Indecomposable operators on Form Hilbert Spaces

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Abstract

The class of orthomodular spaces described by Gross and Künzi based on H. Keller's work is a generalization of classic Hilbert spaces. Let E be an orthomodular space in this class, endowed with a positive form ϕ. As in Hilbert spaces, ϕ induces a topology on E making it a complete space. For every $n \in \mathbb{N}$, we describe definite spaces (E_n, ϕ_n), with $\dim(E_n) = 2^n$ over the base field $K_n = \mathbb{R}((\chi_1, \ldots, \chi_n))$, and we build a family of selfadjoint and indecomposable operators. Later we build an orthomodular definite space (E, ϕ) with infinite dimension and we also prove that the sequence of operators in this family induces a bounded, selfadjoint and indecomposable operator in (E, ϕ).

1 Preliminaries.

In this chapter we will study the fields that we are going to use as base fields of different vector spaces, including finite rank and infinite rank, vector spaces with finite dimension, and finally an orthomodular vector space with infinite dimension.

1.1 Fields of Generalized Power Series.

Let $(G, +)$ be an ordered additive group and $K = \mathbb{R}((G))$ be the field of generalized power series with real coefficients, that is, the set of all maps $\alpha : G \longrightarrow \mathbb{R}$ such that $supp(\alpha) = \{g \in G : \ \alpha(g) \neq 0\}$ is a well ordered set. Addition and multiplication are defined as follows:

$$(\alpha + \beta)(g) = \alpha(g) + \beta(g)$$

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$$(\alpha \cdot \beta)(g) = \sum_{g'+g''=g} \alpha(g') \cdot \beta(g'')$$

for all $\alpha, \beta \in K$ and $g, g', g'' \in G$.

Let τ^g be the characteristic function of $\{g\} \subseteq G$. Then $\alpha \in K$ can be represented by the following expression:

$$\alpha = \sum_{g \in G} a_g \tau^g$$

with $a_g := \alpha(g) \in \mathbb{R}$.

On one hand, it is possible to order K in the following way:

If $\alpha \in K$ then $\alpha = \sum\limits_{g \in G} a_g \tau^g \in K$ with $g_0 = \min supp(\alpha)$, we say that α is positive in the order of K if and only if a_{g_0} is positive as a real number.

On the other hand, the function $v : K \longrightarrow G \cup \{\infty\}$ defined by:

$$v(\alpha) := \min supp(\alpha) \quad \text{para } \alpha \neq 0 \text{ and } v(0) = \infty$$

is a valuation in the sense of Krull, it means:

a) $v(\alpha) = \infty$ if and only if $\alpha = 0$,

b) $v(\alpha \cdot \beta) = v(\alpha) + v(\beta)$, and

c) $v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}$.

This order defined in K is compatible with the valuation in the sense that for all $\alpha, \beta \in K$ if $0 \leq \alpha \leq \beta$ then $v(\beta) \leq v(\alpha)$.

We can also deal with K as a topological field considering

$$U_g(\alpha) = \{x \in K : \ v(x - \alpha) > g\}$$

as a base for open sets at $\alpha \in K$, for any $g \in G$. With this topology K is complete.

1.2 The E_n vector space over the field K_n.

a) **The valued field K_n.** We will consider the additive group $G_n = \bigoplus_{i=1}^{n} \mathbb{Z}$. G_n can be ordered antilexicographically, that is, if $h = (h_1, \ldots, h_n) \neq f = (f_1, \ldots, f_n) \in G_n$ and $k = \max\{j \in \{1, 2, \ldots, n\} : h_i \neq f_i\}$, then $f \leq h$ if and only if $f_k < h_k$. Now we can define $K_n = \mathbb{R}((G_n))$ as the field of generalized power series with real coefficients.

Remark 1.1 *The field $K_1 = \mathbb{R}((G_1))$ is isomorphic to $\mathbb{R}((t_1))$ the field of Laurent series over t_1. Moreover, $K_n \cong \mathbb{R}((t_1, \ldots, t_n))$*

Lemma 1.2 *[2] Let $K_n = \mathbb{R}((G_n))$. For $i = 1, 2, \ldots, n$ let $h_i = (0, .., 1, 0, .., 0)$ and $\chi_i := \tau^{h_i} \in K_n$. Then*

$$\Sigma_n := \{\chi_1^{\epsilon_1} \chi_2^{\epsilon_2} \cdots \chi_n^{\epsilon_n} : \quad \epsilon_1, \ldots, \epsilon_n \in \{0, 1\}\}$$

is a set of representatives of the positive squares classes of K_n.

