Riemann-Stieltjes operators on Hardy spaces in the unit ball of $\mathbb{C}^n$

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Abstract

Let $g : B \to \mathbb{C}^1$ be a holomorphic map of the unit ball B. We study the integral operators

$$T_g f(z) = \int_0^1 f(tz)\Re g(tz)\frac{dt}{t}; \quad L_g f(z) = \int_0^1 \Re f(tz) g(tz)\frac{dt}{t}, \qquad z \in B.$$

The boundedness and compactness of the operators T_g and L_g on the Hardy space H^2 in the unit ball are discussed in this paper.

1 Introduction

Let $B = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball in $\mathbb{C}^n$, $S = \partial B = \{z \in \mathbb{C}^n : |z| = 1\}$ be its boundary, $d\nu$ the normalized Lebesgue measure of B, i.e. $\nu(B) = 1$, and $d\sigma$ the normalized surface measure on ∂B. Let $H(B)$ denote the class of all holomorphic functions on the unit ball. For $f \in H(B)$ with the Taylor expansion $f(z) = \sum_{|\beta|\geq 0} a_\beta z^\beta$, let $\Re f(z) = \sum_{|\beta|\geq 0} |\beta| a_\beta z^\beta$ be the radial derivative of f, where $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ is a multi-index and $z^\beta = z_1^{\beta_1} \cdots z_n^{\beta_n}$. It is well known that $\Re f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)$, see, for example, [22].

The Hardy space $H^p = H^p(B)$ $(0 < p < \infty)$ is defined on B by

$$H^p(B) = \Big\{ f \,|\, f \in H(B) \quad \text{and} \quad ||f||_{H^p} = \sup_{0\leq r<1} M_p(f, r) < \infty \Big\},$$

where

$$M_p(f, r) = \left(\int_{\partial B} |f(r\zeta)|^p d\sigma(\zeta) \right)^{1/p}.$$

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It is well known that $f \in H^2$ if and only if (see [22])

$$||f||_{H^2}^2 \asymp |f(0)|^2 + \int_B |\Re f(z)|^2 (1-|z|^2) d\nu(z) < \infty. \tag{1}$$

The $BMOA$ space consists of all $f \in H^2$ satisfying the condition (see [22])

$$\|f\|_{BMOA}^2 = |f(0)| + \sup \frac{1}{\sigma(Q)} \int_Q |f - f_Q|^2 d\sigma < \infty,$$

where f_Q denotes the averages of f over Q and the supremum is taken over all

$$Q = Q(\xi, \delta) = \{\eta \in S : |1 - \langle \eta, \xi \rangle|^{1/2} < \delta\}$$

for $\xi \in S$ and $0 < \delta \le 2$. The closure in $BMOA$, of the set of all polynomials is called $VMOA$. By [12, 22], we know that $f \in BMOA$ if and only if

$$\sup_{a \in B} \int_B |\Re f(z)|^2 (1-|z|^2) \left(\frac{1-|a|^2}{|1-\langle z, a \rangle|^2} \right)^n d\nu(z) < \infty, \tag{2}$$

and $f \in VMOA$ if and only if

$$\lim_{|a| \to 1} \int_B |\Re f(z)|^2 (1-|z|^2) \left(\frac{1-|a|^2}{|1-\langle z, a \rangle|^2} \right)^n d\nu(z) = 0. \tag{3}$$

Let D be the open unit disk in the complex plane $\mathbb{C}^1$. Denote by $H(D)$ the class of all analytic functions on D. Suppose that $g \in H(D)$. The operator

$$J_g f(z) = \int_0^1 f(tz) z g'(tz) dt = \int_0^z f(\xi) g'(\xi) d\xi, \qquad z \in D,$$

where $f \in H(D)$, was introduced in [13] where Pommerenke showed that J_g is a bounded operator on the Hardy space $H^2(D)$ if and only if $g \in BMOA$. Aleman and Siskasis proved that J_g is a compact operator on the Hardy space $H^2(D)$ if and only if $g \in VMOA$ (see [2]).

