Comparison of some notions of C^k-maps in multi-variable non-archimedean analysis

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Abstract

Various definitions of C^k-maps on open subsets of finite-dimensional vector spaces over a complete valued field have been proposed in the literature. We show that the C^k-maps considered by Schikhof and De Smedt coincide with those of Bertram, Glöckner and Neeb. By contrast, Ludkovsky's C^k-maps need not be C^k in the former sense, at least in positive characteristic. We also compare various types of Hölder differentiable maps on finite-dimensional and metrizable spaces.

1 Introduction

Various concepts of C^k-maps on subsets of finite-dimensional vector spaces have been used in the literature on non-archimedean analysis. Schikhof's textbook [18] gave a comprehensive discussion of the single-variable calculus of C^k-maps over a complete ultrametric field $\mathbb{K}$, and suggested a definition of multi-variable C^k-maps (in §84), which was then elaborated by De Smedt [4]. Ludkovsky introduced a notion of C^k-map between open subsets of locally convex spaces over a finite extension $\mathbb{K}$ of $\mathbb{Q}_p$ (see [13, Definition 2.3] and [14, Part I, Definition 2.3] for the case of Banach spaces, [14, Part II, Remark 4.4] for the general case). Recently, Bertram, Glöckner and Neeb [2] introduced a notion of C^k-map between open subsets of arbitrary (Hausdorff) topological vector spaces over a (non-discrete) topological field $\mathbb{K}$.

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While the definition of C^k-maps by Schikhof and De Smedt is based on the existence of continuous extensions to certain partial difference quotients, the definition of Bertram et al. and Ludkovsky's definition are based on continuous extendibility of certain iterated directional difference quotients. The primary goal of this paper is to compare these notions of C^k-maps, and some related concepts. To describe our main results, let E and F be topological vector spaces over a topological field $\mathbb{K}$, $k \in \mathbb{N}_0$, and $f\colon U \to F$ be a map on an open set $U \subseteq E$. We start with a special case of Theorem 3.1, which generalizes a result for functions of a single variable obtained in [2, Proposition 6.9].

Theorem A. *If $E = \mathbb{K}^d$ for some $d \in \mathbb{N}$, then f is C^k in the sense of Bertram et al. if and only if f is C^k in the sense of Schikhof and De Smedt.*

If $\mathbb{K}$ is a valued field, then variants of the two approaches just discussed can be used to define k times Hölder differentiable maps with Hölder exponent $\sigma \in \,]0,1]$ ($C^{k,\sigma}$-maps, for short). As a special case of Theorem 3.1, we have:

Theorem B. *If $E = \mathbb{K}^d$ for some $d \in \mathbb{N}$, then f is $C^{k,\sigma}$ in the sense of Bertram et al. if and only if f is $C^{k,\sigma}$ in the sense of Schikhof and De Smedt.*

By contrast, the mappings introduced by S. V. Ludkovsky differ from the preceding ones, if his definition is used for fields of positive characteristic. We show by example (see Theorem 4.7):

Theorem C. *For each local field $\mathbb{K}$ of positive characteristic, there exists a map $f\colon \mathbb{O} \to \mathbb{K}$ on $\mathbb{O} := \{z \in \mathbb{K}\colon |z| \leq 1\}$ which is C^∞ in Ludkovsky's sense, but not C^2 in Schikhof's sense.*

We also provide alternative characterizations of $C^{k,\sigma}$-maps (in the sense of Bertram et al.) on open subsets of metrizable spaces, for $\sigma \in \,]0,1]$. Theorem 5.1 establishes the following characterization. It is our technically most difficult result, and its proof relies heavily on a tool of convenient differential calculus [12], which has been adapted to non-archimedian analysis in [8].

Theorem D. *If $\mathbb{K}$ is $\mathbb{R}$ or an ultrametric field and E is metrizable, then f is $C^{k,\sigma}$ if and only if $f \circ \gamma\colon \mathbb{K}^{k+1} \to F$ is $C^{k,\sigma}$, for each smooth map $\gamma\colon \mathbb{K}^{k+1} \to U$.*

Note that neither E nor F need to be locally convex here. An analogous characterization of C^k-maps was given earlier in [2, Theorem 12.4]. As a consequence of Theorem D, the simplified description of $C^{k,\sigma}$-maps on finite-dimensional spaces via partial difference quotients can also be used to deal with Hölder differentiable maps on metrizable spaces. This may be useful on the way towards ultrametric (and non-locally convex) analogues of Boman's Theorem (cf. [3, Theorem 2] and [12, Theorem 12.8]), which characterizes $C^{k,\sigma}$-maps on open subsets of finite-dimensional (or metrizable) real locally convex spaces as those maps which are $C^{k,\sigma}$ along smooth curves. While the preceding result provided a reduction to finite-dimensional domains, our next result (Theorem 6.6) reduces to the case of a one-dimensional range.

Theorem E. *If $\mathbb{K} \neq \mathbb{C}$ is locally compact, E is metrizable and F is locally convex and Mackey complete, then f is $C^{k,\sigma}$ if and only if f is weakly $C^{k,\sigma}$, i.e., $\lambda \circ f\colon U \to \mathbb{K}$ is $C^{k,\sigma}$, for each continuous linear functional $\lambda\colon F \to \mathbb{K}$.*

We remark that yet another approach to C^k-maps of several variables has been proposed by De Smedt in [5]. The C^1-maps in the sense of [5] coincide with the strictly differentiable maps defined in [9]. It is known that strictly differentiable maps on open subsets of $\mathbb{K}^n$ coincide with C^1-maps in the sense of Bertram et al., for each complete valued field $\mathbb{K}$ (see [9, Proposition C.1]).

The present studies are part of a larger project, the goal of which is to transfer the main ideas of infinite-dimensional real differential calculus and non-linear functional analysis into non-archimedian analysis (and analysis over arbitrary topological fields). A survey of the results obtained so far, with applications to Lie groups and dynamical systems, can be found in [7].

2 Main concepts, terminology and notation

In this section, we compile terminology and notation concerning differential calculus over topological fields, together with basic facts. Most of these facts are easy to take on faith, and we recommend to skip the proofs on a first reading. If desired, the proofs can be looked up in Appendix A.

All topological fields occurring in this article are assumed Hausdorff and non-discrete; all topological vector spaces are assumed Hausdorff. Given a field $\mathbb{K}$, as usual we write $\mathbb{K}^\times := \mathbb{K} \setminus \{0\}$ for its group of invertible elements. A valued field is a field $\mathbb{K}$, equipped with an absolute value $|.|\colon \mathbb{K} \to [0,\infty[$ which defines a non-discrete topology on $\mathbb{K}$. If $|.|$ satisfies the ultrametric inequality, we call $(\mathbb{K}, |.|)$ an *ultrametric field*. Totally disconnected, locally compact topological fields will be referred to as *local fields*. It is well known that each locally compact field admits an absolute value defining its topology. We fix such an absolute value, and thus consider $\mathbb{K}$ as a valued field. On $\mathbb{R}$ and $\mathbb{C}$, we shall always use the usual absolute value. We write $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

C^k-maps in the sense of Bertram, Glöckner and Neeb

We recall the approach to C^k-maps between open subsets of topological vector spaces over a topological field developed in [2] (and its extension to maps on non-open domains from [9]). More information concerning this approach can be found in the survey [7]. Cf. [1] for applications of the corresponding differential calculus over topological rings in differential geometry. We are mostly interested in mappings on open domains, but some results will hold more generally.

Let E and F be topological vector spaces over a topological field $\mathbb{K}$ and $f\colon U \to F$ be a map, defined on a subset $U \subseteq E$ with dense interior. Then the directional difference quotient

$$f^{]1[}(x,y,t) \ := \ \frac{f(x+ty)-f(x)}{t}$$

makes sense for all (x, y, t) in the subset

$$U^{]1[} \ := \ \{(x,y,t) \in U \times E \times \mathbb{K}^\times \colon x+ty \in U\}$$

of $E \times E \times \mathbb{K}$. To define directional derivatives, we need to allow also the value $t = 0$. Hence, we consider

$$U^{[1]} := \{(x,y,t) \in U \times E \times \mathbb{K} \colon x + ty \in U\} .$$

Then $U^{[1]} = U^{]1[} \cup (U \times E \times \{0\})$, as a disjoint union. If U is open, then $U^{[1]}$ is an open subset of the topological $\mathbb{K}$-vector space $E^{[1]} = E \times E \times \mathbb{K}$. In the general case, $U^{[1]} \subseteq E^{[1]}$ has dense interior. Recursively, we define $U^{[k]} := (U^{[1]})^{[k-1]}$ and $U^{]k[} := (U^{]1[})^{]k-1[}$ for $2 \leq k \in \mathbb{N}$. Then $U^{]k[}$ is dense in $U^{[k]}$ (see [9, Remark 1.6]).

Definition 2.1. The map $f \colon U \to F$ is called C^1_{BGN} if f is continuous (i.e., C^0, or C^0_{BGN}), and there exists a continuous map $f^{[1]} \colon U^{[1]} \to F$ extending $f^{]1[} \colon U^{]1[} \to F$. Given $k \in \mathbb{N}$ with $k \geq 2$, we say that f is C^k_{BGN} if f is C^1_{BGN} and $f^{[1]} \colon U^{[1]} \to F$ is C^{k-1}_{BGN}. We define $f^{[k]} := (f^{[1]})^{[k-1]} \colon U^{[k]} \to F$ in this case. The map f is C^∞_{BGN} if it is C^k_{BGN} for all $k \in \mathbb{N}_0$.

Since $U^{]1[}$ is dense in $U^{[1]}$, $f^{[1]}$ is unique if it exists (and likewise each $f^{[k]}$).

2.2. For example, every continuous linear map $\lambda \colon E \to F$ is C^∞_{BGN} with $\lambda^{[1]}(x,y,t) = \lambda(y)$ for all $(x,y,t) \in E \times E \times \mathbb{K}$ (whence also $\lambda^{[1]}$ is continuous linear). Also each continuous multilinear map is C^∞_{BGN} (see [2]).

2.3. (Chain Rule). If E, F and H are topological $\mathbb{K}$-vector spaces, $U \subseteq E$ and $V \subseteq F$ are subsets with dense interior, and $f \colon U \to V \subseteq F$, $g \colon V \to H$ are C^k_{BGN}-maps, then also the composition $g \circ f \colon U \to H$, $x \mapsto g(f(x))$ is C^k_{BGN}. If $k \geq 1$, we have $(\widehat{T}f)(x,y,t) := (f(x), f^{[1]}(x,y,t), t) \in V^{[1]}$ for all $(x,y,t) \in U^{[1]}$, and

$$(g \circ f)^{[1]}(x,y,t) = g^{[1]}(f(x), f^{[1]}(x,y,t), t) . \tag{1}$$

Thus $(g \circ f)^{[1]} = g^{[1]} \circ \widehat{T}f$ with $\widehat{T}f \colon U^{[1]} \to V^{[1]}$ (see [2, Proposition 3.1 and Proposition 4.5], also [9, § 1]).

We recall from [2, Lemma 4.9] and [9, § 1] that being C^k is a local property.

Lemma 2.4. *Let E and F be topological $\mathbb{K}$-vector spaces, and $f \colon U \to F$ be a map, defined on a subset $U \subseteq E$ with dense interior. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. If there is an open cover $(U_i)_{i \in I}$ of U such that $f|_{U_i} \colon U_i \to F$ is C^k_{BGN} for each $i \in I$, then f is C^k_{BGN}.* ∎

C^k-maps in the sense of Schikhof and De Smedt

In this section, we give a definition of C^k-maps of several variables based on continuous extensions to certain partial difference quotient maps, which generalizes special cases considered by Schikhof [18, § 84] and De Smedt [4]. Our notation differs from the one used in [4] and [18], because we find it more convenient to use multi-indices in higher dimensions.

2.5. Until Remark 2.16, let $\mathbb{K}$ be a topological field, $d \in \mathbb{N}$, $U \subseteq \mathbb{K}^d$ be an open subset (where $\mathbb{K}^d$ is equipped with the product topology), and F be a topological $\mathbb{K}$-vector space. As usual, for $i \in \{1, \ldots, d\}$ we set $e_i := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{K}^d$, with i-th entry 1.

2.6. As usual, given a "multi-index" $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, we write $|\alpha| := \sum_{i=1}^d \alpha_i$. The definition of a C^k-map $f\colon U \to F$ in the sense of Schikhof and De Smedt will involve a certain continuous extension $f^{<\alpha>}$ of a partial difference quotient map $f^{>\alpha<}$, for each multi-index $\alpha \in \mathbb{N}_0^d$ such that $|\alpha| \leq k$. It is convenient to define the domains $U^{<\alpha>}$ and $U^{>\alpha<}$ of these mappings first. They will be subsets of $\mathbb{K}^{d+|\alpha|}$. It is useful to write elements $x \in \mathbb{K}^{d+|\alpha|}$ in the form $x = (x^{(1)}, x^{(2)}, \ldots, x^{(d)})$, where $x^{(i)} \in \mathbb{K}^{1+\alpha_i}$ for $i \in \{1, \ldots, d\}$. We write $x^{(i)} = (x_0^{(i)}, x_1^{(i)}, \ldots, x_{\alpha_i}^{(i)})$ with $x_j^{(i)} \in \mathbb{K}$ for $j \in \{0, \ldots, \alpha_i\}$.

2.7. Given $\alpha \in \mathbb{N}_0^d$, we now define $U^{<\alpha>}$ as the set of all $x \in \mathbb{K}^{d+|\alpha|}$ such that, for all $i_1 \in \{0, 1, \ldots, \alpha_1\}, \ldots, i_d \in \{0, 1, \ldots, \alpha_d\}$, we have

$$(x_{i_1}^{(1)}, \ldots, x_{i_d}^{(d)}) \in U\,.$$

We let $U^{>\alpha<}$ be the set of all $x \in U^{<\alpha>}$ such that, for all $i \in \{1, \ldots, d\}$ and $0 \leq j < k \leq \alpha_i$, we have $x_j^{(i)} \neq x_k^{(i)}$. It is easy to see that $U^{<\alpha>}$ and $U^{>\alpha<}$ are open in $\mathbb{K}^{d+|\alpha|}$ and $U^{>\alpha<}$ is dense in $U^{<\alpha>}$.

Example 2.8. If $U = U_1 \times \cdots \times U_d$ with open sets $U_i \subseteq \mathbb{K}$, then simply

$$U^{<\alpha>} \;=\; U_1^{1+\alpha_1} \times U_2^{1+\alpha_2} \times \cdots \times U_d^{1+\alpha_d}\,. \tag{2}$$

Only this case (in fact only special cases thereof) was considered in [4] and [18].