Remark 1.3 *As K_n is an ordered field, it is possible to order Σ_n from the biggest to the smallest representative as follow:*

$$\Sigma_n = \{\tau_1 = 1, \tau_2 = \chi_1, \tau_3 = \chi_2, \dots, \tau_{2^n} = \chi_1\chi_2\cdots\chi_n\}$$

b) **The vector space E_n.**

Definition 1.4 $E_n = \{(\xi_1, \dots, \xi_{2^n}) \in K_n^{2^n}\}$ *is the K_n-vector space with* $\dim(E_n) = 2^n$.

We write $e_1^n, e_2^n, \dots, e_{2^n}^n$ as the vectors of the canonical base of E_n. E_n is endowed with a symmetric positive form $\phi_n : E_n \times E_n \longrightarrow K_n$ defined by:

For $i, j \in \{1, \dots, 2^n\}$:

a) If $i \neq j$, then $\phi_n(e_i^n, e_j^n) = 0$, and

b) $\phi_n(e_i^n, e_i^n) = \tau_i \in \Sigma_n$ (see 1.3).

c) **Operators in $L(E_n)$.**

Definition 1.5 *Let (E, ϕ) be a vector space endowed with an inner product and*
$B : E \longrightarrow E$ a linear operator, we say that B is decomposable if E is an orthogonal addition of two non-trivial invariant subspaces: $E = E_1 \oplus E_2$. In that case E admits an orthogonal base in which the matrix of B is decomposed into two blocks

$$\mathcal{B} = \begin{pmatrix} \mathcal{B}_1 & 0 \\ 0 & \mathcal{B}_2 \end{pmatrix}.$$

We deal with decompositions of operators B which are selfadjoint, that is, $\phi(B(x), y) = \phi(x, B(y))$ for all $x, y \in E$. Such an operator is decomposable if and only if E admits a non-trivial subspace E_1, for then $E_2 = E_1^{\perp}$ is also invariant.

We are interested about proving the decomposability or indecomposability of selfadjoint linear operators in E_n. Next lemma explains why the dimension of the space E_n as a power of 2.

Lemma 1.6 *([2], Lemma 2.6) Let K_n be the base field and (E, ϕ) a positive defined vector space. If* $\dim(E)$ *is not a power of 2, then any selfadjoint linear operator $B : E \longrightarrow E$ is decomposable.*

In [2] Keller and Ochsenius had built for every $n \in \mathbb{N}$ a selfadjoint, indecomposable linear operator defined in (E_n, ϕ_n) and in [7] is proved that this collection induces an operator in infinite dimension also selfadjoint and indecomposable. The aim of this paper is to generalize this construction.

Lemma 1.7 *Let E be a vector space with finite dimension over a field K. If $B : E \longrightarrow E$ is a decomposable operator, then the characteristic polynomial is reducible.*

Proof: The proof is straightforward and will be omitted.

2 The K-vector space E.

As this work can be divided into two parts, on one hand the part developed in finite dimension and, on the other hand the infinite dimension one, we are going to indicate some previous results about the vector space with infinite dimension (see[7]).

Let $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}$ ordered antilexicographically, and $K = \mathbb{R}((G))$ the field of generalized power series over $\mathbb{R}$. As in the preliminaries, K is a non-archimedean valued field in the sense of Krull, Henselian, topological and ordered, consistently with the valuation. Let Σ be a complete set of representatives of the positive square classes of $\dot{K}$, ordered decreasingly:

$$\Sigma = \{\tau_1 = \tau^{(0,0,\ldots)}, \tau_2 = \tau^{(1,0,0,\ldots)}, \tau_3 = \tau^{(0,1,0,0,\ldots)}, \tau_4 = \tau^{(1,1,0,0,\ldots)}, \ldots\}$$

Now, we are ready to define the K-vector space E (see[4]):

$$E := \{(\xi_i)_{i\in\mathbb{N}} \in K^{\mathbb{N}} : \sum_{i=1}^{\infty} \xi_i^2 \tau_i \text{ converges in the valuation topology } \},$$