The following integral operator was recently introduced and studied in [20]

$$I_g f(z) = \int_0^z f'(\xi) g(\xi) d\xi.$$

The operator J_g, I_g acting on various function spaces have been studied recently in [1, 2, 3, 10, 15, 20] (see, also the references therein).

The operators J_g, I_g can be extended to the unit ball. Suppose that $g : B \to \mathbb{C}^1$ is a holomorphic map of the unit ball, for a holomorphic function $f : B \to \mathbb{C}^1$, define

$$T_g f(z) = \int_0^1 f(tz) \frac{dg(tz)}{dt} = \int_0^1 f(tz) \Re g(tz) \frac{dt}{t}, \qquad z \in B.$$

This operator is called Riemann-Stieltjes operator (or Extended-Cesàro operator), which was introduced in [5], and studied in [5, 6, 7, 9, 16, 17].

Here, we extend the operator I_g for the case of holomorphic functions on the unit ball as follows (see also [9])

$$L_g f(z) = \int_0^1 \Re f(tz) g(tz) \frac{dt}{t}, \qquad z \in B.$$

The purpose of this paper is to study the boundedness and compactness of operators T_g and L_g on the Hardy space H^2, which extend the results of [2, 13]. Moreover, our method is different to their's. Below are our main results.

Theorem 1. *Suppose that g is a holomorphic function on B. Then*

1. $T_g : H^2 \to H^2$ *is bounded if and only if* $g \in BMOA$.
2. $L_g : H^2 \to H^2$ *is bounded if and only if*

$$\sup_{a\in B}\int_B \left(\frac{1-|a|^2}{|1-\langle a,z\rangle|^2}\right)^{n+2}|g(z)|^2(1-|z|^2)d\nu(z)<\infty. \tag{4}$$

Theorem 2. *Suppose that g is a holomorphic function on B. Then*

1. $T_g : H^2 \to H^2$ *is compact if and only if* $g \in VMOA$.
2. $L_g : H^2 \to H^2$ *is compact if and only if*

$$\lim_{|a|\to 1}\int_B \left(\frac{1-|a|^2}{|1-\langle a,z\rangle|^2}\right)^{n+2}|g(z)|^2(1-|z|^2)d\nu(z)=0. \tag{5}$$

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the next. The notation $a \preceq b$ means that there is a positive constant C such that $a \le Cb$. If both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2 Auxiliary Results

In this section, we state some auxiliary results which are incorporated in the following lemmas.

Lemma 1. ([5]) *For every $f, g \in H(B)$ it holds*

$$\Re[T_g(f)](z) = f(z)\Re g(z) \quad \text{and} \quad \Re[L_g(f)](z) = \Re f(z)g(z).$$

For $\zeta \in S$ and $r > 0$, the nonisotropic metric ball $S(\zeta, r)$ is defined to be

$$Q_r(\zeta) = \{z \in B : |1-\langle z,\zeta\rangle|^{1/2} < r\}.$$

A positive Borel measure μ on B is called a γ-Carleson measure if there exists a constant $C > 0$ such that

$$\mu(Q_r(\zeta)) \le Cr^\gamma$$

for all $\zeta \in S$ and $r > 0$.

A positive Borel measure μ on B is called a vanishing γ-Carleson measure if

$$\lim_{r\to 0}\frac{\mu(Q_r(\zeta))}{r^\gamma}=0$$

for all $\zeta\in S$ and $r>0$.

A well-known result about the γ-Carleson measure and vanishing γ-Carleson measure characterization is the following lemma (see [18, 19, 22]).

Lemma 2. *Let μ be a positive Borel measure on B. Then μ is a γ-Carleson measure if and only if*

$$\sup_{a\in B}\int_B\left(\frac{1-|a|^2}{|1-\langle a,z\rangle|^2}\right)^\gamma d\mu(z)<\infty.$$

μ is a vanishing γ-Carleson measure if and only if

$$\lim_{|a|\to 1}\int_B\left(\frac{1-|a|^2}{|1-\langle a,z\rangle|^2}\right)^\gamma d\mu(z)=0.$$

The following lemma can be found in [21].