Remark 2.9. A simple induction on $|\alpha|$ shows that the sets $U^{<\alpha>}$ can be defined alternatively by recursion on $|\alpha|$, as follows: Set $U^{<0>} := U$. Given $\alpha \in \mathbb{N}_0^d$ such that $|\alpha| \geq 1$, pick $\beta \in \mathbb{N}_0^d$ such that $\alpha = \beta + e_i$ for some $i \in \{1, \ldots, d\}$. Then $U^{<\alpha>}$ is the set of all elements $x \in \mathbb{K}^{d+|\alpha|}$ such that

$$(x^{(1)}, \ldots, x^{(i-1)}, x_0^{(i)}, x_1^{(i)}, \ldots, x_{\alpha_i-1}^{(i)}, x^{(i+1)}, \ldots, x^{(d)}) \in U^{<\beta>}$$

holds as well as

$$(x^{(1)}, \ldots, x^{(i-1)}, x_{\alpha_i}^{(i)}, x_1^{(i)}, \ldots, x_{\alpha_i-1}^{(i)}, x^{(i+1)}, \ldots, x^{(d)}) \;\in\; U^{<\beta>}\,.$$

We now define certain mappings $f^{>\alpha<}\colon U^{>\alpha<} \to F$ and show afterwards that they can be interpreted as partial difference quotient maps.

Definition 2.10. We set $f^{>0<} := f$. Given a multi-index $\alpha \in \mathbb{N}_0^d$ such that $|\alpha| \geq 1$, we define $f^{>\alpha<}(x)$ for $x \in U^{>\alpha<}$ as the sum

$$\sum_{j_1=0}^{\alpha_1} \cdots \sum_{j_d=0}^{\alpha_d} \left(\prod_{k_1 \neq j_1} \frac{1}{x_{j_1}^{(1)} - x_{k_1}^{(1)}} \cdot \ldots \cdot \prod_{k_d \neq j_d} \frac{1}{x_{j_d}^{(d)} - x_{k_d}^{(d)}} \right) f(x_{j_1}^{(1)}, \ldots, x_{j_d}^{(d)}), \tag{3}$$

using the notational conventions from **2.6**. The products are taken over all $k_\ell \in \{0, \ldots, \alpha_\ell\}$ such that $k_\ell \neq j_\ell$, for $\ell \in \{1, \ldots, d\}$.

The map $f^{>\alpha<}$ has important symmetry properties.

Lemma 2.11. *Assume that* $\alpha \in \mathbb{N}_0^d$, $i \in \{1, \ldots, d\}$ *and* π *is a permutation of* $\{0, 1, \ldots, \alpha_i\}$. *Then* $(x^{(1)}, \ldots, x^{(i-1)}, x^{(i)}_{\pi(0)}, \ldots, x^{(i)}_{\pi(\alpha_i)}, x^{(i+1)}, \ldots, x^{(d)}) \in U^{>\alpha<}$ *for each* $x \in U^{>\alpha<}$, *and*

$$f^{>\alpha<}(x^{(1)}, \ldots, x^{(i-1)}, x^{(i)}_{\pi(0)}, \ldots, x^{(i)}_{\pi(\alpha_i)}, x^{(i+1)}, \ldots, x^{(d)}) \;=\; f^{>\alpha<}(x)\,. \tag{4}$$

The next lemma shows that $f^{>\alpha<}$ can indeed be interpreted as a partial difference quotient map.

Lemma 2.12. *For all* $i \in \{1, \ldots, d\}$ *and* $x \in U^{>e_i<}$, *the element* $f^{>e_i<}(x)$ *is given by*

$$\frac{f(x^{(1)}, \ldots, x^{(i-1)}, x^{(i)}_0, x^{(i+1)}, \ldots, x^{(d)}) - f(x^{(1)}, \ldots, x^{(i-1)}, x^{(i)}_1, x^{(i+1)}, \ldots, x^{(d)})}{x^{(i)}_0 - x^{(i)}_1}.$$

If $\alpha \in \mathbb{N}_0^d$ *such that* $|\alpha| \geq 2$, *let* $\beta \in \mathbb{N}_0^d$ *be a multi-index such that* $\alpha = \beta + e_i$ *for some* $i \in \{1, \ldots, d\}$. *Then* $f^{>\alpha<}(x)$ *is given by*

$$\frac{1}{x^{(i)}_0 - x^{(i)}_{\alpha_i}} \cdot \Big(f^{>\beta<}(x^{(1)}, \ldots, x^{(i-1)}, x^{(i)}_0, x^{(i)}_1, \ldots, x^{(i)}_{\alpha_i - 1}, x^{(i+1)}, \ldots, x^{(d)}) \\ - f^{>\beta<}(x^{(1)}, \ldots, x^{(i-1)}, x^{(i)}_{\alpha_i}, x^{(i)}_1, \ldots, x^{(i)}_{\alpha_i - 1}, x^{(i+1)}, \ldots, x^{(d)}) \Big) \tag{5}$$

for all $x \in U^{>\alpha<}$.

Definition 2.13. We say that f is C^0_{SDS} if it is continuous, and define $f^{<0>} := f$ in this case. Recursively, given an integer $k \geq 1$ we say that f is C^k_{SDS} if f is C^{k-1}_{SDS} and, for each multi-index $\alpha \in \mathbb{N}_0^d$ such that $|\alpha| = k$, there exists a continuous map $f^{<\alpha>} \colon U^{<\alpha>} \to F$ such that $f^{<\alpha>}|_{U^{>\alpha<}} = f^{>\alpha<}$. As usual, f is called C^∞_{SDS} if f is C^k_{SDS} for each $k \in \mathbb{N}_0$.

Since $U^{>\alpha<}$ is dense in $U^{<\alpha>}$, the continuous extension $f^{<\alpha>}$ of $f^{>\alpha<}$ is unique whenever it exists. We readily deduce from Lemma 2.11:

Lemma 2.14. *Let* f *be a* C^k*-mapping for some* $k \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^d$ *with* $|\alpha| = k$, $i \in \{1, \ldots, d\}$, *and* π *be a permutation of* $\{0, 1, \ldots, \alpha_i\}$. *Then*

$$(x^{(1)}, \ldots, x^{(i-1)}, x^{(i)}_{\pi(0)}, \ldots, x^{(i)}_{\pi(\alpha_i)}, x^{(i+1)}, \ldots, x^{(d)}) \;\in\; U^{<\alpha>} \tag{6}$$

for each $x \in U^{<\alpha>}$, *and*

$$f^{<\alpha>}(x^{(1)}, \ldots, x^{(i-1)}, x^{(i)}_{\pi(0)}, \ldots, x^{(i)}_{\pi(\alpha_i)}, x^{(i+1)}, \ldots, x^{(d)}) \;=\; f^{<\alpha>}(x)\,. \tag{7}$$

The following variant of Lemma 2.12 is available for $f^{<\alpha>}$.

Lemma 2.15. *Let* f *be a* C^k_{SDS}*-map for an integer* $k \geq 2$, $\alpha \in \mathbb{N}_0^d$ *such that* $|\alpha| = k$, *and* $\beta \in \mathbb{N}_0^d$ *such that* $\alpha = \beta + e_i$ *for some* $i \in \{1, \ldots, d\}$. *Then* $f^{<\alpha>}(x)$ *is given by*

$$\frac{1}{x^{(i)}_0 - x^{(i)}_{\alpha_i}} \cdot \Big(f^{<\beta>}(x^{(1)}, \ldots, x^{(i-1)}, x^{(i)}_0, x^{(i)}_1, \ldots, x^{(i)}_{\alpha_i - 1}, x^{(i+1)}, \ldots, x^{(d)}) \\ - f^{<\beta>}(x^{(1)}, \ldots, x^{(i-1)}, x^{(i)}_{\alpha_i}, x^{(i)}_1, \ldots, x^{(i)}_{\alpha_i - 1}, x^{(i+1)}, \ldots, x^{(d)}) \Big) \tag{8}$$

for all $x \in U^{<\alpha>}$ *such that* $x^{(i)}_0 \neq x^{(i)}_{\alpha_i}$.

Remark 2.16. If $U \subseteq \mathbb{K}^d$ is a subset (possibly with empty interior) of the form $U = U_1 \times \cdots \times U_d$, where $U_i \subseteq \mathbb{K}$ is a non-empty subset without isolated points for $i \in \{1, \ldots, d\}$, then Definition 2.13 can be used just as well to define C^k_{SDS}-maps $f \colon U \to F$.

If $d = 1$, we also write $f^{<j>}$ in place of $f^{<je_1>}$, as in [2, § 6].

Seminorms and gauges

Gauges on topological vector spaces over valued fields were introduced in [9] as a substitute for continuous seminorms when dealing with a general topological vector space, the topology of which need not come from a family of continuous seminorms (cf. [10, § 6.3] for the real case). We only recall some essentials here; see [9] for further information.

Definition 2.17. Let E be a topological vector space over a valued field $(\mathbb{K}, |.|)$. A *gauge on* E is a map $q \colon E \to [0, \infty[$ (also written $\|.\|_q := q$) such that $q(tx) = |t|q(x)$ for all $t \in \mathbb{K}$ and $x \in E$, and which is continuous at 0. Thus $B^q_r(0)$ is a zero-neighbourhood for each $r > 0$, where $B^q_r(x) := \{y \in E \colon \|y - x\|_q < r\}$ for all $x \in E$ and $r > 0$. We also define $\overline{B}^q_r(x) := \{y \in E \colon \|y - x\|_q \leq r\}$. If $(E, \|.\|)$ is a normed space, we relax notation and write $B^E_r(x) := B^{\|\cdot\|}_r(x)$.

In [9], only upper semicontinuous gauges q were considered, i.e., it was required that $B^q_r(0)$ is an *open* 0-neighbourhood, for each $r > 0$.

Remark 2.18. Typical examples of gauges are Minkowski functionals μ_U of balanced, open 0-neighbourhoods U in a topological vector space E over a valued field $\mathbb{K}$; these are upper semicontinuous (see [9, Remark 1.21]). Here $U \subseteq E$ is called *balanced* if $tU \subseteq U$ for all $t \in \mathbb{K}$ such that $|t| \leq 1$. The Minkowski functional is $\mu_U \colon E \to [0, \infty[$, $x \mapsto \inf\{|t| \colon t \in \mathbb{K}^\times \text{ with } x \in tU\}$.

Remark 2.19. Note that gauges need not satisfy the triangle inequality. But we still have a certain substitute: Given a gauge $q \colon E \to [0, \infty[$, there always exists a gauge $p \colon E \to [0, \infty[$ such that

$$\|x + y\|_q \;\leq\; \|x\|_p + \|y\|_p \quad \text{for all } x, y \in E \tag{9}$$

(cf. [9, Lemma 1.29]). We shall refer to (9) as the *fake triangle inequality.*

As in the case of continuous seminorms, it frequently suffices to consider a sufficiently large set of gauges:

Definition 2.20. A set Γ of gauges on a topological $\mathbb{K}$-vector space E is called a *fundamental system of gauges* if each 0-neighbourhood in E contains some finite intersection of balls of the form $B^q_r(0)$, with $q \in \Gamma$ and $r > 0$.

Cf. [9, Lemma 1.24] for the next lemma, which is useful to determine fundamental systems of gauges.

Lemma 2.21. *Let $p, q\colon E \to [0,\infty[$ be gauges on a topological vector space E over a valued field $\mathbb{K}$. If there exist $r, s > 0$ such that $B^q_s(0) \subseteq B^p_r(0)$, then*

$$p \ \leq rs^{-1}|a|^{-1}q\,,$$

for each $a \in \mathbb{K}^\times$ such that $|a| < 1$. In particular, $p \leq Cq$ for some $C > 0$. ∎

Remark 2.22. Combining Remark 2.18 and Lemma 2.21, it is easy to see that upper semicontinuous gauges form a fundamental system of gauges, for each topological vector space over a valued field (cf. also [9, Remark 1.21]). In the real case, continuous gauges form a fundamental system (cf. [10, § 6.4]).

Examples 2.23. Given $r \in \,]0,1]$, a gauge $q\colon E \to [0,\infty[$ is called an r-seminorm if $q(x+y)^r \leq q(x)^r + q(y)^r$ for all $x, y \in E$. If, furthermore, $q(x) = 0$ if and only if $x = 0$, then q is called an r-norm (cf. [10, § 6.3] for the real case). For examples of r-normed spaces over $\mathbb{R}$ and more general non-locally convex real topological vector spaces, the reader is referred to [10, § 6.10] and [11]. For $\mathbb{K}$ a valued field, the simplest examples are the spaces $\ell^p(\mathbb{K})$ of all $x = (x_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ such that $\|x\|_p := \sqrt[p]{\sum_{n=1}^\infty |x_n|^p} < \infty$, for $p \in \,]0,1[$. Then $\|.\|_p$ is a p-norm on $\ell^p(\mathbb{K})$ defining a Hausdorff vector topology on this space (and thus $\{\|.\|_p\}$ is a fundamental system of gauges).

Bounded sets and bounded maps

Let E be a topological vector space over a topological field $\mathbb{K}$. Recall that a subset $B \subseteq E$ is called *bounded* if, for each 0-neighbourhood $U \subseteq E$, there exists a 0-neighbourhood $V \subseteq \mathbb{K}$ such that $VB \subseteq U$. If $\mathbb{K}$ is a valued field, we can test boundedness using gauges.

Lemma 2.24. *Let E be a topological vector space over a valued field $\mathbb{K}$. Then a set $B \subseteq E$ is bounded if and only if $q(B) \subseteq \mathbb{R}$ is bounded, for each gauge q on E.*

It suffices to show $\sup \|B\|_q < \infty$ for q in a fundamental system of gauges. In Section 5, C^k-maps with bounded difference quotients will play a vital role.

Definition 2.25. Let E be a topological vector space over a topological field $\mathbb{K}$.

(a) If X is a topological space, then $BC(X,E)$ denotes the set of all continuous maps $\gamma\colon X \to E$ whose image $\gamma(X)$ is bounded in E. Then $BC(X,E)$ is a vector subspace of E^X.

(b) If $k \in \mathbb{N}_0 \cup \{\infty\}$ and $U \subseteq \mathbb{K}$ is a subset without isolated points, we let $BC^k(U,E)$ be the space of all C^k_{SDS}-mappings $\gamma\colon U \to E$ such that $\gamma^{<j>} \in BC(U^{j+1},E)$ for all $j \in \mathbb{N}_0$ such that $j \leq k$.

We mention that $BC^k(U,E)$ can be made a topological vector space [8, Definition 1.2], but we shall not use this topology. If $\mathbb{K}$ is a valued field, $j \in \mathbb{N}_0$ with $j \leq k$ and q a gauge on E, we define

$$\|\gamma^{<j>}\|_{q,\infty} \ := \ \sup\{\|\gamma^{<j>}(x)\|_q \colon x \in U^{j+1}\} \quad \text{for } \gamma \in BC^k(U,E). \tag{10}$$

Hölder continuity

Using gauges, we now define a version of Hölder continuous maps and record their basic properties.

Definition 2.26. Let E and F be topological vector spaces over a valued field $\mathbb{K}$, and $U \subseteq E$ be a subset. A map $f\colon U \to F$ is called *Hölder continuous of exponent* $\sigma \in \,]0,\infty[$ (or $C^{0,\sigma}$) if, for every $x_0 \in U$ and gauge q on F, there exists a gauge p on E and a neighbourhood $V \subseteq U$ of x_0 such that

$$\|f(y) - f(x)\|_q \;\leq\; \Big(\|y - x\|_p\Big)^\sigma \quad \text{for all } \; x, y \in V. \tag{11}$$

$C^{0,1}$-maps are also called *Lipschitz continuous.*

We remind the reader that Hölder exponents $\sigma > 1$ are meaningful in non-archimedian analysis (see [18, Exercise 26.B] for an instructive example). We are mostly interested in Hölder exponents $\sigma \in \,]0,1]$, but some of the results are valid also for $\sigma > 1$.