E is a K-vector space with addition and scalar multiplication by components. It is possible endow E with a positive form $\phi : E \times E \longrightarrow K$ by:

$$\phi((\xi_i)_{i\in\mathbb{N}}, (\eta_i)_{i\in\mathbb{N}}) = \sum_{i=1}^{\infty} \xi_i \eta_i \tau_i.$$

Remark 2.1 *It is important to realize that for every vector in the canonical base of E, $e_i = (0, \ldots, 1, 0, \ldots)$, we have that $\phi(e_i, e_i) = \tau_i$, and as τ_i is not a square in G, except when $i = 1$, it is clear that the base is orthogonal but not orthonormal.*

Definition 2.2 *Let L be a vector space over an arbitrary skew field endowed with a hermitian form ϕ. Then (E, ϕ) is called orthomodular if*

$$X \subseteq L, \quad X = X^{\perp\perp} \Rightarrow L = X \oplus X^{\perp}$$

holds true.

Theorem 2.3 ([4], Theorem 1) *Let G, K and E as before, then:*

a) (E, ϕ) is an orthomodular space.

b) The map $||\cdot|| : E \longrightarrow G$ defined by $x \mapsto v(\phi(x, x))$ is a non-archimedean norm. The form ϕ is continuous in the norm topology. E is complete in the norm topology.

c) A linear subspace U of E is closed in the norm topology if and only if it is orthogonally closed.

d) If x, y are orthogonal, $x \perp y$, then $||x|| \neq ||y||$. Consequently (E, ϕ) is not isometric to any proper subspace.

3 Building a sequence of indecomposable and selfadjoint operators in finite dimension.

Now that we have finished with the preliminaries we can start building our operators.

Let $\mathcal{I}_n$ be the identity matrix with dimension $2^n \times 2^n$. Then we define T_1 by its matrix, over the canonical base, as follows:

$$\mathcal{T}_1 := \begin{pmatrix} 0 & \chi_1 \\ 1 & 0 \end{pmatrix},$$

and for $n \geq 1$, T_{n+1} recursively

$$\mathcal{T}_{n+1} := \begin{pmatrix} \mathcal{T}_n & \chi_{n+1} \cdot (\mathcal{T}_n - \mathcal{I}_n) \\ & \\ \mathcal{T}_n - \mathcal{I}_n & \mathcal{T}_n \end{pmatrix}.$$

The first step is to prove that $\{T_n\}_{n \in \mathbb{N}}$ is selfadjoint. In order to do this we have the next theorem.

Theorem 3.1 *Let $A, B, C : (E_n, \phi_n) \longrightarrow (E_n, \phi_n)$ be three operators with matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}$ respectively. Then $D : (E_{n+1}, \phi_{n+1}) \longrightarrow (E_{n+1}, \phi_{n+1})$ defined by*

$$\mathcal{D} = \begin{pmatrix} \mathcal{A} & \chi_{n+1} \cdot \mathcal{B} \\ & \\ \mathcal{B} & \mathcal{C} \end{pmatrix}$$

is selfadjoint with respect to (E_{n+1}, ϕ_{n+1}) every time that A, B and C are selfadjoint in (E_n, ϕ_n).

Proof: The proof follows directly from the hypothesis.

Therefore,

Corollary 3.2 *For $n \geq 1$, $T_n : (E_n, \phi_n) \longrightarrow (E_n, \phi_n)$ is a selfadjoint operator.*

Proof: The proof will be omitted, since it is enough to use induction and the previous theorem.

Now, it is our aim to prove that every operator in the sequence is indecomposable. In order to obtain it we have some previous lemma.

Lemma 3.3 *[8] Let E be a vector space with finite dimension over a field K endowed with an inner product ϕ. If $B : E \longrightarrow E$ is selfadjoint, decomposable linear operator, then its characteristic polynomial is reducible in $K[x]$.*

Lemma 3.4 *For all $n \geq 1$, the operator T_n does not have eigenvalues in K_n.*

Proof: We prove by induction that every operator T_n does not have eigenvalues in K_n but it has them in $\overline{K}_n = K_n(\sqrt{\chi_1}, \sqrt{\chi_2}, \ldots, \sqrt{\chi_n})$, and here they are all different.