Lemma 3. *Suppose that $0<p\le q<\infty$, α is real, and μ is a positive Borel measure on B. Then for any nonnegative integer k with $\alpha+kp>-1$, the following conditions are equivalent.*

1. *There exists a constant C (independent of f) such that*

$$\left(\int_B|\Re^k f(z)|^q d\mu(z)\right)^{1/q}\le C\left(\int_B|f(z)|^p d\nu_\alpha(z)\right)^{1/p}$$

for all $f\in A^p(\nu_\alpha)$.

2. *There is a constant $C>0$ such that*

$$\mu(Q_r(\zeta))\le Cr^{(n+1+\alpha+kp)q/p}$$

for all $r>0$ and $\zeta\in S$.

Lemma 4. ([8]) *Suppose that μ is a positive Borel measure on B. Then the following conditions are equivalent.*

1. *There exists a constant C such that*

$$\left(\int_B|f(z)|^2 d\mu(z)\right)^{1/2}\le C\|f\|_{H^2}$$

for all $f\in H^2$.

2. *There is a constant $C>0$ such that*

$$\mu(Q_r(\zeta))\le Cr^n$$

for all $r>0$ and $\zeta\in S$.

The following criterion for compactness follows by standard arguments similar, for example, to those outlined in Proposition 3.11 of [4].

Lemma 5. *The operator T_g (or L_g) : $H^2 \to H^2$ is compact if and only if T_g (or L_g) : $H^2 \to H^2$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in H^2 which converges to zero uniformly on compact subsets of B, we have $\|T_g f_k\|_{H^2} \to 0$ (corresp. $\|L_g f_k\|_{H^2} \to 0$) as $k \to \infty$.*

3 Proofs of the main results

Proof of Theorem 1. It is easy to see that $T_g f(0) = 0$. By (1.1) and Lemma 1, we have

$$\begin{aligned}\|T_g f\|^2_{H^2} &\asymp \int_B |\Re(T_g f)(z)|^2(1-|z|^2)d\nu(z)\\ &= \int_B |\Re g(z)|^2 |f(z)|^2(1-|z|^2)d\nu(z) = \int_B |f(z)|^2 d\mu_1(z),\end{aligned}$$

where

$$d\mu_1(z) = |\Re g(z)|^2(1-|z|^2)d\nu(z).$$

By Lemma 4 we see that $T_g : H^2 \to H^2$ is bounded if and only if

$$\mu_1(Q_r(\zeta)) \le Cr^n. \tag{6}$$

By Lemma 2, (6) is equivalent to

$$\sup_{a\in B}\int_B \left(\frac{1-|a|^2}{|1-\langle a,z\rangle|^2}\right)^n |\Re g(z)|^2(1-|z|^2)d\nu(z) < \infty,$$

i.e. $g \in BMOA$.

Similarly,

$$\|L_g f\|^2_{H^2} \asymp \int_B |\Re f(z)|^2 d\mu_2(z), \tag{7}$$

where

$$d\mu_2(z) = |g(z)|^2(1-|z|^2)d\nu(z).$$

Taking $p = q = 2, k = 1, \alpha = -1$ in Lemma 3, we see that $L_g : H^2 \to H^2$ is bounded if and only if

$$\mu_2(Q_r(\zeta)) \le Cr^{n+2}. \tag{8}$$

By Lemma 2, (8) is equivalent to

$$\sup_{a\in B}\int_B \left(\frac{1-|a|^2}{|1-\langle a,z\rangle|^2}\right)^{n+2} |g(z)|^2(1-|z|^2)d\nu(z) < \infty,$$

as desired.

Proof of Theorem 2. We give the proof of *(a)*. The proof of *(b)* is similar and will be omitted.

