of all bounded functions $X \longrightarrow K$ that are constant on each B_i. We claim that $D \subset BC^n(X)$ and that D, equipped with the induced topology, is linearly homeomorphic to $\ell^\infty(I)$. (Then we have a contradiction since by 1.2 $\ell^\infty(I)$ is not of countable type, proving the lemma.)

Clearly $D \subset C^n(X)$ (in fact, $D \subset C^\infty(X)$ by 3.2(v)). Further, if $f \in D$, $x, y \in X$, $x \neq y$ we have $\frac{f(x)-f(y)}{(x-y)^n} = 0$ if $x, y \in B_i$ for some $i \in I$. Whereas if $x \in B_i$, $y \in B_j$ for $i \neq j$ then $|x - y| \geq r$ and $|f(x) - f(y)| \leq \|f\| = \sup\{|f(a_i)| : i \in I\}$. So we find

$${}_n\|f\| = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^n} \leq r^{-n} \, \|f\|,$$

implying by 3.3 that $f \in BC^n(X)$ and that $\| \ \|$ is equivalent to $\| \ \|_n$ on D. Then it is clear that $f \mapsto (f(a_i))_{i \in I}$ is a surjective linear homeomorphism $D \longrightarrow \ell^\infty(I)$, which finishes the proof.

Lemma 4.4 *Let $BC^n(X)$ be of countable type for some $n \in \{0, 1, \dots\}$. Then X is compact.*

Proof. Suppose X is not compact; we derive a contradiction. By 4.3 X is precompact, so X is not closed in K, let $a \in \overline{X} \setminus X$. From compactness of $\overline{X}$ and the ultrametric property one easily derives that the set $\{|x - a| : x \in X\}$ is discrete with 0 as an accumulation point, say $\{r_1, r_2, \dots\}$, where $r_1 > r_2 > \dots$ and $\lim_m r_m = 0$.

For $m \in \mathbb{N}$, let $R_m := \{x \in X : |x - a| = r_m\}$. Then $R_1, R_2, \dots$ is an infinite clopen covering of X. Choose $a_m \in R_m$ for each m. Let D be the space of all $f \in BC^n(X)$ that are constant on each R_m and for which $\lim_m f(a_m) = 0$. Let

$\lambda_1, \lambda_2, \ldots \in K$ be such that $|\lambda_m| = r_m^{-n}$ for each m. As ℓ^∞ is not of countable type (1.2), we are done once we can prove that

$$f \overset{T}{\longmapsto} (\lambda_1\, f(a_1), \lambda_2\, f(a_2), \ldots)$$

is a linear homeomorphism of D onto ℓ^∞.

First we are going to see that for $f \in D$,

$$\sup_m \frac{|f(a_m)|}{r_m^n} \leq \|f\|_n \leq \max(r_1^n, 1)\ \sup_m \frac{|f(a_m)|}{r_m^n}$$

(which shows that T maps D homeomorphically into ℓ^∞). To prove the first inequality, let $m \in \mathbb{N}$; we show that $\frac{|f(a_m)|}{r_m^n} \leq \|f\|_n$. We may suppose $f(a_m) \neq 0$ so, since $\lim_j f(a_j) = 0$, there is a $j > m$ for which $|f(a_j)| < |f(a_m)|$, so that $|f(a_m)| = |f(a_m) - f(a_j)|$. Also, $|a_m - a_j| = \max(|a_m - a|, |a_j - a|) = |a_m - a| = r_m$. Thus, we obtain $\frac{|f(a_m)|}{r_m^n} = \frac{|f(a_m) - f(a_j)|}{|a_m - a_j|^n}$ which, by 3.3, is $\leq \|f\|_n$. For the second inequality observe that, for $f \in D$,

$$\begin{aligned} {}_n\|f\| &= \sup_{m>k} \frac{|f(a_m) - f(a_k)|}{r_k^n} \leq \\ &\leq \sup_{m>k} \max\left(\frac{|f(a_m)|}{r_m^n}\, \frac{r_m^n}{r_k^n}, \frac{|f(a_k)|}{r_k^n} \right). \end{aligned}$$

Now $\frac{r_m^n}{r_k^n} \leq 1$ so that the previous expression is $\leq \sup_{m>k} \max\left(\frac{|f(a_m)|}{r_m^n}, \frac{|f(a_k)|}{r_k^n}\right) \leq \sup_m \frac{|f(a_m)|}{r_m^n}$. Further,

$$\|f\| = \sup_m \frac{|f(a_m)|}{r_m^n}\, r_m^n \leq \sup_m \frac{|f(a_m)|}{r_m^n}\, r_1^n.$$

Hence, by using 3.3,

$$\|f\|_n \leq \max(r_1^n, 1)\ \sup_m \frac{|f(a_m)|}{r_m^n},$$

and we obtain the desired second inequality.

To finish the proof it has only to be shown that T is surjective i.e. we have to show that if $(\mu_1, \mu_2, \ldots)$ is a bounded sequence in K then the function f that has the value $\frac{\mu_m}{\lambda_m}$ on each R_m, lies in D.