If $n = 1$,

$$\mathcal{T}_1 = \begin{pmatrix} 0 & \chi_1 \\ 1 & 0 \end{pmatrix}$$

and its characteristic polynomial is $p_1(\lambda) = \lambda^2 - \chi_1$. Then its eigenvalues are $\lambda_1 = \sqrt{\chi_1}$ and $\lambda_2 = -\sqrt{\chi_1}$. Those elements do not belong to K_1, because in that case χ_1 should be a square and that is not true.

For $n \geq 1$,

$$\mathcal{T}_{n+1} = \begin{pmatrix} \mathcal{T}_n & \chi_{n+1} \cdot (\mathcal{T}_n - \mathcal{I}_n) \\ \mathcal{T}_n - \mathcal{I}_n & \mathcal{T}_n \end{pmatrix}.$$

Let $p_n(\lambda)$ be the characteristic polynomial of $\mathcal{T}_n$ and C_n an extension of K_n that has all the 2^n roots of $p_n(\lambda)$, $\{\lambda_1, \ldots, \lambda_{2^n}\}$. By hypothesis the 2^n roots of $p_n(\lambda)$ are different and they do not belong to K_n. But in C_n, $\mathcal{T}_n$ can be diagonalized and:

$$R_n^{-1} \cdot \mathcal{T}_n \cdot R_n = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{2^n} \end{pmatrix},$$

for some invertible matrix R_n with coefficients in C_n.

Now, we have to find all the roots of $p_{n+1}(\lambda)$.
We call U_n and U_n^* to:

$$U_n = \begin{pmatrix} R_n & 0 \\ 0 & R_n \end{pmatrix} \text{ and } U_n^* = \begin{pmatrix} R_n^{-1} & 0 \\ 0 & R_n^{-1} \end{pmatrix}.$$

Then:

$p_{n+1}(\lambda)$

$= det(T_{n+1} - \lambda \cdot \mathcal{I}_{n+1})$

$= det(U_n^* \cdot \mathcal{T}_{n+1} \cdot U_n - \lambda \cdot \mathcal{I}_{n+1})$

$$= \begin{vmatrix} R_n^{-1} \cdot \mathcal{T}_n \cdot R_n - \lambda \cdot \mathcal{I}_n & \chi_{n+1} \cdot R_n^{-1} \cdot (\mathcal{T}_n - \mathcal{I}_n) \cdot R_n \\ R_n^{-1} \cdot (\mathcal{T}_n - \mathcal{I}_n) \cdot R_n & R_n^{-1} \cdot \mathcal{T}_n \cdot R_n - \lambda \cdot \mathcal{I}_n \end{vmatrix}$$

$$= \begin{vmatrix} \lambda_1 - \lambda & 0 & \ldots & 0 & (\lambda_1 - 1)\chi_{n+1} & 0 & \ldots & 0 \\ 0 & \lambda_2 - \lambda & \ldots & 0 & 0 & (\lambda_2 - 1)\chi_{n+1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{2^n} - \lambda & 0 & 0 & \cdots & (\lambda_{2^n} - 1)\chi_{n+1} \\ \lambda_1 - 1 & 0 & \ldots & 0 & \lambda_1 - \lambda & 0 & \ldots & 0 \\ 0 & \lambda_2 - 1 & \ldots & 0 & 0 & \lambda_2 - \lambda & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{2^n} - 1 & 0 & 0 & \cdots & \lambda_{2^n} - \lambda \end{vmatrix}$$

and solving this system we obtain:

$$p_{n+1}(\lambda) = [(\lambda-\lambda_1)^2-(\lambda_1-1)^2\chi_{n+1}]\,[(\lambda-\lambda_2)^2-(\lambda_2-1)^2\chi_{n+1}] \cdots [(\lambda-\lambda_{2^n})^2-(\lambda_{2^n}-1)^2\chi_{n+1}]$$

Therefore the eigenvalues of $\mathcal{T}_{n+1}$ are all the solutions of the equations

$$(\lambda - \lambda_i)^2 - (\lambda_i - 1)^2 \cdot \chi_{n+1} = 0, \quad \text{with } i = 1, \ldots, 2^n.$$

That means that they have the form:

$$\lambda_i^{(l)} = \lambda_i + l\sqrt{\chi_{n+1}} \cdot (\lambda_i - 1)$$

where $l \in \{-1, 1\}$ and $i \in \{1, \ldots, 2^n\}$.