First, suppose that $T_g : H^2 \to H^2$ is compact. Let $(a_k)_{k\in\mathbb{N}}$ be a sequence in B such that $\lim_{k\to\infty}|a_k| = 1$. Set

$$f_k(z) = \left(\frac{1-|a_k|^2}{(1-\langle z,a_k\rangle)^2}\right)^{\frac{n}{2}} \qquad (z\in\overline{B},\ k\in\mathbb{N}). \tag{9}$$

By [14, Proposition 1.4.10] $f_k \in H^2$, $k\in\mathbb{N}$, moreover, there is a constant C such that $\sup_{k\in\mathbb{N}}\|f_k\|^2_{H^2}\le C$. On the other hand, it is easy to see that f_k converges to 0 uniformly on compact subsets of B as $k\to\infty$. By Lemma 5, we have that $T_g f_k \to 0$ in H^2 as $k\to\infty$. Hence

$$\begin{aligned}
&\lim_{k\to\infty}\int_B \left(\frac{1-|a_k|^2}{|1-\langle z,a_k\rangle|^2}\right)^n |\Re g(z)|^2(1-|z|^2)\,d\nu(z)\\
=\;& \lim_{k\to\infty}\int_B |\Re(T_g f_k)|^2(1-|z|^2)\,d\nu\\
\asymp\;& \lim_{k\to\infty}\|T_g f_k\|^2_{H^2} = 0.
\end{aligned}$$

This implies that $g\in VMOA$.

Conversely, suppose that $g\in VMOA$. Then $T_g : H^2\to H^2$ is bounded by Theorem 1. Moreover, for every fixed $\varepsilon>0$, there exist an $\eta_0\in(0,1)$ such that

$$\int_B \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^n d\mu_1(z) < \varepsilon \tag{10}$$

for all $a\in B$ with $\eta_0<|a|<1$. Let $r_0 = 1-\eta_0$. For $\zeta\in S, r\in(0,r_0)$, let $a=(1-r)\zeta$. Then $a\in B$, $\eta_0<|a|<1$,

$$|1-\langle z,a\rangle| < 2r \quad\text{and}\quad 1-|a|^2\ge r$$

for each $z\in Q_r(\zeta)$. Hence

$$\left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^n \ge \left(\frac{r}{(2r)^2}\right)^n = (4r)^{-n} \tag{11}$$

for each $z\in Q_r(\zeta)$. By (10) and (11), we have

$$\begin{aligned}
\frac{\mu_1(Q_r(\zeta))}{4^n r^n} &\le \int_{Q_r(\zeta)}\left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^n d\mu_1(z)\\
&\le \int_B \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^n d\mu_1(z) < \varepsilon
\end{aligned}\tag{12}$$

for all $r\in(0,r_0)$ and $\zeta\in S$. Let $\varepsilon>0$ be fixed and $\widetilde{\mu_1}\equiv\mu_1|_{B\setminus(1-r_0)\overline{B}}$. As in the proof of Theorem 1.1 of [11], we see that there exists a constant $C>0$ such that

$$\widetilde{\mu_1}(Q_r(\zeta)) \le C\varepsilon r^n. \tag{13}$$

Suppose that $(f_k)_{k\in\mathbb{N}}$ is a sequence in H^2 such that converges to 0 uniformly on compact subsets of B and $\sup_{k\in\mathbb{N}} \|f_k\|_{H^2} < L$. By Lemma 1, we have

$$\begin{aligned} \|T_g f_k\|^2_{H^2} &\asymp \int_B |\Re g(z)|^2 |f_k(z)|^2 (1-|z|^2) d\nu(z) \\ &= \int_B |f_k(z)|^2 d\widetilde{\mu_1}(z) + \int_{(1-\delta_0)\overline{B}} |f_k(z)|^2 d\mu_1(z). \end{aligned} \tag{14}$$

Using (13) and the method of Theorem 1.1 of [11], there exists a positive constant C such that

$$\int_B |f_k|^2 d\widetilde{\mu_1} \le C\epsilon \|f_k\|^2_{H^2} \le CL\varepsilon, \tag{15}$$

for each $k \in \mathbb{N}$. Since f_k converges to 0 uniformly on $(1-\delta_0)\overline{B}$, the second term in (14) can be made small enough for k sufficiently large. Hence, we obtain

$$\lim_{k\to\infty} \int_{(1-\delta_0)\overline{B}} |f_k(z)|^2 d\mu_1(z) = 0. \tag{16}$$

Combining with (14), (15) and (16), we see that $\|T_g f_k\|_{H^2} \to 0$ as $k \to \infty$. Applying Lemma 5, we obtain that $T_g : H^2 \to H^2$ is compact.

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