Doubly ruled submanifolds in space forms

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Abstract

In this paper we extend a classical result, namely, the one that states that the only doubly ruled surfaces in $\mathbb{R}^3$ are the hyperbolic paraboloid and the hyperboloid of one sheet, in three directions: for all space forms, for any dimensions of the rulings and manifold, and to the conformal realm. We show that all this can be reduced, with the help of quite natural constructions, to just one simple example, the rank one real matrices. We also give the affine classification in Euclidean space. To deal with the conformal case, we make use of recent developments on Ribaucour transformations.

The spirit of this work relies on a very classical result: the only (non-planar) doubly ruled surfaces in $\mathbb{R}^3$ are (open subsets of) two quadrics, the hyperbolic paraboloid and the hyperboloid of one sheet. A discussion on doubly ruled surfaces in $\mathbb{R}^3$ can be found in the beautiful book [**5**], Ch.1, §3. Our purpose here is to give similar classifications in much broader contexts.

The natural spaces to look for doubly ruled submanifolds are the ones provided with as many as possible totally geodesic or totally umbilical submanifolds. By the main result in [**6**], these spaces must have constant sectional curvature. So, let M^m be an m-dimensional (immersed) connected submanifold of a space form $\mathbb{Q}_c^{m+p}$, i.e, a simply-connected complete Riemannian manifold of constant sectional curvature c. We say that $M^m \subset \mathbb{Q}_c^{m+p}$ is *doubly ruled* (resp. *doubly conformally ruled*) if M^m has two nontrivial smooth foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ whose leaves are totally geodesic (resp. totally umbilical) in $\mathbb{Q}_c^{m+p}$ and transversal, that is, $T\mathcal{F}_1 + T\mathcal{F}_2 = TM$ along M^m. Observe that $\mathcal{F}_0 = \mathcal{F}_1 \cap \mathcal{F}_2$ is a smooth foliation whose leaves are also totally geodesic (resp. totally umbilical) in $\mathbb{Q}_c^{m+p}$. Alternatively, we will say that $M^{n+k+r} \subset \mathbb{Q}_c^{n+k+r+p}$ is a $(n+r, k+r)$-ruled (resp. conformally ruled) submanifold,

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if it is doubly ruled (resp. conformally ruled) as above, with $n + r = \operatorname{rank}\mathcal{F}_1$, $k + r = \operatorname{rank}\mathcal{F}_2$, and hence $r = \operatorname{rank}\mathcal{F}_0$. For simplicity, we always assume that $n \geq k$.

Let us introduce a basic and quite natural example inside the space of real matrices.

The maximal cones. For $n \geq k \in \mathbb{N}$, let $\mathcal{C}^{n,k}$ be the cone given by

$$\mathcal{C}^{n,k} = \{S \in \mathbb{R}^{(n+1)\times(k+1)} : \operatorname{rank} S = 1\}.$$

Since $S \in \mathcal{C}^{n,k}$ has all its columns (rows) collinear, there are $x \in \mathbb{R}^{n+1}_* = \mathbb{R}^{n+1} \setminus \{0\}$ and $y \in \mathbb{R}^{k+1}_* = \mathbb{R}^{k+1} \setminus \{0\}$ such that $S = x^t y$ as a product of matrices, where A^t will always denote the transpose of A. Because this way of writing such an S is unique up to a real factor, it is clear that $\mathcal{C}^{n,k}$ is a smooth $(n + k + 1)$-dimensional submanifold of $\mathbb{R}^{(n+1)\times(k+1)} = \mathbb{R}^{n+k+1+nk}$ diffeomorphic to $(\mathbb{R}^{n+1}_* \times \mathbb{R}^{k+1}_*)/\mathbb{R}_*$, under the equivalence relation $(x, y) \sim (\lambda x, \lambda^{-1} y)$, $\lambda \in \mathbb{R}_*$. In fact, $\mathcal{C}^{n,k}$ is an embedded algebraic submanifold of degree two, since an open dense subset of $\mathcal{C}^{n,k}$ can be parametrized by the set of equations $S_{11}S_{ij} = S_{1j}S_{i1}$, $S_{11} \neq 0$, for $2 \leq i \leq n+1$, $2 \leq j \leq k + 1$; see Remark 15. Moreover, we have two well defined submersions:

$$s_C : \mathcal{C}^{n,k} \to \mathbb{RP}^n,\quad c(x^t y) = [x], \qquad \text{and} \qquad s_R : \mathcal{C}^{n,k} \to \mathbb{RP}^k,\quad r(x^t y) = [y].$$