Lemma 2.27. *Let E, F and H be topological vector spaces over a valued field $\mathbb{K}$, $U \subseteq E$ and $V \subseteq F$ be subsets, $f\colon U \to V \subseteq F$ and $g\colon V \to H$ be maps, and $\sigma, \tau > 0$. Then the following holds:*

(a) *If f is $C^{0,\sigma}$, then f is continuous.*

(b) *If f is $C^{0,\sigma}$ and $\sigma \geq \tau$, then f is also $C^{0,\tau}$.*

(c) *If f is $C^{0,\sigma}$ and g is $C^{0,\tau}$, then $g \circ f$ is $C^{0,\sigma\cdot\tau}$.*

(d) *If U has dense interior and f is C^1_{BGN}, then f is Lipschitz continuous.*

In connection with Part (d) of the preceding lemma, note that $\mathrm{id}\colon \mathbb{K} \to \mathbb{K}$, $x \mapsto x$ is C^∞_{BGN} but not $C^{0,\sigma}$ for any $\sigma > 1$.

If f is not Hölder continuous, then pairs of points with pathological behaviour can always be chosen in a given dense set. This will become essential later.

Lemma 2.28. *Let E and F be topological vector spaces over a valued field $\mathbb{K}$, $f\colon U \to F$ be a continuous mapping on a subset $U \subseteq E$, $D \subseteq U$ be a dense subset, and $\sigma > 0$. If f is not $C^{0,\sigma}$, then there exists $x_0 \in U$ and a gauge q on F such that, for each neighbourhood $V \subseteq U$ of x_0 and gauge p on E, there exist $x, y \in V \cap D$ such that $\|f(x) - f(y)\|_q > (\|x - y\|_p)^\sigma$.*

Two approaches to Hölder differentiable maps

We define k times Hölder differentiable maps and record some properties.

Definition 2.29. Let $\mathbb{K}$ be a valued field, E and F be topological $\mathbb{K}$-vector spaces, $U \subseteq E$ be a subset with dense interior, and $f\colon U \to F$ be a mapping. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\sigma > 0$. We say that f is $C^{k,\sigma}_{BGN}$ if f is C^k_{BGN} and $f^{[j]}\colon U^{[j]} \to F$ is $C^{0,\sigma}$ for all $j \in \mathbb{N}_0$ such that $j \leq k$ (where $f^{[0]} := f$).

Remark 2.30. Note that, if $\sigma \in]0,1]$, then $f^{[j]}$ is C^1_{BGN} for $j < k$ and hence automatically $C^{0,\sigma}$, by Lemma 2.27 (b) and (d). In particular, if $\sigma \in]0,1]$, then f is $C^{\infty,\sigma}_{BGN}$ if and only if f is C^∞_{BGN}. And if k is finite, then only $f^{[k]}$ requires attention.

Remark 2.31. k times Lipschitz differentiable mappings ($C^{k,1}$-maps) form a particularly nice class of maps (see [9]). Notably, [9, Theorem 5.2] provides an implicit function theorem for $C^{k,1}$-maps from arbitrary topological vector spaces to Banach spaces, for each valued field and $k \in \mathbb{N} \cup \{\infty\}$.

We need some basic information on mappings into direct products.

Lemma 2.32. *Let E be a topological $\mathbb{K}$-vector space over a valued field $\mathbb{K}$, $(F_i)_{i\in I}$ be a family of topological $\mathbb{K}$-vector spaces and $f\colon U \to F$ be a map into $F := \prod_{i\in I} F_i$, defined on a non-empty subset $U \subseteq E$ with dense interior. Let $k \in \mathbb{N}_0$ and $\sigma > 0$; for $i \in I$, let $\mathrm{pr}_i\colon F \to F_i$ be the projection. Then f is $C^{k,\sigma}_{BGN}$ if and only if each of its components $f_i := \mathrm{pr}_i \circ f\colon U \to F_i$ is $C^{k,\sigma}_{BGN}$.*

The Chain Rule is available in the following form.

Lemma 2.33. *Let $\mathbb{K}$ be a valued field, E, F and H be topological $\mathbb{K}$-vector spaces, $U \subseteq E$ and $V \subseteq F$ be subsets with dense interior, $\sigma \in]0,1]$, $\tau > 0$, $f\colon U \to V$ be $C^{k,\sigma}_{BGN}$, and $g\colon V \to H$ be $C^{k,\tau}_{BGN}$. Then $g \circ f\colon U \to H$ is $C^{k,\sigma\cdot\tau}_{BGN}$.*

The following variant even holds if $\sigma > 1$:

Lemma 2.34. *Let $\mathbb{K}$ be a valued field, E, F and H be topological $\mathbb{K}$-vector spaces, $\lambda\colon F \to H$ be continuous linear, $U \subseteq E$ be a subset with dense interior, $\sigma > 0$ and $f\colon U \to F$ be a $C^{k,\sigma}_{BGN}$-map, where $k \in \mathbb{N}_0$. Then $\lambda \circ f\colon U \to H$ is $C^{k,\sigma}_{BGN}$, and $(\lambda \circ f)^{[k]} = \lambda \circ f^{[k]}$.*

Lemma 2.35. *Let E and F be topological vector spaces over a valued field $\mathbb{K}$, and $f\colon U \to F$ be a mapping, defined on a subset $U \subseteq E$ with dense interior. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\sigma \in]0,1]$. If there exists an open cover $(U_i)_{i\in I}$ of U such that $f|_{U_i}\colon U_i \to F$ is $C^{k,\sigma}_{BGN}$ for each $i \in I$, then f is $C^{k,\sigma}_{BGN}$.*

Hölder differentiable maps can also be defined using the approach of Schikhof and De Smedt.

Definition 2.36. Let $\mathbb{K}$ be a valued field, $d \in \mathbb{N}_0$, F be a topological $\mathbb{K}$-vector space and $f\colon U \to F$ be a mapping, where $U \subseteq \mathbb{K}^d$ is open or $U = U_1 \times \cdots \times U_d$ for certain sets $U_1, \ldots, U_d \subseteq \mathbb{K}$ without isolated points. Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\sigma > 0$. We say that f is $C^{k,\sigma}_{SDS}$ if f is C^k_{SDS} and $f^{<\alpha>}\colon U^{<\alpha>} \to F$ is $C^{0,\sigma}$ for all $\alpha \in \mathbb{N}_0^d$ such that $|\alpha| \leq k$ (where $f^{<0>} := f$).

We mention that for maps from open sets into real or complex locally convex spaces, a simpler description of $C^{k,\sigma}_{BGN}$-maps is available.

Theorem 2.37. *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, F be a locally convex topological $\mathbb{K}$-vector space, $k \in \mathbb{N}_0 \cup \{\infty\}$, $\sigma \in]0,1]$ and $f\colon U \to F$ be a map, defined on an open subset U of a topological $\mathbb{K}$-vector space E. Then f is $C^{k,\sigma}_{BGN}$ if and only if f is $C^{0,\sigma}$, the iterated (real, resp., complex) directional derivatives*

$$d^j f(x, v_1, \ldots, v_j) := (D_{v_1} \cdots D_{v_j} f)(x)$$

exist for all $j \in \mathbb{N}$ such that $j \leq k$, $x \in U$ and $v_1, \ldots, v_j \in E$, and furthermore all of the maps $d^j f\colon U \times E^j \to F$ so obtained are $C^{0,\sigma}$. ■

The proof of Theorem 2.37 can be found in Appendix C of the preprint version of this article.

3 C^k_{BGN}-maps and C^k_{SDS}-maps coincide

In this section, we show that the approach of Bertram, Glöckner and Neeb and the approach of Schikhof and De Smedt give rise to the same classes of C^k-maps and $C^{k,\sigma}$-maps on open domains (and more generally). Throughout the section, $\mathbb{K}$ is a topological field, $d \in \mathbb{N}$ and $f\colon U \to F$ a map to a topological $\mathbb{K}$-vector space F, where $U \subseteq \mathbb{K}^d$ is open or of the form $U = U_1 \times \cdots \times U_d$ for certain sets $U_1, \ldots, U_d \subseteq \mathbb{K}$ with dense interior.

Theorem 3.1. *The following holds for each $k \in \mathbb{N}_0 \cup \{\infty\}$:*

(a) *f is C^k_{SDS} if and only if f is C^k_{BGN}.*

(b) *If $\mathbb{K}$ is a valued field and $\sigma \in \,]0,1]$, then f is $C^{k,\sigma}_{SDS}$ if and only if f is $C^{k,\sigma}_{BGN}$.*

Various lemmas are useful for the proof of Theorem 3.1.

Lemma 3.2. *Let $k \in \mathbb{N}_0$.*

(a) *If $f\colon \mathbb{K}^d \supseteq U \to F$ is C^1_{SDS} and $f^{<e_i>}$ is C^k_{SDS} for each $i \in \{1, \ldots, d\}$, then f is C^{k+1}_{SDS}.*

(b) *Let $\mathbb{K}$ be a valued field and $\sigma > 0$. If $f\colon \mathbb{K}^d \supseteq U \to F$ is $C^{1,\sigma}_{SDS}$ and $f^{<e_i>}$ is $C^{k,\sigma}_{SDS}$ for each $i \in \{1, \ldots, d\}$, then f is $C^{k+1,\sigma}_{SDS}$.*

Proof. Given $\alpha \in \mathbb{N}_0^d$ such that $1 \le |\alpha| \le k+1$, there is $i \in \{1, \ldots, d\}$ such that $\alpha_i > 0$. Then $\beta := \alpha - e_i \in \mathbb{N}_0^d$ and $|\beta| = |\alpha| - 1 \le k$. Write $\beta = (\beta_1, \ldots, \beta_d)$ and set $\beta' := (\beta_1, \ldots, \beta_i, 0, \beta_{i+1}, \ldots, \beta_d) \in \mathbb{N}_0^{d+1}$. For $x \in U^{>\alpha<}$, using the notational conventions from **2.6**, we have

$$f^{>\alpha<}(x) = (f^{>e_i<})^{>\beta'<}(x^{(1)}; \ldots; x^{(i-1)}; x^{(i)}_0, x^{(i)}_2, \ldots, x^{(i)}_{\alpha_i}; x^{(i)}_1; x^{(i+1)}; \ldots; x^{(d)}),$$

as is clear from the definitions. Thus

$$f^{<\alpha>}(x) := (f^{<e_i>})^{<\beta'>}(x^{(1)}; \ldots; x^{(i-1)}; x^{(i)}_0, x^{(i)}_2, \ldots, x^{(i)}_{\alpha_i}; x^{(i)}_1; x^{(i+1)}; \ldots; x^{(d)})$$

for $x \in U^{<\alpha>}$ defines a continuous (resp., $C^{0,\sigma}$-) extension $f^{<\alpha>}\colon U^{<\alpha>} \to F$ of $f^{>\alpha<}$, whenever $|\alpha| \le k+1$. Therefore, f is C^{k+1}_{SDS} (resp., $C^{k+1,\sigma}_{SDS}$). ■

The next lemma establishes one implication in Theorem 3.1 (a) and (b). Note that σ need not be ≤ 1 here.

Lemma 3.3. *If f is C^k_{BGN} for some $k \in \mathbb{N}_0$ (resp., $C^{k,\sigma}_{BGN}$ if $\mathbb{K}$ is a valued field and $\sigma > 0$), then f is C^k_{SDS} (resp., $C^{k,\sigma}_{SDS}$).*

Proof. The proof is by induction on $k \in \mathbb{N}_0$. The case $k = 0$ is trivial. If $k \geq 1$, let $i \in \{1, \ldots, n\}$. For each $x \in U^{>e_i<}$, we then have

$$f^{>e_i<}(x) \;=\; f^{[1]}(x^{(1)}, \ldots, x^{(i-1)}, x_1^{(i)}, x^{(i+1)}, \ldots, x^{(d)}; e_i;\, x_0^{(i)} - x_1^{(i)})\,.$$

Thus

$$f^{<e_i>}(x) \;:=\; f^{[1]}(x^{(1)}, \ldots, x^{(i-1)}, x_1^{(i)}, x^{(i+1)}, \ldots, x^{(d)}; e_i;\, x_0^{(i)} - x_1^{(i)}) \tag{12}$$

for $x \in U^{<e_i>}$ defines a continuous extension of $f^{>e_i<}$ and hence f is C^1_{SDS}, with $f^{<e_i>}$ as just described. Note that the right hand side of (12) expresses $f^{<e_i>}$ as a composition of the C^{k-1}_{BGN}-map (resp., $C^{k-1,\sigma}_{BGN}$-map) $f^{[1]}$ and the restriction of a map $\mathbb{K}^{d+1} \to \mathbb{K}^d \times \mathbb{K}^d \times \mathbb{K}$ which is continuous affine-linear and hence C^∞_{BGN}. By the Chain Rule (resp., Lemma 2.33), $f^{<e_i>}$ is C^{k-1}_{BGN} (resp., $C^{k-1,\sigma}_{BGN}$) and hence C^{k-1}_{SDS} (resp., $C^{k-1,\sigma}_{SDS}$), by induction. Hence f is C^{k+1}_{SDS} (resp., $C^{k+1,\sigma}_{SDS}$), by Lemma 3.2. ∎

Lemma 3.4. *If f is C^k_{SDS} for some $k \in \mathbb{N}_0$ (resp., if $\mathbb{K}$ is a valued field and f is $C^{k,\sigma}_{SDS}$ for some $\sigma > 0$), then $f^{<\alpha>}$ is $C^{k-|\alpha|}_{SDS}$ (resp., $C^{k-|\alpha|,\sigma}_{SDS}$) for each $\alpha \in \mathbb{N}_0^d$ such that $|\alpha| \leq k$.*

Remark 3.5. The proof of Lemma 3.4 will give the following formula for $(f^{<\alpha>})^{<\beta>}$, if $\alpha \in \mathbb{N}_0^d$ such that $|\alpha| \leq k$ and $\beta \in \mathbb{N}_0^{d+|\alpha|}$ such that $|\beta| \leq k - |\alpha|$:

$$(f^{<\alpha>})^{<\beta>} \;=\; f^{<\alpha+\bar{\beta}>}\,, \tag{13}$$

where $\bar{\beta} \in \mathbb{N}_0^d$ is defined by $\bar{\beta}_j := \sum_{i=s_j}^{s_{j+1}-1} \beta_i$ for $j \in \{1, \ldots, d\}$, with $s_j := j + \sum_{i=1}^{j-1} \alpha_i$ and $s_{d+1} := d + |\alpha| + 1$.