As $\sqrt{\chi_{n+1}} \notin K_{n+1}(\lambda_1, \ldots, \lambda_{2^n})$, then we have $\lambda_i^{(l)} \notin K_{n+1}$. Thus, $\mathcal{T}_{n+1}$ has no eigenvalues in K_{n+1}, and all the 2^{n+1} roots of $p_{n+1}(\lambda)$ are different. ■

Classically in Galois theory we have that:

Lemma 3.5 *Let F be a field and $r_1, \ldots, r_m$, in the algebraic closure of F, the m different roots of a polynomial $p(x) \in F[x]$ such that $deg\{p\} = m$. Also suppose that $r_i \notin F$ for $i = 1, \ldots, m$. Let $K = F(r_1, \ldots, r_m)$ be an extension of F and $G = \mathcal{G}al_F(K)$. Then $p(x)$ is irreducible over $F[x]$ if and only if for every pair r_i, r_j there is an automorphism $g \in G$ such that $g(r_i) = r_j$.*

We have shown how to prove the irreducibility of a polynomial, now we will show what happens when we change the base field.

Lemma 3.6 *Let $s(x) \in K_n[x]$ be a polynomial with $deg(s) > 1$. If $s(x)$ is irreducible in $K_n[x]$, then so it is over $K_{n+1}[x]$.*

Proof: The proof relies directly on the fact that K_{n+1} is a transcendental extension of K_n.

Finally we are ready to prove the main theorem in this section:

Theorem 3.7 *For all $n \geq 1$, the operator T_n is indecomposable.*

Proof: By Lemma 3.3, it is enough to prove that its characteristic polynomial, $p_n(\lambda)$, is irreducible. And now we proceed by induction:

For $n = 1$, we have that $p_1(\lambda) = \lambda^2 - \chi_1$ is irreducible because $\sqrt{\chi_1} \notin K_1$.

For $n \geq 1$, let suppose that $p_n(x)$ is irreducible over K_n. Let consider J_n as the splitting field of $p_n(x)$ over K_{n+1} and let G' be the Galois group $Gal_{K_{n+1}}(J_n)$. By 3.6 $p_n(x)$ is irreducible over $K_{n+1}[x]$ and by lemma 3.5 we have for every pair of roots λ_i, λ_j of $p_n(x)$ there is $g' \in G'$ such that $g'(\lambda_i) = \lambda_j$. Let $C_{n+1} = K_{n+1}(\lambda_1, \ldots, \lambda_{2^n}, \sqrt{\chi_{n+1}})$ the splitting field of $f_{n+1}(x) = x^2 - \chi_{n+1}$ over J_n. Then $|C_{n+1} : K_{n+1}| = |C_{n+1} : J_n| \cdot |J_n : K_{n+1}| = 2 \cdot 2^n = 2^{n+1}$. We can notice that C_{n+1} is also the splitting field of $p_{n+1}(x)$ over K_{n+1} and, for every $g' \in G'$ there is $g \in G = Gal_{K_{n+1}}(C_{n+1})$ such that $g|_{J_n} = g'$.

Our purpose is to prove the irreducibility of $p_{n+1}(x)$ over $K_{n+1}[x]$. By Lemma 3.5 is enough to prove that for every pair of roots of this polynomial it is possible to find an automorphism mapping one root into the other one, by Lemma 3.4 every root of $p_{n+1}(x)$ has the following form:

$$\lambda_i^{(l)} = \lambda_i + l\sqrt{\chi_{n+1}}(\lambda_i - 1)$$

In order to prove the theorem consider two roots $\lambda_i^{(l)}, \lambda_j^{(l')}$ and $g' \in G'$ such that $g'(\lambda_i) = \lambda_j$, and the homomorphism $g : C_{n+1} \longrightarrow C_{n+1}$ such that $g|_{J_n} = g'$ and $g(\sqrt{\chi_{n+1}}) = ll'\sqrt{\chi_{n+1}}$. Then g defined as before is an automorphism of $Gal_{K_{n+1}}(C_{n+1})$ and:

$$\begin{aligned} g(\lambda_i^{(l)}) &= g(\lambda_i + l\sqrt{\chi_{n+1}}(\lambda_i - 1)) \\ &= g(\lambda_i) + lg(\sqrt{\chi_{n+1}})(g(\lambda_i) - 1) \\ &= g'(\lambda_i) + lg(\sqrt{\chi_{n+1}})(g'(\lambda_i) - 1) \\ &= \lambda_j + l^2 l'\sqrt{\chi_{n+1}}(\lambda_j - 1) \\ &= \lambda_j + l'\sqrt{\chi_{n+1}}(\lambda_j - 1) \\ &= \lambda_j^{(l')} \end{aligned}$$