Clearly $f \in C^\infty(X)$, hence it is in $C^n(X)$. Also, $\lim_m f(a_m) = \lim_m \frac{\mu_m}{\lambda_m} = 0$ as $(\mu_1, \mu_2, \ldots)$ is bounded and $|\lambda_m^{-1}| = r_m^n \to 0$. It remains to see that $f \in BC^n(X)$ i.e. that $\|f\|_n < \infty$. Now f is clearly bounded, so by 3.3 it suffices to prove that ${}_n\|f\| < \infty$. Let $x, y \in X$, $x \neq y$. We may suppose $x \in R_m$, $y \in R_k$ with $m > k$. Then $|x - y|^n = r_k^n$ and $|f(x) - f(y)| \leq \max(|f(a_k)|, |f(a_m)|)$. So we get

$$\begin{aligned} \frac{|f(x) - f(y)|}{|x - y|^n} &\leq \max\left(\frac{|f(a_k)|}{r_k^n}, \frac{|f(a_m)|}{r_m^n}\, \frac{r_m^n}{r_k^n} \right) \leq \\ &\leq \sup_m \frac{|f(a_m)|}{r_m^n} = \sup_m \left| \frac{\mu_m}{\lambda_m}\, \lambda_m \right| = \sup_m |\mu_m| \end{aligned}$$

so that ${}_n\|f\| \leq \sup_m |\mu_m| < \infty$.

Combining 4.2 and 4.4 we obtain the following conclusion.

Theorem 4.5 *Let $n \in \{0, 1, \ldots\}$. Then $BC^n(X)$ is of countable type if and only if X is compact.*

5 When $BC^\infty(X)$ is of countable type?

The space $BC^\infty(X) := \bigcap\{BC^n(X) : n \in \{0, 1, \ldots\}\}$, equipped with the locally convex topology induced by the norms $\| \ \|_n$ $(n \in \{0, 1, \ldots\})$, is easily seen to be a Fréchet space. We will see that, contrary to the previous case, precompactness of X is enough to ensure that $BC^\infty(X)$ is of countable type. The reason is the following.

Lemma 5.1 *Each $f \in BC^\infty(X)$ extends uniquely to an $\overline{f} \in BC^\infty(\overline{X})$, where $\overline{X}$ is the closure of X in K. The map $f \mapsto \overline{f}$ is a linear homeomorphism of $BC^\infty(X)$ onto $BC^\infty(\overline{X})$.*

Proof. For $(x_1, \ldots, x_{n+1}, a_1, \ldots, a_{n+1}) \in \bigtriangledown^{2n+2}X$ we have for all $f : X \longrightarrow K$ (see [3] 29.2)

$$\Phi_n f(x_1, \ldots, x_{n+1}) - \Phi_n f(a_1, \ldots, a_{n+1}) = \sum_{j=1}^{n+1} (x_j - a_j)\, \Phi_{n+1} f(a_1, \ldots, a_j, x_j, \ldots, x_{n+1}).$$

As $f \in C^\infty(X)$ we can, for each $n \in \{0, 1, \ldots\}$, extend this by continuity:

$$\overline{\Phi}_n f(x_1, \ldots, x_{n+1}) - \overline{\Phi}_n f(a_1, \ldots, a_{n+1}) = \sum_{j=1}^{n+1} (x_j - a_j)\, \overline{\Phi}_{n+1} f(a_1, \ldots, a_j, x_j, \ldots, x_{n+1}) \tag{3}$$

for all $(x_1, \ldots, x_{n+1})$, $(a_1, \ldots, a_{n+1}) \in X^{n+1}$.

Denoting the canonical norm on K^{n+1} by $\| \ \|_\infty$ we obtain from (3) the following inequality.

$$\left|\overline{\Phi}_n f(u) - \overline{\Phi}_n f(v)\right| \leq \|u - v\|_\infty \|\overline{\Phi}_{n+1} f\| \qquad (u, v \in X^{n+1}).$$

Thus, $\overline{\Phi}_n f$ is Lipschitz, hence uniformly continuous on X^{n+1} and therefore can uniquely be extended to a (bounded) continuous function h_n on $\overline{X}^{n+1}$. By continuity $h_n = \overline{\Phi}_n h_0$ for all $n \in \{0, 1, \ldots\}$. Hence, $h_0 \in BC^\infty(\overline{X})$ and we can take $\overline{f} := h_0$, which proves the first statement. The second one is now immediate.

Theorem 5.2 *$BC^\infty(X)$ is of countable type if and only if X is precompact.*

Proof. Let X be precompact. Then $\overline{X}$ is compact and by 4.5 $BC^n(\overline{X})$ is of countable type for each $n \in \{0, 1, \ldots\}$ and, by 1.1, so are $(BC^\infty(\overline{X}), \| \ \|_n)$ $(n \in \{0, 1, \ldots\})$. So $BC^\infty(\overline{X})$ is of countable type, hence so is $BC^\infty(X)$ by 5.1.

Conversely, suppose X is not precompact. With the same reasoning as in 4.3, we find an infinite set I, a subspace D of $BC^\infty(X)$ and a linear map $D \longrightarrow \ell^\infty(I)$ that is a surjective homeomorphism when we endow D with any of the norms $\| \ \|_n$. Then D, with the topology induced by $BC^\infty(X)$, is linearly homeomorphic to $\ell^\infty(I)$. As $\ell^\infty(I)$ is not of countable type (1.2), neither is $BC^\infty(X)$ by 1.1.

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