Proof of Lemma 3.4. Given $\alpha \in \mathbb{N}_0^d$, we first show by induction on $\ell \in \{0, \ldots, k - |\alpha|\}$ that

$$(f^{>\alpha<})^{>\beta<} \;=\; f^{>\alpha+\bar{\beta}<}\,, \tag{14}$$

for each $\beta \in \mathbb{N}_0^{d+|\alpha|}$ such that $|\beta| = \ell$, where $\bar{\beta}$ (and $s_1, \ldots, s_{d+1}$) are as in Remark 3.5. The case $\ell = 0$ being trivial, let us assume now that (14) holds for some β with $|\beta| < k - |\alpha|$. For each $i \in \{1, \ldots, d + |\alpha|\}$, we have to show that (14) holds with β replaced by $\gamma := \beta + e_i$ and $\bar{\beta}$ replaced by the corresponding $\bar{\gamma}$. There is a unique $j \in \{1, \ldots, d\}$ such that $s_j \leq i < s_{j+1}$. Then $\bar{\gamma} = \bar{\beta} + e_j$. Given $x \in (U^{>\alpha<})^{>\gamma<} = (U^{>\alpha<})^{>\beta+e_i<} \subseteq (\mathbb{K}^{d+|\alpha|})^{>\beta+e_i<}$, we abbreviate $y := (x^{(1)}, \ldots, x^{(s_j-1)})$ and $z := (x^{(s_j+1)}, \ldots, x^{(d+|\alpha|)})$. Define $t := x_0^{(i)} - x_{\beta_i+1}^{(i)}$. Then $t \cdot (f^{>\alpha<})^{>\gamma<}(x)$ is given by

$$\begin{aligned}
&(f^{>\alpha<})^{>\beta<}(x^{(1)};\ldots;x^{(i-1)};x^{(i)}_0,x^{(i)}_1,\ldots,x^{(i)}_{\beta_i};x^{(i+1)};\ldots;x^{(d+|\alpha|)})\\
&-(f^{>\alpha<})^{>\beta<}(x^{(1)};\ldots;x^{(i-1)};x^{(i)}_{\beta_i+1},x^{(i)}_1,\ldots,x^{(i)}_{\beta_i};x^{(i+1)};\ldots;x^{(d+|\alpha|)})\\
=\;&f^{>\alpha+\bar\beta<}(x^{(1)};\ldots;x^{(i-1)};x^{(i)}_0,x^{(i)}_1,\ldots,x^{(i)}_{\beta_i};x^{(i+1)};\ldots;x^{(d+|\alpha|)})\\
&-f^{>\alpha+\bar\beta<}(x^{(1)};\ldots;x^{(i-1)};x^{(i)}_{\beta_i+1},x^{(i)}_1,\ldots,x^{(i)}_{\beta_i};x^{(i+1)};\ldots;x^{(d+|\alpha|)})\\
=\;&f^{>\alpha+\bar\beta<}(y;x^{(s_j)};\ldots;x^{(i-1)};x^{(i)}_0,x^{(i)}_1,\ldots,x^{(i)}_{\beta_i};x^{(i+1)},\ldots,x^{(s_{j+1}-1)};z)\\
&-f^{>\alpha+\bar\beta<}(y;x^{(s_j)};\ldots;x^{(i-1)};x^{(i)}_{\beta_i+1},x^{(i)}_1,\ldots,x^{(i)}_{\beta_i};x^{(i+1)};\ldots;x^{(s_{j+1}-1)};z)\\
=\;&f^{>\alpha+\bar\beta<}(y;x^{(i)}_0,x^{(i)}_1,\ldots,x^{(i)}_{\beta_i};x^{(s_j)};\ldots;x^{(i-1)};x^{(i+1)};\ldots;x^{(s_{j+1}-1)};z)\\
&-f^{>\alpha+\bar\beta<}(y;x^{(i)}_{\beta_i+1},x^{(i)}_1,\ldots,x^{(i)}_{\beta_i};x^{(s_j)};\ldots;x^{(i-1)};x^{(i+1)};\ldots;x^{(s_{j+1}-1)};z)\\
=\;&tf^{>\alpha+\bar\gamma<}(y;x^{(i)}_0,x^{(i)}_1,\ldots,x^{(i)}_{\beta_i};x^{(s_j)};\ldots;x^{(i-1)};x^{(i+1)};\ldots;x^{(s_{j+1}-1)};x^{(i)}_{\beta_i+1};z)\\
=\;&tf^{>\alpha+\bar\gamma<}(y;x^{(s_j)};\ldots;x^{(s_{j+1}-1)};z)\;=\;tf^{>\alpha+\bar\gamma<}(x)\,,
\end{aligned}$$

using (14) for the first equality and the symmetry properties of $f^{>\alpha+\bar\beta<}$ and $f^{>\alpha+\bar\gamma<}$ (as in Lemma 2.11) for the third and penultimate equality. This completes the inductive proof of (14).

As a consequence of (14), the continuous (resp., $C^{0,\sigma}$-) map $f^{<\alpha+\bar\beta>}$ extends the map $(f^{>\alpha<})^{>\beta<}$. It hence also extends $(f^{<\alpha>})^{>\beta<}$, for each $\beta\in\mathbb{N}_0^{d+|\alpha|}$ with $|\beta|\leq k-|\alpha|$. Hence $f^{<\alpha>}$ is $C^{k-|\alpha|}_{SDS}$ (resp., $C^{k-|\alpha|,\sigma}_{SDS}$) and (13) holds. ■

Proof of Theorem 3.1, completed. It remains to show that if f is C^k_{SDS} (resp., $C^{k,\sigma}_{SDS}$ with $\sigma\in\,]0,1]$), then f is C^k_{BGN} (resp., $C^{k,\sigma}_{BGN}$). We assume first that $U=U_1\times\cdots\times U_d$ for certain subsets $U_i\subseteq\mathbb{K}$. The proof is by induction on $k\in\mathbb{N}_0$. The case $k=0$ being trivial, assume now that f is C^k_{SDS} (resp., $C^{k,\sigma}_{SDS}$) for some $k\geq 1$. For each $(x,y,t)\in U^{]1[}$, we have

$$\begin{aligned}
f^{]1[}(x,y,t)\;&=\;\frac{f(x+ty)-f(x)}{t}\;=\;\sum_{j=1}^d\frac{f(x+t\sum_{i=1}^j y_ie_i)-f(x+t\sum_{i=1}^{j-1}y_ie_i)}{t}\\
&=\;\sum_{j=1}^d y_jf^{<e_j>}(x_1+ty_1;\ldots;x_{j-1}+ty_{j-1};x_j,x_j+ty_j;x_{j+1},\ldots,x_d),
\end{aligned}\tag{15}$$

because $a_j:=\frac{f(x+t\sum_{i=1}^j y_ie_i)-f(x+t\sum_{i=1}^{j-1}y_ie_i)}{t}=y_j\frac{f(x+t\sum_{i=1}^j y_ie_i)-f(x+t\sum_{i=1}^{j-1}y_ie_i)}{y_jt}$ coincides with $b_j:=y_jf^{<e_j>}(x_1+ty_1;\ldots;x_{j-1}+ty_{j-1};x_j,x_j+ty_j;x_{j+1},\ldots,x_d)$ if $y_j\neq 0$, while both a_j and b_j vanish if $y_j=0$. Since the right hand side of (15) defines a continuous (resp., $C^{0,\sigma}$-) map on all of $U^{[1]}$, we see that f is C^1_{BGN} (resp., $C^{1,\sigma}_{BGN}$) with

$$f^{[1]}(x,y,t)\;=\;\sum_{j=1}^d y_jf^{<e_j>}(x_1+ty_1;\ldots;x_{j-1}+ty_{j-1};x_j,x_j+ty_j;x_{j+1},\ldots,x_d)\tag{16}$$

for all $(x,y,t)\in U^{[1]}$. Here $f^{<e_j>}$ is a C^{k-1}_{SDS}-map (resp., a $C^{k-1,\sigma}_{SDS}$-map) on

$$U^{<e_j>}\;=\;U_1\times\cdots\times U_{j-1}\times U_j\times U_j\times U_{j+1}\times\cdots\times U_d$$

by Lemma 3.4 and hence a C^{k-1}_{BGN}-map (resp., a $C^{k-1,\sigma}_{BGN}$-map), by induction. Formula (16) now shows that $f^{[1]}$ is built up from $f^{<e_j>}$ and various smooth maps, whence $f^{[1]}$ is C^{k-1}_{BGN} (resp., $C^{k-1,\sigma}_{BGN}$), by the Chain Rule (resp., Lemma 2.33). Hence f is C^k_{BGN} (resp., $C^{k,\sigma}_{BGN}$). This finishes the proof if $U = U_1 \times \cdots \times U_d$.

If U is open but not necessarily of the form $U_1 \times \cdots \times U_d$, then every point $x \in U$ has an open neighbourhood V of the form $V = V_1 \times \cdots \times V_d$ for certain open subsets $V_1, \ldots, V_d \subseteq \mathbb{K}$. It is clear that $f|_V$ is C^k_{SDS} (resp., $C^{k,\sigma}_{SDS}$) if so is f. Thus $f|_V$ is C^k_{BGN} (resp., $C^{k,\sigma}_{BGN}$) by the case already settled, and thus f is C^k_{BGN} (resp., $C^{k,\sigma}_{BGN}$) as these properties can be checked locally (see Lemma 2.4, resp., Lemma 2.35). ■

In the following, we shall often refer to C^k_{BGN}-maps as C^k-maps, and to $C^{k,\sigma}_{BGN}$-maps as $C^{k,\sigma}$-maps.

We mention that both the definition of C^k_{BGN}-maps and the definition of C^k_{SDS}-maps suggest a natural definition of a vector topology on the space $C^k(U,F)$ of all C^k-maps $U \to F$: We write $C^k(U,F)_{BGN}$ for $C^k(U,F)$, equipped with the initial topology with respect to the maps $C^k(U,F) \to C(U^{[j]},F)$, $f \mapsto f^{[j]}$, for $j \in \mathbb{N}_0$ such that $j \leq k$, where $C(U^{[j]},F)$ is equipped with the compact-open topology. We write $C^k(U,F)_{SDS}$ for $C^k(U,F)$, equipped with the initial topology with respect to the maps $C^k(U,F) \to C(U^{<\alpha>},F)$, $f \mapsto f^{<\alpha>}$, for $\alpha \in \mathbb{N}_0^d$ such that $|\alpha| \leq k$. Then the following holds:

Theorem 3.6. *$C^k(U,F)_{BGN} = C^k(U,F)_{SDS}$ as a topological $\mathbb{K}$-vector space.*

The proof of Theorem 3.6 can be found in Appendix B of the preprint version of this article. It exploits that $f^{<\alpha>}$ can be expressed in terms of $f^{[j]}$ with $j := |\alpha|$, while $f^{[j]}$ can be expressed in terms of the maps $f^{<\beta>}$ with $\beta \in \mathbb{N}_0^d$ such that $|\beta| \leq j$.

4 Comparison with Ludkovsky's concepts

In this section, we give a definition of C^k-maps following an idea of Ludkovsky, and show that such maps need not be C^k_{BGN} (nor C^k_{SDS}) in the case of ground fields of positive characteristic.

4.1. To define C^k-maps in Ludkovsky's sense, we find it useful to introduce the following notations for U an open subset of a topological vector space E over a topological field $\mathbb{K}$: We define $\Phi_1(U) := U^{]1[}$ and $\overline{\Phi}_1(U) := U^{[1]}$. Given an integer $k \geq 2$, we let $\overline{\Phi}_k(U)$ be the set of all $(x, \xi_1, \ldots, \xi_k, t_1, \ldots, t_k) \in U \times E^k \times \mathbb{K}^k$ such that $(x, \xi_1, \ldots, \xi_{k-1}, t_1, \ldots, t_{k-1}) \in \overline{\Phi}_{k-1}(U)$ holds as well as

$$(x + t_k\xi_k, \xi_1, \ldots, \xi_{k-1}, t_1, \ldots, t_{k-1}) \in \overline{\Phi}_{k-1}(U).$$

Finally, we let $\Phi_k(U)$ be the set of all $(x, \xi_1, \ldots, \xi_k, t_1, \ldots, t_k) \in \overline{\Phi}_k(U)$ such that $t_k \neq 0$.

Definition 4.2. Let E and F be topological vector spaces over a topological field $\mathbb{K}$, and $f\colon U \to F$ be a map on an open subset $U \subseteq E$. We say that f is C^1_{Lud} if f is C^1_{BGN}, i.e., if the directional difference quotient map $\Phi_1(f) := f^{]1[}$ admits a continuous extension $\overline{\Phi}_1(f) := f^{[1]}$ to $\overline{\Phi}_1(U) = U^{[1]}$. Recursively, having said when f is a C^{k-1}_{Lud}-map and having defined a map $\overline{\Phi}_{k-1}(f)\colon \overline{\Phi}_{k-1}(U) \to F$ in this case, we say that f is C^k_{Lud} if f is C^{k-1}_{Lud} and if the map $\Phi_k(f)\colon \Phi_k(U) \to F$ taking $(x, \xi_1, \dots, \xi_k, t_1, \dots, t_k)$ to

$$\frac{\overline{\Phi}_{k-1}(x + t_k\xi_k, \xi_1, \dots, \xi_{k-1}, t_1, \dots, t_{k-1}) - \overline{\Phi}_{k-1}(x, \xi_1, \dots, \xi_{k-1}, t_1, \dots, t_{k-1})}{t_k}$$

admits a continuous extension $\overline{\Phi}_k(f)\colon \overline{\Phi}_k(U) \to F$. We say that f is C^∞_{Lud} if f is C^k_{Lud} for each $k \in \mathbb{N}$.

Remark 4.3. Initially, Ludkovsky defined C^k-maps only for certain ultrametric fields of characteristic 0 and E, F locally convex, but of course the preceding definition is meaningful in the stated generality.[1] Furthermore, he only required the existence of $\overline{\Phi}_1(f)$ locally on a neighbourhood of $(x, 0, 0)$ for each given point $x \in U$ (and similarly for $\overline{\Phi}_k(f)$). We find it more convenient to define $\overline{\Phi}_k(f)$ globally on $\overline{\Phi}_k(U)$ (which is equivalent to the local existence). This is also the approach in [15].

Remark 4.4. If f is C^k_{Lud}, we define

$$d^j f(x, \xi_1, \dots, \xi_j) \;:=\; \overline{\Phi}_j(f)(x, \xi_1, \dots, \xi_j, 0, \dots, 0)$$

for $x \in U$, $j \in \mathbb{N}$ with $j \leq k$, and $\xi_1, \dots, \xi_j \in E$. We also write $df(x, \xi) := d^1 f(x, \xi)$. Then $d^j f\colon U \times E^j \to F$ is continuous, being a partial map of $\overline{\Phi}_j(f)$ (i.e., a map obtained from $\overline{\Phi}_j(f)$ by fixing some of its arguments). Furthermore, $f^{(j)}(x) := d^j f(x, \bullet)\colon E^j \to F$ is a symmetric j-linear map, by a reasoning similar to that used to prove [2, Lemma 4.8].[2] We remark that Ludkovsky initially made the j-linearity of the maps $f^{(j)}(x)$ part of his definition of a C^k-map; by the preceding, this requirement is redundant and can be omitted.

Remark 4.5. We mention that Definition 4.2 captures the basic idea of Ludkovsky's approach, but differs slightly from his original definition which imposes additional boundedness conditions. In the example discussed in Theorem 4.7 below, the domain $\mathbb{O}$ will be an open and compact set, whence these additional conditions will be satisfied automatically. In his most recent preprints (like [15]), Ludkovsky also omits the boundedness conditions.