By Lemma 3.5 we have that $p_{n+1}(x)$ is irreducible over $K_{n+1}[x]$.

Then, by induction, for all $n \geq 1$ the polynomial $p_n(x)$ is irreducible and by Lemma 3.3 the operator T_n is indecomposable. ∎

4 An indecomposable operator over E.

Now we build an operator defined in E such that it is bounded, selfadjoint and indecomposable. But first we must study the residual spaces of E.

4.1 Residual Spaces.

The isolated subgroups of $G = \bigoplus_{i=1}^{\infty}\mathbb{Z}$ are $\Delta_0 \subset \Delta_1 \ldots$ defined for every $n \in \mathbb{N} \cup \{0\}$ by $\Delta_n = \bigoplus_{i=1}^{\infty} D_i$ where $D_i = \mathbb{Z}$ if $i \leq n$ and $D_i = \{0\}$ if $i > n$. Each of these subgroups has associated a valuation ring R_n defined by:

$$R_n = \{\xi \in K : \ v(\xi) \geq \delta \text{ for some } \delta \in \Delta_n\}$$

and its unique maximal ideal is

$$J_n = \{\xi \in K : \ v(\xi) > \delta \text{ for all } \delta \in \Delta_n\}.$$

Then $\hat{K}_n := R_n/J_n$ (with $\theta_n : R_n \longrightarrow \hat{K}_n$, the canonical projection) is the residual field related to Δ_n.

Lemma 4.1 *For all $n \geq 1$ $\hat{K}_n \cong K_n$.*

Remark 4.2 *In this section we analyze the residual spaces of E as well as the residual fields of K. Concerning the latter ones, it is possible to prove that they are isomorphic to the fields that we have deal with in Section 2.2. For that reason it is natural to define E_n in the way that it has been done.*

Note that

$$M_n := \{x \in E : \ ||x|| \geq \delta \text{ for some } \delta \in \Delta_n\}$$

is a module over R_n and

$$S_n := \{x \in E : \ ||x|| > \delta \text{ for all } \delta \in \Delta_n\}$$

is a submodule.

Then $\hat{E}_n := M_n/S_n$ (with $\pi_n : M_n \longrightarrow \hat{E}_n$, the canonical projection) is a $\hat{K}_n$-vector space by defining scalar multiplication in the following way:

$$\pi_n(\xi x) := \theta_n(\xi)\pi_n(x)$$

when $x \in M_n$ and $\xi \in R_n$.

$\hat{E}_n$ is endowed with an inner product defined by:

$$\hat{\phi}_n(\pi_n(x), \pi_n(y)) := \theta_n(\phi(x, y))$$

with $x, y \in M_n$.

Lemma 4.3 *$\hat{E}_n$ is isomorphic to E_n, and we call the isomorphism π_n.*

It is also possible project a subspace $U \subseteq E$ under π_n to $\hat{E}_n$ by:

$$\pi_n(U) := \{\pi_n(x) : \ \ x \in U \cap M_n\}.$$

Lemma 4.4 [6] *If two sub-spaces $U, W \subseteq E$ are orthogonal, $U \perp W$, then $\pi_n(U) \perp \pi_n(W)$ and $\pi_n(U \oplus W) = \pi_n(U) \oplus \pi_n(W)$.*

4.2 Bounded linear operators in E.

Now we define an infinite matrix that will represent a linear operator. For this operator it is our aim to prove that it is bounded, selfadjoint and indecomposable.

Definition 4.5 *A linear operator $B : E \longrightarrow E$ is bounded if there is $g \in G$ such that*

$$||B(x)|| - ||x|| \geq g$$

for all $x \in E \setminus \{0\}$.