Remark 4.6. It is clear that each C^k_{BGN}-map is also C^k_{Lud}; a suitable partial map of $f^{[j]}$ serves as the continuous extension $\overline{\Phi}_j(f)$ of $\Phi_j(f)$, for each $j \in \mathbb{N}$ such that $j \leq k$. To make this more precise, let us write $E^{[j]} = E \times H_j \times \mathbb{K}$, where H_j collects all factors in the middle. Explicitly, we have $H_1 := E$, $H_j := H_{j-1} \times \mathbb{K} \times E^{[j-1]}$ if $j \geq 2$. Let 0_{H_j} be the zero element in H_j. Then a simple induction on $k \in \mathbb{N}$ shows that, if f is C^k_{BGN}, then f is C^k_{Lud}, with

$$\overline{\Phi}_k(x, \xi_1, \dots, \xi_k, t_1, \dots, t_k) \;:=\; f^{[k]}(x, \xi_1, t_1; \xi_2, 0_{H_1}, t_2; \dots; \xi_k, 0_{H_{k-1}}, t_k)$$

for $(x, \xi_1, \dots, \xi_k, t_1, \dots, t_k) \in \overline{\Phi}_k(U)$ giving the continuous extension of $\Phi_k(f)$ to a map on $\overline{\Phi}_k(U)$.

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[1] In the meantime, Ludkovsky uses his approach also in positive characteristic [15].

[2] In the example discussed in Theorem 4.7 below, the j-linearity will be obvious.

Let $\mathbb{K}$ be a local field of positive characteristic now. Thus, up to isomorphism, $\mathbb{K} = \mathbb{F}_q(\!(X)\!)$ is a field of formal Laurent series over a finite field $\mathbb{F}_q$ with $q = p^\ell$ elements for some $\ell \in \mathbb{N}$ and prime p. We let $\mathbb{O} := \mathbb{F}_q[\![X]\!]$ be the ring of formal power series, which is an open, compact subring of $\mathbb{K}$. Its elements are of the form $x = \sum_{k=0}^\infty a_k X^k$, where $a_k \in \mathbb{F}_q$. Recall that if $x \neq 0$, then its absolute value is given by $|x| = q^{-k}$, where $k \in \mathbb{N}_0$ is chosen minimal such that $a_k \neq 0$. We define a map

$$f\colon \mathbb{O} \to \mathbb{K}\,, \quad \sum_{k=0}^\infty a_k X^k \mapsto \sum_{k=0}^\infty a_k X^{[\frac{3}{2}k]}\,, \tag{17}$$

where $[r]$ denotes the Gauß bracket (integer part) of a real number $r \geq 0$.

Theorem 4.7. *The map $f\colon \mathbb{O} \to \mathbb{K}$ defined in* (17) *is C^∞_{Lud}, but not C^2_{BGN}.*

Proof. It is useful to note first that f is a homomorphism of additive groups, i.e., $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{O}$. For $x, y \in \mathbb{O}$, we have

$$|x-y|^{\frac{3}{2}} \;\leq\; |f(x-y)| \;\leq\; q|x-y|^{\frac{3}{2}}\,,$$

where $|f(x-y)| = |f(x) - f(y)|$. As a consequence, f is C^1_{BGN} with derivative $f'(x) = 0$ for all $x \in \mathbb{O}$ (see [6, Lemma 2.1]; cf. [18, Theorem 29.12]). Furthermore, f is not C^2_{BGN} because it does not admit a second order Taylor expansion (see [6, Lemma 2.2]). We now show that f is C^∞_{Lud}. First, we note that f is C^1_{Lud} because it is C^1_{BGN}, with

$$\overline{\Phi}_1(f)(x,\xi,0) \;=\; df(x,\xi) \;=\; f'(x)\xi \;=\; 0 \quad \text{for all } x \in \mathbb{O} \text{ and } \xi \in \mathbb{K}. \tag{18}$$

Using that f is a homomorphism, for all $x \in \mathbb{O}$, $\xi \in \mathbb{K}$ and $t \in \mathbb{K}^\times$ such that $x + t\xi \in \mathbb{O}$, we obtain

$$\overline{\Phi}_1(f)(x,\xi,t) = \frac{f(x+t\xi) - f(x)}{t} = \frac{f(x) + f(t\xi) - f(x)}{t} = \frac{f(t\xi)}{t}. \tag{19}$$

Since the right hand side of (19) is independent of x, we obtain for all $x \in \mathbb{O}$, $\xi_1, \xi_2 \in \mathbb{K}$ and $t_1, t_2 \in \mathbb{K}^\times$ such that $(x, \xi_1, \xi_2, t_1, t_2) \in \Phi_2(\mathbb{O})$:

$$\Phi_2(f)(x,\xi_1,\xi_2,t_1,t_2) = \frac{\overline{\Phi}_1(f)(x + t_2\xi_2, \xi_1, t_1) - \overline{\Phi}_1(f)(x,\xi_1,t_1)}{t_2} = 0\,. \tag{20}$$

By (18), we also have $\Phi_2(f)(x,\xi_1,\xi_2,t_1,t_2) = 0$ for all $x \in \mathbb{O}$, $\xi_1, \xi_2 \in \mathbb{K}$, $t_1 = 0$ and $t_2 \in \mathbb{K}^\times$ such that $(x, \xi_1, \xi_2, 0, t_2) \in \Phi_2(\mathbb{O})$. Thus

$$\overline{\Phi}_2(f)\colon \overline{\Phi}_2(\mathbb{O}) \to F\,, \quad (x,\xi_1,\xi_2,t_1,t_2) \mapsto 0$$

is a continuous map which extends $\Phi_2(f)$, and thus f is C^2_{Lud} with $\overline{\Phi}_2(f) = 0$. It now readily follows by induction that f is C^k_{Lud} for each $k \geq 2$, with $\overline{\Phi}_k(f) = 0$. ∎

Remark 4.8. It would be interesting to clarify whether C^k_{Lud}-maps between locally convex spaces over an ultrametric field $\mathbb{K}$ of characteristic 0 (as originally considered by Ludkovsky) coincide with C^k_{BGN}-maps (for $k \geq 2$), notably for $\mathbb{K} = \mathbb{Q}_p$. It is also unknown whether C^k_{Lud}-maps into real non-locally convex spaces are C^k_{BGN}. The author conjectures that neither is the case, but has not found counterexamples so far. [15, Corollary 20] claims that $C^k_{BGN} = C^k_{Lud}$ in the case of ultrametric fields of zero characteristic, but the author was not convinced by the reasoning.

5 Hölder differentiable maps on metrizable spaces

This section is devoted to the proof of the following characterization of $C^{\ell,\sigma}_{BGN}$-maps on open subsets of metrizable spaces.

Theorem 5.1. *Let $(\mathbb{K}, |.|)$ be $\mathbb{R}$ or an ultrametric field. Let E and F be topological $\mathbb{K}$-vector spaces and $f\colon U \to F$ be a map, defined on an open subset $U \subseteq E$. Let $\ell \in \mathbb{N}_0$ and $\sigma \in \,]0,1]$. If E is metrizable, then f is $C^{\ell,\sigma}$ if and only if $f\circ\gamma\colon \mathbb{K}^{\ell+1} \to F$ is $C^{\ell,\sigma}$, for each C^∞-map $\gamma\colon \mathbb{K}^{\ell+1} \to U$.*

The proof of Theorem 5.1 heavily relies on tools developed in [8], which are variants of standard methods of differential calculus in real locally convex spaces (cf. [12]). To describe these tools, we need the auxiliary notion of a "calibration" on a topological vector space E over a valued field $\mathbb{K}$.

Definition 5.2. A sequence $(q_n)_{n\in\mathbb{N}_0}$ of gauges on E is called a *calibration* if

$$(\forall n \in \mathbb{N}_0)(\forall x, y \in E) \quad q_n(x+y) \;\leq\; q_{n+1}(x) + q_{n+1}(y)\,. \tag{21}$$

If q is a gauge on E, then there always exists a calibration $(q_n)_{n\in\mathbb{N}_0}$ such that $q_0 = q$ (cf. Remark 2.19); we then say that q extends to $(q_n)_{n\in\mathbb{N}_0}$.

Remark 5.3. If $(q_n)_{n\in\mathbb{N}_0}$ is a calibration, then $q_n \leq q_{n+1}$ for each $n \in \mathbb{N}_0$ because $q_n(x) = q_n(x+0) \leq q_{n+1}(x) + q_{n+1}(0) = q_{n+1}(x)$ for each $x \in E$. Also note that if $q\colon E \to [0,\infty[$ is a continuous seminorm, then $(q)_{n\in\mathbb{N}_0}$ is a calibration. If $(q_n)_{n\in\mathbb{N}_0}$ is any calibration extending the seminorm q, then $q_n \geq q$ for each n, by the preceding remark. Thus $(q)_{n\in\mathbb{N}_0}$ is the smallest calibration extending q.

The following two lemmas are the main results of [8]. They are variants of [12, Lemma 12.2]. In the first lemma, $\mathbb{O} := \overline{B}{}^{\mathbb{K}}_1(0)$.

Lemma 5.4 (Ultrametric General Curve Lemma)**.** *Let E be a topological vector space over an ultrametric field $\mathbb{K}$, $\rho \in \mathbb{K}^\times$ with $|\rho| < 1$ and $(\gamma_n)_{n\in\mathbb{N}}$ be a family of smooth maps $\gamma_n \in BC^\infty(\rho^n\mathbb{O}, E)$ which become small sufficiently fast in the sense that, for each gauge q on E, there exists a calibration $(q_n)_{n\in\mathbb{N}_0}$ extending q such that*

$$(\forall a > 0)\,(\forall k, m \in \mathbb{N}_0) \quad \lim_{n\to\infty} a^n \|\gamma_n^{<k>}\|_{q_{n+m},\infty} \;=\; 0\,. \tag{22}$$

Then there exists a smooth map $\gamma\colon \mathbb{K} \to E$ with $\mathrm{im}(\gamma) = \{0\} \cup \bigcup_{n\in\mathbb{N}} \mathrm{im}(\gamma_n)$, such that $\gamma(\rho^{n-1} + t) = \gamma_n(t)$ for all $n \in \mathbb{N}$ and $t \in \rho^n\mathbb{O}$. ■

Remark 5.5. Let E in Lemma 5.4 be metrizable and suppose that there exists a calibration $(p_n)_{n\in\mathbb{N}_0}$ such that $\{p_n\colon n \in \mathbb{N}_0\}$ is a fundamental system of gauges, and $C > 0$ such that

$$(\forall k \in \mathbb{N}_0)\,(\forall n \geq k) \quad \|\gamma_n^{<k>}\|_{p_{2n},\infty} \;\leq\; Cn^{-n}\,. \tag{23}$$

Then the hypothesis (22) of Lemma 5.4 is satisfied: Any q extends to a suitable calibration via $q_n := rp_{n+n_0}$ for $n \in \mathbb{N}$, with $r > 0$ and $n_0 \in \mathbb{N}_0$ sufficiently large.

Lemma 5.6 (Real Case of General Curve Lemma). *Let E be a real topological vector space and $(s_n)_{n\in\mathbb{N}}$ as well as $(r_n)_{n\in\mathbb{N}}$ be sequences of positive reals with $\sum_{n=1}^{\infty} s_n < \infty$ and $r_n \geq s_n + \frac{2}{n^2}$ for each $n \in \mathbb{N}$. Furthermore, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of smooth maps $\gamma_n\colon [-r_n, r_n] \to E$ which become small sufficiently fast in the sense that, for each gauge q on E, there exists a calibration $(q_n)_{n\in\mathbb{N}_0}$ extending q such that*

$$(\forall k, \ell, m \in \mathbb{N}_0) \qquad \lim_{n\to\infty} n^{\ell} \|\gamma_n^{<k>}\|_{q_{n+m},\infty} = 0\,. \tag{24}$$

Then there exists a curve $\gamma \in BC^{\infty}(\mathbb{R}, E)$ with $\operatorname{im}(\gamma) \subseteq [0,1] \cdot \bigcup_{n\in\mathbb{N}} \operatorname{im}(\gamma_n)$ and a convergent sequence $(t_n)_{n\in\mathbb{N}}$ of real numbers such that $\gamma(t_n+t) = \gamma_n(t)$ for all $n \in \mathbb{N}$ and $t \in [-s_n, s_n]$. ■

Again, (23) enables to manufacture calibrations satisfying (24).

In our applications, the maps γ_n are restrictions of affine-linear maps to balls. The following simple lemma will help us to verify the hypotheses of the General Curve Lemmas in this case.

Lemma 5.7. *Consider the map $\gamma\colon \overline{B}_r^{\mathbb{K}}(0) \to E$, $x \mapsto xa+b$, where E is a topological vector space over a valued field $\mathbb{K}$, $a, b \in E$ and $r > 0$. Let q_1 and q_2 be gauges on E with $q_1(x + y) \leq q_2(x) + q_2(y)$ for all $x, y \in E$. Then $\|\gamma\|_{q_1,\infty} \leq r\|a\|_{q_2} + \|b\|_{q_2}$, $\|\gamma^{<1>}\|_{q_1,\infty} = \|a\|_{q_1}$ and $\|\gamma^{<k>}\|_{q_1,\infty} = 0$ for $k \geq 2$.*

Proof. Since $\|\gamma(x)\|_{q_1} = \|xa + b\|_{q_1} \leq |x| \cdot \|a\|_{q_2} + \|b\|_{q_2} \leq r \cdot \|a\|_{q_2} + \|b\|_{q_2}$ for each $x \in \overline{B}_r^{\mathbb{K}}(0)$, the first inequality holds. The remaining assertions follow from the observations that $\gamma^{<1>}(x, y) = a$ for all $x, y \in \overline{B}_r^{\mathbb{K}}(0)$ and $\gamma^{<k>} = 0$ for all $k \geq 2$. ■

Another simple observation will be used.

Lemma 5.8. *Let $(\mathbb{K}, |.|)$ be either $\mathbb{R}$ or an ultrametric field. Let E and F be topological $\mathbb{K}$-vector spaces and $f\colon U \to F$ be a map, defined on an open subset $U \subseteq E$. Let $\ell, d \in \mathbb{N}_0$ and $\sigma \in\,]0,1]$. If $f \circ \gamma\colon \mathbb{K}^d \to F$ is $C^{\ell,\sigma}$, for each C^{∞}-map $\gamma\colon \mathbb{K}^d \to U$, then also $f \circ \gamma\colon V \to F$ is $C^{\ell,\sigma}$, for each C^{∞}-map $\gamma\colon V \to U$ defined on an open subset $V \subseteq \mathbb{K}^d$.*

Proof. Given $x_0 \in V$, there exists a smooth map $\kappa\colon \mathbb{K}^d \to V$ such that $\kappa|_W = \mathrm{id}_W$ for some open neighbourhood $W \subseteq V$ of x_0. In fact, if $\mathbb{K}$ is ultrametric, we can choose an open, closed neighbourhood $W \subseteq V$ of x_0 and define $\kappa(x) := x$ if $x \in W$, $\kappa(x) := x_0$ if $x \in V \setminus W$. In the real case, we can manufacture κ by standard arguments, using a cut-off function. Then $\eta := \gamma \circ \kappa\colon \mathbb{K}^d \to U$ is smooth and hence $f \circ \eta$ is $C^{\ell,\sigma}$. Then $(f \circ \gamma)|_W = (f \circ \eta)|_W$ is $C^{\ell,\sigma}$. Hence $f \circ \gamma$ is locally $C^{\ell,\sigma}$ and thus $C^{\ell,\sigma}$, by Lemma 2.35. ■

Proof of Theorem 5.1. If f is $C^{\ell,\sigma}$, then $f \circ \gamma$ is $C^{\ell,\sigma}$ for each C^{∞}-map $\gamma\colon \mathbb{K}^{\ell+1} \to U$, by Lemma 2.33 and Remark 2.30. To prove the converse direction, we first assume that $\mathbb{K}$ is an ultrametric field. We start with the case $\ell = 0$. If f is not $C^{0,\sigma}$, then the condition formulated in Definition 2.26 is violated by some $x_0 \in U$. Hence, there exists a gauge q on F such that, for each neighbourhood $V \subseteq U$ of x_0 and gauge p on E, there are $x, y \in V$ such that $\|f(x) - f(y)\|_q > (\|x - y\|_p)^{\sigma}$. After a translation, we may assume that $x_0 = 0$. Pick a gauge q_0 on E such that

$B_1^{q_0}(0) \subseteq U$, and extend it to a calibration $(q_n)_{n\in\mathbb{N}_0}$ on E such that $\{q_n : n \in \mathbb{N}_0\}$ is a fundamental system of gauges. Also, pick $\rho \in \mathbb{K}^\times$ such that $|\rho| < 1$. After replacing $q_1, q_2, \ldots$ by large multiples if necessary, we may assume that

$$q_0 \;\leq\; \Big(\tfrac{1}{2} + \tfrac{1}{|\rho|}\Big)^{-1} q_1 \,. \tag{25}$$

Applying the above property of q for a given $n \in \mathbb{N}$ to $V := \overline{B}^{q_{2n+3}}_{\frac{1}{2}n^{-n}|\rho|^n}(0)$ and $p := n^{\frac{1}{\sigma}} n^n q_{2n+2}$, we find $x_n, y_n \in E$ such that

$$\|x_n\|_{q_{2n+3}}, \|y_n\|_{q_{2n+3}} \;\leq\; \frac{1}{2} n^{-n} |\rho|^n \tag{26}$$

and

$$\|f(x_n) - f(y_n)\|_q \;>\; n \cdot n^{\sigma n} \Big(\|x_n - y_n\|_{q_{2n+2}}\Big)^\sigma \,. \tag{27}$$

Case 1. If $\|x_n - y_n\|_{q_{2n+2}} \neq 0$, let k_n be the unique integer such that

$$|\rho|^{k_n} \;\leq\; n^n \|x_n - y_n\|_{q_{2n+2}} \;<\; |\rho|^{k_n - 1} \,. \tag{28}$$

Since $n^n \|x_n - y_n\|_{q_{2n+2}} \leq n^n(\|x_n\|_{q_{2n+3}} + \|y_n\|_{q_{2n+3}}) \leq |\rho|^n$ by (26), we have $k_n \geq n$.

Case 2. If $\|x_n - y_n\|_{q_{2n+2}} = 0$ holds, we choose the integer $k_n \geq n$ so large that $\|f(x_n) - f(y_n)\|_q \geq n(|\rho|^{k_n})^\sigma$.

In either case, we define

$$\gamma_n \colon \overline{B}^{\mathbb{K}}_{|\rho|^n}(0) \to E \,, \quad \gamma_n(t) \;:=\; x_n + \frac{t}{\rho^{k_n}} (y_n - x_n) \,.$$

By Lemma 5.7, we then have $\gamma_n^{<k>} = 0$ for $k \geq 2$, furthermore

$$\|\gamma_n^{<1>}\|_{q_{2n},\infty} \;=\; \frac{\|x_n - y_n\|_{q_{2n}}}{|\rho|^{k_n}} \;\leq\; \frac{\|x_n - y_n\|_{q_{2n+2}}}{|\rho|^{k_n}} \;<\; \frac{n^{-n}}{|\rho|}$$

by definition of k_n, and finally

$$\|\gamma_n\|_{q_{2n},\infty} \leq |\rho^n| \frac{\|x_n - y_n\|_{q_{2n+1}}}{|\rho|^{k_n}} + \|x_n\|_{q_{2n+1}} < \tfrac{n^{-n}}{|\rho|} + \tfrac{1}{2} n^{-n} |\rho|^n < \Big(\tfrac{1}{2} + \tfrac{1}{|\rho|}\Big) n^{-n} \,,$$

entailing that $\|\gamma_n\|_{q_0,\infty} < 1$ (see (25)) and thus $\operatorname{im} \gamma_n \subseteq B_1^{q_0}(0) \subseteq U$. In view of the preceding, (23) in Remark 5.5 is satisfied by the calibration $(q_n)_{n\in\mathbb{N}_0}$ with $C = \frac{1}{2} + \frac{1}{|\rho|}$. Therefore the General Curve Lemma (Lemma 5.4) provides a smooth map $\gamma \colon \mathbb{K} \to E$ with $\gamma(\mathbb{K}) \subseteq U$ such that $\gamma(t) = \gamma_n(t - \rho^{n-1})$ for each $n \in \mathbb{N}$ and $t \in \overline{B}^{\mathbb{K}}_{|\rho|^n}(\rho^{n-1})$. In particular, $\gamma(\rho^{n-1}) = \gamma_n(0) = x_n$ and $\gamma(\rho^{n-1} + \rho^{k_n}) = \gamma_n(\rho^{k_n}) = y_n$ for each $n \in \mathbb{N}$. Hence

$$\|f(\gamma(\rho^n)) - f(\gamma(\rho^{n-1} + \rho^{k_n}))\|_q = \|f(x_n) - f(y_n)\|_q \;>\; n \cdot n^{\sigma n} (\|x_n - y_n\|_{q_{2n+2}})^\sigma \tag{29}$$

$$= n \cdot n^{\sigma k} \left(\frac{\|x_n - y_n\|_{q_{2n+2}}}{|\rho^{k_n}|} \right)^\sigma |\rho^{k_n}|^\sigma \;\geq\; n |\rho^{k_n}|^\sigma \tag{30}$$

in Case 1, using (27) in (29) and then (28) for the final inequality. In Case 2, we have

$$\|f(\gamma(\rho^n)) - f(\gamma(\rho^{n-1}+\rho^{k_n}))\|_q \;=\; \|f(x_n)-f(y_n)\|_q \;\geq\; n\cdot (|\rho|^{k_n})^\sigma \tag{31}$$

by choice of k_n. Thus (31) holds for each n, and hence $f\circ\gamma$ is not $C^{0,\sigma}$. In fact, if $f\circ\gamma$ were $C^{0,\sigma}$, there would be a 0-neighbourhood $J\subseteq\mathbb{K}$ and a gauge g on $\mathbb{K}$ such that $\|f(\gamma(t))-f(\gamma(s))\|_q\leq (\|t-s\|_g)^\sigma$ for all $s,t\in J$. As a consequence of Lemma 2.21, there is $C>0$ such that $g\leq C|.|$. Hence $\|f(\gamma(t))-f(\gamma(s))\|_q\leq C^\sigma|t-s|^\sigma$ for all $s,t\in J$, which contradicts (31).

The general case: Let ℓ be a positive integer now. If $f\circ\gamma$ is $C^{\ell,\sigma}$ for each smooth map $\gamma\colon \mathbb{K}^{\ell+1}\to U$, then $f\circ\gamma$ is C^ℓ in particular and hence f is C^ℓ, by [2, Theorem 12.4]. To prove that f is $C^{\ell,\sigma}$, it only remains to show that $f^{[\ell]}$ is $C^{0,\sigma}$. We assume that $f^{[\ell]}$ is not $C^{0,\sigma}$ and derive a contradiction. Since $U^{]\ell[}$ is dense in the domain $U^{[\ell]}$ of the continuous map $f^{[\ell]}$, Lemma 2.28 shows that there exists $x_0\in U^{[\ell]}$ and a gauge q on F such that, for each neighbourhood $V\subseteq U^{[\ell]}$ of x_0 and gauge p on $E^{[\ell]}$, there are $x,y\in V\cap U^{]\ell[}$ such that $\|f^{[\ell]}(x)-f^{[\ell]}(y)\|_q>(\|x-y\|_p)^\sigma$. We now pick $x_n,y_n\in U^{]\ell[}$ as above in the case $\ell=0$, applied to $f^{[\ell]}$ instead of f, and obtain a smooth curve $\gamma\colon\mathbb{K}\to U^{[\ell]}$ such that $\gamma(\rho^{n-1})=x_n$ and $\gamma(\rho^{n-1}+\rho^n)=y_n$. Applying [2, Lemma 12.3] with $m:=1$, $V:=\mathbb{K}$, $D:=\{\rho^{n-1}\colon n\in\mathbb{N}\}\cup\{\rho^{n-1}+\rho^n\colon n\in\mathbb{N}\}$ and $X_0:=\{0\}$, we obtain a smooth map $\Gamma\colon W\to U$, defined on an open subset $W\subseteq\mathbb{K}^{\ell+1}$, an open neighbourhood Y of 0 in $\mathbb{K}$, and a smooth map $g\colon Y\to W^{[\ell]}$ such that

$$(\forall t\in D\cap Y)\quad f^{[\ell]}(\gamma(t)) \;=\; (f\circ\Gamma)^{[\ell]}(g(t))\,. \tag{32}$$

There is $N\in\mathbb{N}$ such that $\rho^{n-1}\in Y$ and $\rho^{n-1}+\rho^n\in Y$ for each integer $n\geq N$. The hypothesis implies that $f\circ\Gamma$ is $C^{\ell,\sigma}$ (see Lemma 5.8). As a consequence, $(f\circ\Gamma)^{[\ell]}$ is $C^{0,\sigma}$ and hence also $(f\circ\Gamma)^{[\ell]}\circ g$ is $C^{0,\sigma}$. However, by construction of γ and (32), for each $n\geq N$ we have

$$\begin{aligned}
&\|(f\circ\Gamma)^{[\ell]}(g(\rho^{n-1}))-(f\circ\Gamma)^{[\ell]}(g(\rho^{n-1}+\rho^n))\|_q\\
&\quad=\;\|f^{[\ell]}(\gamma(\rho^{n-1}))-f^{[\ell]}(\gamma(\rho^{n-1}+\rho^n))\|_q \;=\; \|f^{[\ell]}(x_n)-f^{[\ell]}(y_n)\|_q\\
&\quad\geq\; n|\rho^n|^\sigma\,,
\end{aligned}$$

arguing as in (30) to pass to the last line. Hence $(f\circ\Gamma)^{[\ell]}\circ g$ is not $C^{0,\sigma}$, contradicting the preceding. This closes the proof in the ultrametric case.

Now assume that $\mathbb{K}=\mathbb{R}$, and pick $r\in\,]0,1[$. If f is not $C^{0,\sigma}$, then there exists $x_0\in U$ and a gauge q on F such that, for each neighbourhood $V\subseteq U$ of x_0 and gauge p on E, there are $x,y\in V$ such that $\|f(x)-f(y)\|_q>(\|x-y\|_p)^\sigma$. After a translation, we may assume that $x_0=0$. Take a gauge q_0 on E such that $\overline{B}^{q_0}_1(0)\subseteq U$, and extend it to a calibration $(q_n)_{n\in\mathbb{N}_0}$ on E such that $\{q_n\colon n\in\mathbb{N}_0\}$ is a fundamental system of gauges. We may assume that $q_0\leq\frac{2}{7}q_1$. Applying the above property of q for a given $n\in\mathbb{N}$ to $V:=\overline{B}^{q_{2n+3}}_{\frac{1}{2}n^{-n}r^n}(0)$ and $p:=n^{\frac{1}{\sigma}}n^n q_{2n+2}$, we find $x_n,y_n\in E$ such that

$$\|x_n\|_{q_{2n+3}},\|y_n\|_{q_{2n+3}}\;\leq\;\frac{1}{2}n^{-n}r^n$$

and $\|f(x_n)-f(y_n)\|_q>n\cdot n^{\sigma n}\Big(\|x_n-y_n\|_{q_{2n+2}}\Big)^\sigma$.

Case 1: If $\|x_n - y_n\|_{q_{2n+2}} \neq 0$, define $s_n := n^n \|x_n - y_n\|_{q_{2n+2}} \leq r^n$. *Case* 2: If $\|x_n - y_n\|_{q_{2n+2}} = 0$, choose $s_n \in \,]0, r^n]$ such that $\|f(x_n) - f(y_n)\|_q \geq n \cdot (s_n)^\sigma$. In either case, we define $r_n := s_n + \frac{2}{n^2}$ and

$$\gamma_n \colon [-r_n, r_n] \to E\,, \quad \gamma_n(t) \;:=\; x_n + \frac{t}{s_n}(y_n - x_n)\,.$$

By Lemma 5.7, we then have $\gamma_n^{<k>} = 0$ for $k \geq 2$, furthermore $\|\gamma_n^{<1>}\|_{q_{2n},\infty} = \frac{\|y_n - x_n\|_{q_{2n}}}{s_n} \leq n^{-n}$ by definition of s_n, and finally $\|\gamma_n\|_{q_{2n},\infty} < \frac{7}{2} n^{-n}$ because

$$\|\gamma_n(x)\|_{q_{2n}} \leq \|x_n\|_{q_{2n+1}} + r_n \frac{\|y_n - x_n\|_{q_{2n+1}}}{s_n} \leq \tfrac{1}{2} n^{-n} r^n + \left(r^n + \tfrac{2}{n^2}\right) n^{-n} \leq \tfrac{7}{2} n^{-n},$$

entailing that $\|\gamma_n\|_{q_0,\infty} \leq 1$ and thus $\operatorname{im} \gamma_n \subseteq \overline{B}_1^{q_0}(0) \subseteq U$. In view of the preceding, (23) in Remark 5.5 is satisfied with $C = \frac{7}{2}$. Therefore the General Curve Lemma (Lemma 5.6) provides a smooth map $\gamma \colon \mathbb{R} \to E$ with $\gamma(\mathbb{R}) \subseteq [0,1]\overline{B}_1^{q_0}(0) = \overline{B}_1^{q_0}(0) \subseteq U$, and a convergent sequence $(t_n)_{n\in\mathbb{N}}$ of reals such that $\gamma(t_n + t) = \gamma_n(t)$ for each $n \in \mathbb{N}$ and $t \in \mathbb{R}$ such that $|t| \leq s_n$. In particular, $\gamma(t_n) = \gamma_n(0) = x_n$ and $\gamma(t_n + s_n) = \gamma_n(s_n) = y_n$ for each $n \in \mathbb{N}$. Hence

$$\begin{aligned} \|f(\gamma(t_n)) - f(\gamma(t_n + s_n))\|_q &= \|f(x_n) - f(y_n)\|_q \;>\; n \cdot n^{\sigma n} (\|x_n - y_n\|_{q_{2n+2}})^\sigma \\ &= n \cdot n^{\sigma k} \left(\frac{\|x_n - y_n\|_{q_{2n+2}}}{s_n} \right)^\sigma (s_n)^\sigma \;=\; n \cdot (s_n)^\sigma \end{aligned}$$

in Case 1. In Case 2, we have

$$\|f(\gamma(t_n)) - f(\gamma(t_n + s_n))\|_q \;=\; \|f(x_n) - f(y_n)\|_q \;\geq\; n \cdot (s_n)^\sigma \tag{33}$$

by choice of s_n. Thus (33) holds for each n, and hence $f \circ \gamma$ is not $C^{0,\sigma}$.