Lemma 4.6 [6](3.1) *A map $B_0 : \{e_i : \ \ i \in \mathbb{N}\} \longrightarrow E$ can be extended to a bounded linear operator $B : E \longrightarrow E$ if and only if the set $\{||B_0(e_i)|| - ||e_i|| : \ \ i \in \mathbb{N}\}$ is bounded from below.*

4.3 The operator T.

In chapter 2 we built a sequence of selfadjoint, indecomposable and bounded operators over the residual spaces (E_n, ϕ_n) of (E, ϕ). Now we will see how this sequence induces a linear operator defined on E.

Definition 4.7 *Let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence of matrices where $P_n = (p^n_{ij})$, and such that for all $n \in \mathbb{N}$*

a) $P_n \in Mat_{2^n}(K_n)$,

b) *if* $i, j \leq 2^n$ *then* $p_{ij}^{n+1} = p_{ij}^n$.

We define the ***final matrix*** *as the sequence* $\{P_n\}_{n\in\mathbb{N}}$ *given by the infinite matrix* P *defined by:*

$$P = (p_{ij})$$

where, for given $i, j \in \mathbb{N}$ *we take any* $m \in \mathbb{N}$ *such that*$i, j \leq 2^m$, *and we define* $p_{ij} := p_{ij}^m$.

We define the matrix $\mathcal{T}$ as the final matrix of the sequence $\{\mathcal{T}_n\}_{n\in\mathbb{N}}$ from chapter 2 and we will prove that it represents a bounded, selfadjoint and indecomposable operator in (E, ϕ).

Using Theorem 4.6 and the next lemma we conclude that $\mathcal{T}$ induces a bounded operator in E.

Lemma 4.8 *For all* $n \in \mathbb{N}$ *it is true that* $||T_n(e_i^n)|| - ||e_i^n|| = v(\chi_1) > 0$ *for* $i \in \{1, \ldots, 2^n\}$.

Proof: Using induction a proof for this lemma is readily built.

Theorem 4.9 $\mathcal{T}$ *defines a bounded operator* T *in* E.

Proof: In fact we prove that T is bounded from zero. Let us suppose that there is $i \in \mathbb{N}$ such that $||T(e_i)|| - ||e_i|| < 0$. Let k be such that $v(t_{ki}^2 \tau_k) = ||T(e_i)||$. Then there is $n \in \mathbb{N}$ such that $i, k < 2^n$, therefore $v((t_{ki}^n)^2 \tau_k) = ||T_n(e_i^n)||$ and $||e_i|| = ||e_i^n||$, then

$0 > ||T(e_i)|| - ||e_i|| = ||T_n(e_i^n)|| - ||e_i^n||$, arriving to a contradiction with the previous lemma. ■

From Theorem 4.9 we have that for all $n \in \mathbb{N} \cup \{0\}$, $T(M_n) \subseteq M_n$ and $T(S_n) \subseteq S_n$, then T induces an operator

$$\begin{array}{rccc} \hat{T}_n : & \hat{E}_n & \longrightarrow & \hat{E}_n \\ & \pi_n(x) & \longmapsto & \pi_n(T(x)) \quad (x \in M_n) \end{array}$$

Lemma 4.10 $T : (E, \phi) \longrightarrow (E, \phi)$ *is selfadjoint.*

Therefore T is a selfadjoint and bounded operator in (E, ϕ). Now, in order to close the chapter we will show that no closed subspace of E is left invariant by T.

Theorem 4.11 T *is indecomposable.*

Proof: Let suppose that T leaves invariant a nontrivial closed subspace U. By orthomodularity it is possible to write $E = U \oplus U^\perp$. As U and $U^\perp$ are not trivial there exist $x \in U$ and $z \in U^\perp$, both non zero and $n \in \mathbb{N}$ such that $\pi_n(x), \pi_n(z) \neq 0$.

We write $E_n \cong \hat{E}_n = \pi_n(E) = \pi_n(U) \oplus \pi_n(U^\perp)$. By Lemma 4.4, $\pi_n(U)$ and $\pi_n(U^\perp)$ are different to $\{0\}$, then $\pi_n(U)$ is invariant under $\hat{T}_n$ arriving to a contradiction with Theorem 3.7. Hence T is indecomposable. ■

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