The general case: If ℓ is a positive integer and f is not $C^{\ell,\sigma}$ although $f \circ \gamma$ is $C^{\ell,\sigma}$ for each smooth map $\gamma \colon \mathbb{K}^{\ell+1} \to U$, we reach a contradiction along the lines of the ultrametric case. First, applying the case $\ell = 0$ to $f^{[\ell]}$ instead of f, we find a gauge q on F and $x_n, y_n \in U^{]\ell[}$, positive reals s_n such that $\sum_{n=1}^\infty s_n < \infty$, a smooth curve $\gamma \colon \mathbb{R} \to U^{[\ell]}$ and a convergent sequence $(t_n)_{n\in\mathbb{N}}$ of reals such that $\gamma(t_n) = x_n$, $\gamma(t_n + s_n) = y_n$ and $\|f^{[\ell]}(x_n) - f^{[\ell]}(y_n)\|_q \geq n(s_n)^\sigma$ for each $n \in \mathbb{N}$. Let $t_\infty := \lim_{n\to\infty} t_n$. Applying [2, Lemma 12.3] with $m := 1$, $V := \mathbb{K}$, $D := \{t_n \colon n \in \mathbb{N}\} \cup \{t_n + s_n \colon n \in \mathbb{N}\}$ and $X_0 := \{t_\infty\}$, we obtain a smooth map $\Gamma \colon W \to U$, defined on an open subset $W \subseteq \mathbb{R}^{\ell+1}$, an open neighbourhood Y of t_∞ in $\mathbb{R}$, and a smooth map $g \colon Y \to W^{[\ell]}$ such that (32) holds. There is $N \in \mathbb{N}$ such that $t_n \in Y$ and $t_n + s_n \in Y$ for all $n \geq N$. The hypothesis implies that $f \circ \Gamma$ is $C^{\ell,\sigma}$ (Lemma 5.8). As a consequence, $(f \circ \Gamma)^{[\ell]}$ is $C^{0,\sigma}$ and hence also $(f \circ \Gamma)^{[\ell]} \circ g$ is $C^{0,\sigma}$. However, by construction of γ and (32), we have $\|(f \circ \Gamma)^{[\ell]}(g(t_n)) - (f \circ \Gamma)^{[\ell]}(g(t_n + s_n))\|_q = \|f^{[\ell]}(\gamma(t_n)) - f^{[\ell]}(\gamma(t_n + s_n))\|_q = \|f^{[\ell]}(x_n) - f^{[\ell]}(y_n)\|_q \geq n(s_n)^\sigma$ for each $n \geq N$, whence $(f \circ \Gamma)^{[\ell]} \circ g$ is not $C^{0,\sigma}$, which is absurd. ∎

6 Weakly Hölder differentiable maps

If $\mathbb{K}$ is a topological field and E a topological $\mathbb{K}$-vector space, we let E' be the space of all continuous linear functionals $\lambda\colon E \to \mathbb{K}$.

Definition 6.1. Let E and F be topological vector spaces over a valued field $\mathbb{K}$ and $f\colon U \to F$ be a map on a subset $U \subseteq E$. Let $\sigma > 0$. We say that f is *weakly* $C^{0,\sigma}$ if $\lambda \circ f\colon U \to \mathbb{K}$ is $C^{0,\sigma}$ for each $\lambda \in F'$. If U has dense interior and $k \in \mathbb{N} \cup \{\infty\}$, we say that f is *weakly* $C^{k,\sigma}$ if $\lambda \circ f\colon U \to \mathbb{K}$ is $C^{k,\sigma}$ for each $\lambda \in F'$.

Remark 6.2. Note that each $C^{k,\sigma}$-map is weakly $C^{k,\sigma}$ (cf. Lemma 2.34).

Remark 6.3. Let $f\colon U \to F$ be a weakly $C^{k,\sigma}$-map on a subset $U \subseteq E$ with dense interior and $g\colon V \to U$ be a $C^{k,1}$-map on a subset V with dense interior of a topological $\mathbb{K}$-vector space H (e.g., a C^{k+1}-map). Then $\lambda \circ (f \circ g) = (\lambda \circ f) \circ g$ is $C^{k,\sigma}$ for each $\lambda \in F'$ (by Lemma 2.33) and thus $f \circ g$ is weakly $C^{k,\sigma}$.

Recall that a topological vector space over an ultrametric field is called *locally convex* if its vector topology can be defined by a family of ultrametric seminorms (cf. [16] for further information).

Lemma 6.4. *Let $(E, \|.\|)$ be a normed space over a locally compact field $\mathbb{K}$, F be a locally convex space over $\mathbb{K}$ and $f\colon K \to F$ be a map on a compact set $K \subseteq E$. Let $\sigma > 0$. Then the following conditions are equivalent:*

(a) *f is a $C^{0,\sigma}$-map.*

(b) *For each continuous seminorm q on F, there is $C \in [0, \infty[$ such that*

$$\|f(y) - f(x)\|_q \;\leq\; C\,(\|y - x\|)^\sigma \quad \textit{for all } x, y \in K. \tag{34}$$

(c) *f is weakly $C^{0,\sigma}$.*

Proof. (a)$\Rightarrow$(b): Let q be as in (b). If f is $C^{0,\sigma}$, then for each $z \in K$ there exists an open neighbourhood $U_z \subseteq K$ of z and a gauge p_z on E such that

$$\|f(y) - f(x)\|_q \;\leq\; (\|y - x\|_{p_z})^\sigma \quad \text{for all } x, y \in U_z.$$

Let V_z be an open neighbourhood of z in K with compact closure $\overline{V_z} \subseteq U_z$. There exists a finite subset $\Phi \subseteq K$ such that $K = \bigcup_{z \in \Phi} V_z$. For each $z \in \Phi$, there exists $r_z > 0$ such that $p_z \leq r_z\|.\|$ (cf. Lemma 2.21). Let $r := \max\{r_z\colon z \in \Phi\}$. The sets $\overline{V_z}$ and $K \setminus U_z$ being compact and disjoint, we can define

$$s \;:=\; \sup\{\|y - x\|^{-\sigma}\colon z \in \Phi,\, x \in \overline{V_z},\, y \in K \setminus U_z\} \;\in\; [0, \infty[\,.$$

Then (34) holds with $C := \max\{r^\sigma, 2s \max \|f(K)\|_q\}$. In fact, given $x, y \in K$, there exists $z \in \Phi$ such that $x \in V_z$. If $y \in U_z$, then $\|f(y) - f(x)\|_q \leq (\|y - x\|_{p_z})^\sigma \leq (r_z)^\sigma \|y - x\|^\sigma \leq C\|y - x\|^\sigma$. If $y \notin U_x$, also $\|f(y) - f(x)\|_q \leq \frac{\|f(y)\|_q + \|f(x)\|_q}{\|y - x\|^\sigma}\|y - x\|^\sigma \leq 2s \max \|f(K)\|_q \|y - x\|^\sigma \leq C\|y - x\|^\sigma$.

(b)$\Rightarrow$(a): Given a gauge g on F, by local convexity there exists a continuous seminorm q (which can be chosen ultrametric if $\mathbb{K}$ is a local field) such that $g \leq q$

(cf. Lemma 2.21). Let C be as in (b). Then $p := C^{\frac{1}{\sigma}}\|.\|$ is a gauge on E such that $\|f(y)-f(x)\|_g \leq \|f(y)-f(x)\|_q \leq (\|x-y\|_p)^\sigma$ for all $x, y \in K$. Thus (a) holds.

(a)$\Rightarrow$(c): See Remark 6.2.

(c)$\Rightarrow$(b): Pick $\rho \in \mathbb{K}^\times$ with $|\rho| < 1$ and define $\beta(r) := \rho^k$ for $r \in]0, \infty[$, where $k \in \mathbb{Z}$ is the unique integer such that $|\rho|^{k+1} < r^\sigma \leq |\rho|^k$. Then

$$|\rho| \cdot |\beta(r)| \ < \ r^\sigma \ \leq \ |\beta(r)| \quad \text{for all } r \in]0, \infty[. \tag{35}$$

If f is weakly $C^{0,\sigma}$, define

$$B \ := \ \left\{ \frac{f(y)-f(x)}{\beta(\|y-x\|)} \colon x, y \in K \text{ such that } x \neq y. \right\}$$

We claim that B is bounded. If this is true, then $M := \sup \|B\|_q < \infty$ for each continuous seminorm (or gauge) q on F (see Lemma 2.24) and hence

$$\|f(y)-f(x)\|_q \ = \ \frac{\|f(y)-f(x)\|_q}{|\beta(\|y-x\|)|}\, |\beta(\|y-x\|)| \ \leq \ M\, |\beta(\|y-x\|)| \ \leq \ M|\rho|^{-1}\|y-x\|^\sigma$$

for all $x, y \in K$ such that $x \neq y$, using (35) for the final inequality. Hence (34) holds with $C := M|\rho|^{-1}$.

It remains to show that B is bounded, or equivalently, that $\lambda(B) \subseteq \mathbb{K}$ is bounded for each $\lambda \in F'$ (see [17, Theorem 3.18] for the real case (from which the complex case follows) and [19, Theorem 4.21] for the case where $\mathbb{K}$ is a local field). However, for each $\lambda \in F'$, the map $\lambda \circ f \colon K \to \mathbb{K}$ is $C^{0,\sigma}$ and hence, by (a)$\Rightarrow$(b) already established, there exists $C \in [0, \infty[$ such that

$$|\lambda(f(y)) - \lambda(f(x))| \ \leq \ C\, \|y-x\|^\sigma \quad \text{for all } x, y \in K.$$

But then $\sup |\lambda(B)| \leq C$ (whence $\lambda(B)$ is bounded), since

$$\left|\lambda\Big(\frac{f(y)-f(x)}{\beta(\|y-x\|)}\Big)\right| = \frac{|\lambda(f(y)) - \lambda(f(x))|}{|\beta(\|y-x\|)|} \leq \frac{|\lambda(f(y)) - \lambda(f(x))|}{\|y-x\|^\sigma} \leq C \tag{36}$$

for all $x, y \in K$ such that $x \neq y$, using (35) to obtain the first inequality. ■

Remark 6.5. If $\mathbb{K}$ is a local field in the situation of Lemma 6.4, it suffices to consider *ultrametric* continuous seminorms in (b) (as the proof shows).

Recall that a topological vector space E over a topological field $\mathbb{K}$ is called *sequentially complete* if every Cauchy sequence in E is convergent. We say that E is *Mackey complete* if every Mackey-Cauchy sequence in E is convergent. Here, a sequence $(x_n)_{n \in \mathbb{N}}$ in E is called a *Mackey-Cauchy sequence* if there exists a bounded subset $B \subseteq E$ and elements $\mu_{n,m} \in \mathbb{K}$ such that $x_n - x_m \in \mu_{n,m} B$ for all $n, m \in \mathbb{N}$ and $\mu_{n,m} \to 0$ in $\mathbb{K}$ as both $n, m \to \infty$.

Note that every Mackey-Cauchy sequence also is a Cauchy sequence; hence every sequentially complete topological $\mathbb{K}$-vector space is Mackey complete. In the real locally convex case, Mackey completeness is a (particularly weak) standard completeness property, which is of great usefulness for infinite-dimensional calculus (see [12, notably §2] for an in-depth discussion).

Theorem 6.6. *Let $\mathbb{K} \neq \mathbb{C}$ be a locally compact field, E and F be topological $\mathbb{K}$-vector spaces, $f\colon U \to F$ be a map on an open set $U \subseteq E$, $k \in \mathbb{N}_0 \cup \{\infty\}$ and $\sigma \in \,]0,1]$. If E is metrizable and F is both Mackey complete and locally convex, then f is $C^{k,\sigma}$ if and only if f is weakly $C^{k,\sigma}$.*

Proof. The other implication being trivial, we only need to show that if f is weakly $C^{k,\sigma}$, then f is $C^{k,\sigma}$. As a consequence of Theorem 5.1, f will be $C^{k,\sigma}$ if we can show that $g := f \circ \gamma\colon \mathbb{K}^\ell \to F$ is $C^{k,\sigma}$ for each $\ell \in \mathbb{N}$ and each smooth map $\gamma\colon \mathbb{K}^\ell \to U$. Note that g is weakly $C^{k,\sigma}$ since so is f (see Remark 6.3). Hence, after replacing f with g, we may assume that $U = E = \mathbb{K}^\ell$ for some $\ell \in \mathbb{N}$. We may assume that $k \in \mathbb{N}_0$; the proof is by induction on k.

If $k = 0$ and $f\colon E = \mathbb{K}^\ell \to F$ is weakly $C^{0,\sigma}$, let $x \in E$ and $K \subseteq E$ be a compact neighbourhood of x. Then $f|_K$ is $C^{0,\sigma}$ by Lemma 6.4. Hence f is $C^{0,\sigma}$ locally and hence f is $C^{0,\sigma}$, by Lemma 2.35.

Induction step. If $k \geq 1$ and $f\colon E = \mathbb{K}^\ell \to F$ is weakly $C^{k,\sigma}$, given $x, y \in E$ choose a sequence $(t_n)_{n\in\mathbb{N}}$ of pairwise distinct elements in $\overline{B}_1^{\mathbb{K}}(0) \setminus \{0\}$ with $t_n \to 0$. Set

$$B \;:=\; \left\{ \frac{f^{]1[}(x,y,t_m) - f^{]1[}(x,y,t_n)}{\beta(|t_m - t_n|)} : n, m \in \mathbb{N} \right\},$$

where $\beta\colon \;]0,\infty[\to \mathbb{K}^\times$ is as in the proof of Lemma 6.4 (as well as ρ used to define β). Then $\lambda(B) \subseteq \mathbb{K}$ is bounded for each $\lambda \in F'$ and hence B is bounded (by [17, Theorem 3.18], resp., [19, Theorem 4.21]). In fact, since $\lambda \circ f$ is $C^{1,\sigma}$, it follows that $(\lambda \circ f)^{[1]}$ is $C^{0,\sigma}$. Applying now Lemma 6.4 to the restriction of $(\lambda \circ f)^{[1]}$ to the compact set $\{x\} \times \{y\} \times \overline{B}_1^{\mathbb{K}}(0)$, we find $C \in [0,\infty[$ such that

$$|(\lambda \circ f)^{[1]}(x,y,t) - (\lambda \circ f)^{[1]}(x,y,s)| \;\leq\; C|t-s|^\sigma \quad \text{for all } s,t \in \overline{B}_1^{\mathbb{K}}(0).$$

Repeating the calculation in (36), we find that $\sup|\lambda(B)| \leq C$. Hence B is indeed bounded.

Since $f^{]1[}(x,y,t_m) - f^{]1[}(x,y,t_n) \in \beta(|t_m - t_n|)B$, where B is bounded and $\beta(|t_m - t_n|) \to 0$ as both $n, m \to \infty$, we deduce that $(f^{]1[}(x,y,t_n))_{n\in\mathbb{N}}$ is a Mackey-Cauchy sequence in F and thus convergent; we let $g(x,y,0)$ be its limit. Then $\lambda(g(x,y,0)) = \lim_{n\to\infty}(\lambda \circ f)^{]1[}(x,y,t_n) = (\lambda \circ f)^{[1]}(x,y,0)$ for each λ. Furthermore, trivially $\lambda(g(x,y,t)) = (\lambda \circ f)^{[1]}(x,y,t)$ for $g(x,y,t) := f^{]1[}(x,y,t) = t^{-1}(f(x+ty) - f(x))$ if $(x,y,t) \in E \times E \times \mathbb{K}^\times$. Thus $\lambda \circ g = (\lambda \circ f)^{[1]}$ is $C^{k-1,\sigma}$ for each λ, whence g is $C^{k-1,\sigma}$, by induction. Hence f is $C^{1,\sigma}$, with $f^{[1]} = g$ a $C^{k-1,\sigma}$-map. Thus f is $C^{k,\sigma}$. ∎

A Details for Section 2

In this appendix, proofs are provided for the lemmas of Section 2.

Proof of Lemma 2.11. The assertions are obvious from our definitions of $U^{>\alpha<}$ and $f^{>\alpha<}$. ∎

Proof of Lemma 2.12. The 1-dimensional case of this lemma is well known (see [18, Exercise 29.A]). Having done this exercise (or not), the reader should not have difficulties to work out the details of the following sketch: We start with formula (3) for $f^{>\alpha<}(x)$, and split the sum $\sum_{j_i=0}^{\alpha_i}$ occurring there into the sum $\sum_{j_i=1}^{\beta_i}$, plus the two remaining summands with $j_i = 0$ and $j_i = \alpha_i$, respectively. In the sum $\sum_{j_i=1}^{\beta_i}$, rewrite the factor $\frac{1}{x_{j_i}^{(i)}-x_0^{(i)}} \cdot \frac{1}{x_{j_i}^{(i)}-x_{\alpha_i}^{(i)}}$ of the product involved in the summands as

$$\frac{1}{x_0^{(i)} - x_{\alpha_i}^{(i)}} \cdot \left(\frac{1}{x_{j_i}^{(i)} - x_0^{(i)}} - \frac{1}{x_{j_i}^{(i)} - x_{\alpha_i}^{(i)}} \right).$$

Finally, take $\frac{1}{x_0^{(i)}-x_{\alpha_i}^{(i)}}$ out of the sum and combine the summands to

$$\begin{aligned} &f^{>\beta<}(x^{(1)}, \ldots, x^{(i-1)}, x_0^{(i)}, \ldots, x_{\beta_i}^{(i)}, x^{(i+1)}, \ldots, x^{(d)}) \\ &\qquad -f^{>\beta<}(x^{(1)}, \ldots, x^{(i-1)}, x_1^{(i)}, \ldots, x_{\alpha_i}^{(i)}, x^{(i+1)}, \ldots, x^{(d)}) . \end{aligned}$$

After a reordering of the arguments in the second term with the help of Lemma 2.11, we obtain (5). ■

Proof of Lemma 2.14. The validity of (6) is clear from the definition of $U^{<\alpha>}$. Since $U^{>\alpha<}$ is dense in $U^{<\alpha>}$, if suffices to check (7) for $x \in U^{>\alpha<}$. But then (7) holds by Lemma 2.11. ■

Proof of Lemma 2.15. We let W be the set of all $x \in U^{<\alpha>}$ such that $x_0^{(i)} \neq x_{\alpha_i}^{(i)}$, and define $h(x)$ by (8) for $x \in W$. Then both $f^{<\alpha>}|_W$ and $h\colon W \to F$ are continuous and coincide with $f^{>\alpha<}$ on $U^{>\alpha<}$, by Lemma 2.12. Since $U^{>\alpha<}$ is dense in W, it follows that $f^{<\alpha>}|_W = h$. ■

Proof of Lemma 2.24. If $B \subseteq E$ is bounded and q is a gauge, then there exists $t \in \mathbb{K}^\times$ such that $tB \subseteq B_1^q(0)$. Thus $|t| \cdot \|x\|_q = \|tx\|_q < 1$ for all $x \in B$ and thus $\sup q(B) \leq |t|^{-1}$. Conversely, suppose that $q(B)$ is bounded for each gauge q. If $U \subseteq E$ is a 0-neighbourhood, there exists a gauge q on E such that $B_1^q(0) \subseteq U$. Choose $r > 0$ such that $r \sup q(B) < 1$. Then $B_r^{\mathbb{K}}(0) \cdot B \subseteq B_1^q(0) \subseteq U$, showing that B is bounded. ■

Proof of Lemma 2.27. (a) Given $x_0 \in U$, a gauge q on F, and $\varepsilon > 0$, we choose a gauge p on E and a neighbourhood $W \subseteq U$ of x_0 such that $\|f(y) - f(x)\|_q \leq (\|y - x\|_p)^\sigma$ for all $x, y \in W$. Define $\delta := \varepsilon^{\frac{1}{\sigma}}$. Then $f(W \cap B_\delta^p(x_0)) \subseteq B_{\delta^\sigma}^q(f(x_0)) = B_\varepsilon^q(f(x_0))$, as $\|f(y) - f(x_0)\|_q \leq (\|y - x_0\|_p)^\sigma$ for all $y \in W$. Hence f is continuous at x_0 (see [9, Lemma 1.27 (b)]).

(b) Since f is $C^{0,\sigma}$, given $x_0 \in U$ and a gauge q on F, we find a gauge p on E and a neighbourhood $W \subseteq U$ of x_0 such that $\|f(y) - f(x)\|_q \leq (\|y - x\|_p)^\sigma$ for all $x, y \in W$. Let s be a gauge on E such that $p(x+y) \leq s(x) + s(y)$ for $x, y \in E$. After replacing W with $W \cap B_{1/2}^s(x_0)$, we may assume that $\|y - x\|_p < 1$ for all $x, y \in W$. Then $\|f(y) - f(x)\|_q \leq (\|y - x\|_p)^\sigma = (\|y - x\|_p)^{\sigma-\tau}(\|y - x\|_p)^\tau \leq (\|y - x\|_p)^\tau$ for all $x, y \in W$, whence f is $C^{0,\tau}$.

(c) Let $x_0 \in U$. Given a gauge q on H, there exists a gauge p on F and a neighbourhood $R \subseteq V$ of $f(x_0)$ such that $\|g(y) - g(x)\|_q \leq (\|y - x\|_p)^\tau$ for all $x, y \in R$. There exists a gauge s on E and a neighbourhood $S \subseteq f^{-1}(R)$ of x_0 such that $\|f(y) - f(x)\|_p \leq (\|y - x\|_s)^\sigma$ for all $x, y \in S$. Then $\|g(f(y)) - g(f(x))\|_q \leq (\|f(y) - f(x)\|_p)^\tau \leq (\|y - x\|_s)^{\sigma \cdot \tau}$ for all $x, y \in S$.

(d) See [9, Lemma 2.5 (c)]. ■

Proof of Lemma 2.28. If f is not $C^{0,\sigma}$, then there exists $x_0 \in U$ and a gauge q_0 on F such that, for each neighbourhood $V \subseteq U$ of x_0 and gauge p on E, there are $x, y \in V$ such that $\|f(y) - f(x)\|_{q_0} > (\|y - x\|_p)^\sigma$. Let q be a gauge on F such that $q_0(u + v) \leq q(u) + q(v)$ for all $u, v \in F$. After replacing q with a larger gauge, we may assume that q is upper semicontinuous (cf. Remark 2.18 and Lemma 2.21). We now verify that x_0 and q have the desired properties. To this end, let $V \subseteq U$ be a neighbourhood of x_0 and p_0 be a gauge on E. Let $p \geq p_0$ be an upper semicontinuous gauge. Then there are $x, y \in V$ such that $\varepsilon := \|f(y) - f(x)\|_{q_0} - (\|y - x\|_p)^\sigma > 0$. Choose $r > \|y - x\|_p$ such that $r^\sigma \leq (\|y - x\|_p)^\sigma + \frac{\varepsilon}{2}$. Since $B^p_r(0)$ and $B^q_{\varepsilon/2}(0)$ are open and the relevant maps are continuous, we find $x', y' \in V \cap D$ such that $\|y' - x'\|_p < r$ and $\|f(y) - f(x) - f(y') + f(x')\|_q < \frac{\varepsilon}{2}$. Using the fake triangle inequality, we now obtain

$$\begin{aligned} \|f(y') - f(x')\|_q &\geq \|f(y) - f(x)\|_{q_0} - \|f(y) - f(x) - f(y') + f(x')\|_q \\ &> \|f(y) - f(x)\|_{q_0} - \frac{\varepsilon}{2} = (\|y - x\|_p)^\sigma + \frac{\varepsilon}{2} \\ &\geq (\|y' - x'\|_p)^\sigma \geq (\|y' - x'\|_{p_0})^\sigma , \end{aligned}$$

as desired. ■

Proof of Lemma 2.34. If $k = 0$, then $\lambda \circ f$ is $C^{0,\sigma}$ by Lemma 2.27 (c), exploiting that λ, being continuous linear, is C^∞_{BGN} and hence Lipschitz continuous. If f is $C^{k+1,\sigma}_{BGN}$, then $\lambda \circ f$ is $C^{k,\sigma}_{BGN}$ by induction, with $(\lambda \circ f)^{[k]} = \lambda \circ f^{[k]}$. Furthermore, $\lambda \circ f$ is C^{k+1}_{BGN}, by **2.3**. Now $(\lambda \circ f)^{[k+1]} = ((\lambda \circ f)^{[k]})^{[1]} = \lambda^{[1]} \circ \widehat{T}(f^{[k]}) = \lambda \circ (f^{[k]})^{[1]} = \lambda \circ f^{[k+1]}$ is $C^{0,\sigma}$, using **2.3** for the second equality, **2.2** for the third (cf. also [9, Remark 1.7]). Hence $\lambda \circ f$ is $C^{k+1,\sigma}_{BGN}$ with $(\lambda \circ f)^{[k+1]} = \lambda \circ f^{[k+1]}$ of the desired form. ■

Proof of Lemma 2.32. If f is $C^{k,\sigma}_{BGN}$, then also $f_i = \mathrm{pr}_i \circ f$ is $C^{k,\sigma}_{BGN}$, since pr_i is continuous linear (Lemma 2.34). Conversely, assume that each component f_i of $f \colon U \to \prod_{i \in I} F_i = F$ is $C^{k,\sigma}_{BGN}$. We proceed by induction.

The case $k = 0$. Given a gauge q on F, there exists a finite subset $J \subseteq I$ and balanced, open 0-neighbourhoods $W_j \subseteq F_j$ for $j \in J$ such that $W := \bigcap_{j \in J} \mathrm{pr}_j^{-1}(W_j) \subseteq B^q_1(0)$. We let $s := \mu_W \colon F \to [0, \infty[$ be the Minkowski functional of W (see Remark 2.18), and $s_j \colon F_j \to [0, \infty[$ be the Minkowski functional of W_j, for $j \in J$. Then $s(x) = \max\{s_j(x_j) \colon j \in J\}$ holds for each $x = (x_i)_{i \in I} \in F$. Furthermore, $q(x) \leq s(x)$. In fact, given $x \in F$ and $t \in \mathbb{K}^\times$ such that $x \in tW$, we have $q(x) = q(t(x/t)) = |t| \cdot q(x/t) \leq |t|$ (using that $W \subseteq B^q_1(0)$). Letting $|t| \to \mu_W(x)$, we see that $q(x) \leq \mu_W(x) = s(x)$. For each $j \in J$, there is a gauge p_j on E and a neighbourhood V_j of x_0 in U such that $\|f_j(y) - f_j(x)\|_{s_j} \leq (\|y - x\|_{p_j})^\sigma$ for all $x, y \in V_j$. Set $V := \bigcap_{j \in J} V_j$ and $p(x) := \max\{p_j(x) \colon j \in J\}$ for $x \in E$. Then p is a gauge on E such that $\|f(y) - f(x)\|_q \leq \|f(y) - f(x)\|_s = \max\{\|f_j(y) - f_j(x)\|_{s_j} \colon j \in J\} \leq (\|y - x\|_p)^\sigma$ for all $x, y \in V$.

Induction step. Assume that each component f_i is $C^{k,\sigma}_{BGN}$, where $k \geq 1$. Then f is C^1_{BGN}, with $f^{[1]} = (f_i^{[1]})_{i \in I}$ (cf. [2, Lemma 10.2]). The components $f_i^{[1]}$ of this map are $C^{k-1,\sigma}_{BGN}$, whence $f^{[1]}$ is $C^{k-1,\sigma}_{BGN}$, by induction. Hence f is $C^{k,\sigma}_{BGN}$. ■

Proof of Lemma 2.33. We proceed by induction on $k \in \mathbb{N}_0$. If $k = 0$, then Lemma 2.33 is a special case of Lemma 2.27 (c).

Induction step: If f is $C^{k,\sigma}_{BGN}$ and g is $C^{k,\tau}_{BGN}$ with $k \geq 1$, then $g \circ f$ is C^k_{BGN} and $(g \circ f)^{[1]} = g^{[1]} \circ \widehat{T}f$ with $\widehat{T}f \colon U^{[1]} \to V^{[1]} \subseteq F \times F \times \mathbb{K}$, $\widehat{T}f(x,y,t) := (f(x), f^{[1]}(x,y,t), t)$ (see **2.3**). Here the second component of $\widehat{T}f$ is $C^{k-1,\sigma}_{BGN}$; the final component is continuous linear and hence $C^{\infty,\sigma}_{BGN}$ (since $\sigma \leq 1$); and the first component is a composition of the $C^{k,\sigma}_{BGN}$-map f and (a restriction of) the continuous linear (and hence $C^{\infty,1}_{BGN}$-) mapping $E \times E \times \mathbb{K} \to E$, $(x,y,t) \mapsto x$, whence also the first component is $C^{k-1,\sigma}_{BGN}$, by the case $k-1$ (valid by induction). Now Lemma 2.32 shows that $\widehat{T}f$ is $C^{k-1,\sigma}_{BGN}$, and thus $(g \circ f)^{[1]} = g^{[1]} \circ \widehat{T}f$ is $C^{k-1,\sigma\cdot\tau}_{BGN}$, by induction. Hence $g \circ f$ is $C^{k,\sigma\cdot\tau}_{BGN}$. ∎

Proof of Lemma 2.35. We may assume that $k \in \mathbb{N}_0$; the proof is by induction. If $k = 0$ and $f|_{U_i}$ is $C^{0,\sigma}$ for each $i \in I$, then f is $C^{0,\sigma}$, as is obvious from the definition. Induction step: If $f|_{U_i}$ is $C^{k+1,\sigma}_{BGN}$, then f is $C^{k,\sigma}_{BGN}$ by induction, and furthermore f is C^1_{BGN} (by Lemma 2.4). The sets $U_i^{[1]}$ together with $U^{]1[}$ form an open cover for $U^{[1]}$, and $f^{[1]}|_{U_i^{[1]}} = (f|_{U_i})^{[1]}$ is $C^{k,\sigma}_{BGN}$ for each $i \in I$. For $(x,y,t) \in U^{]1[}$, we have $f^{[1]}(x,y,t) = \frac{f(x+ty)-f(x)}{t}$; since f is $C^{k,\sigma}_{BGN}$, we deduce with Lemma 2.33 from the preceding formula that $f^{[1]}|_{U^{]1[}}$ is $C^{k,\sigma}_{BGN}$. Applying the inductive hypothesis, we see that $f^{[1]}$ is $C^{k,\sigma}_{BGN}$. Hence f is $C^{k+1,\sigma}_{BGN}$. ∎

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