The fiber $C_{[x]} = s_C^{-1}([x]) \subset \mathcal{C}^{n,k}$ (resp. $R_{[y]} = s_R^{-1}([y]) \subset \mathcal{C}^{n,k}$) is the $(k + 1)$-dimensional (resp. $(n + 1)$-dimensional) vector subspace, with the origin removed, of all rank one matrices whose columns (resp. rows) are collinear with x (resp. y). They thus define two smooth totally geodesic foliations C and R of $\mathcal{C}^{n,k}$, whose intersection is

$$C_{[x]} \cap R_{[y]} = \mathbb{R}_*(x^t y). \tag{1}$$

Therefore, $\mathcal{C}^{n,k}$ is a $(n + 1, k + 1)$-ruled submanifold. Since it is easy to see that the intersection of $\mathcal{C}^{n,k}$ with any affine hyperplane that does not pass through the origin, $\mathbb{R}^{n+k+nk}_D = \{A \in \mathbb{R}^{(n+1)\times(k+1)} : \operatorname{trace}(A^t D) = 1\}$, $D \in \mathbb{R}^{(n+1)\times(k+1)} \setminus \{0\}$, is nonempty and transversal by (1), we obtain the family of complete (n, k)-ruled submanifolds of dimension $n + k$ in $\mathbb{R}^{n+k+nk} = \mathbb{R}^{n+k+nk}_D$ given by

$$\mathcal{N}^{n,k}_D = \mathcal{C}^{n,k} \cap \mathbb{R}^{n+k+nk}_D. \tag{2}$$

With the cone $\mathcal{C}^{n,k}$ we can also easily construct 'linear' cylinders in space forms. Given $n \geq k \in \mathbb{N}$, $r \in \mathbb{N} \cup \{0\}$, consider on $\mathbb{R}^{N+1}$ the usual positive definite (resp. Lorentzian) inner product $\langle\ ,\ \rangle$ which turns $\mathbb{E}^{N+1} = (\mathbb{R}^{N+1}, \langle\ ,\ \rangle)$ into the standard flat Euclidean space $\mathbb{R}^{N+1}$ (resp. Lorentzian space $\mathbb{L}^{N+1}$), where, throughout this paper,

$$N = N(n, k, r) = n + k + r + nk.$$

For each $c > 0$, we then have the usual model of the standard sphere of curvature c, $\mathbb{Q}^N_c = \mathbb{S}^N_c = \{Z \in \mathbb{R}^{N+1} : \langle Z, Z\rangle = c^2\}$, and, for $c < 0$, the hyperbolic space of curvature c, $\mathbb{Q}^N_c = \mathbb{H}^N_c = \{Z \in \mathbb{L}^{N+1} : \langle Z, Z\rangle = -c^2, Z_0 > 0\}$. For curvature $c = 0$, consider a fixed affine hyperplane $\mathbb{Q}^N_0 \subset \mathbb{R}^{N+1}$ that does not contain the origin. Again by (1), for any linear isomorphism L of $\mathbb{R}^{N+1}$, $L \in GL(N+1, \mathbb{R})$, all these hypersurfaces of constant sectional curvature intersect the *maximal cone*

$L(\mathcal{C}^{n,k} \times \mathbb{R}^r) \subset \mathbb{R}^{N+1}$ and both foliations $L(C \times \mathbb{R}^r)$ and $L(R \times \mathbb{R}^r)$ transversally. Hence,

$$ {}^c\mathcal{N}_L^{n,k,r} = L(\mathcal{C}^{n,k} \times \mathbb{R}^r) \cap \mathbb{Q}_c^N \subset \mathbb{Q}_c^N $$

are all embedded $(n+r, k+r)$-ruled $(n+k+r)$-dimensional algebraic submanifolds of degree two in $\mathbb{Q}_c^N$, $c \in \mathbb{R}$. Moreover, they are *substantial* (resp. *conformally substantial*), i.e., they are not contained in a proper totally geodesic (resp. totally umbilical) submanifold of the ambient space. Notice that the boundary of ${}^c\mathcal{N}_L^{n,k,r}$ is $L(\{0\} \times \mathbb{R}^r) \cap \mathbb{Q}_c^N$, that coincides with the singular set of its closure.

Our first classification result states that these are the only examples of doubly ruled submanifolds in space forms, even locally, when in maximal codimension.

Theorem 1. *Let $M^{n+k+r} \subset \mathbb{Q}_c^{n+k+r+p}$ be a substantial $(n+r, k+r)$-ruled submanifold, with $n, k \geq 1, r \geq 0$. Then, $p \leq nk$. Moreover, if equality holds, then M^{n+k+r} is congruent to (an open subset of) ${}^c\mathcal{N}_L^{n,k,r} \subset \mathbb{Q}_c^N$, for some $L \in GL(N+1, \mathbb{R})$.*

With the above it is not hard to obtain the affine classification in Euclidean space that easily recovers the classical result for surfaces, for which $n = k = 1$ and $r = 0$. We set $\text{sign}(r) = 0$ for $r = 0$, $\text{sign}(r) = 1$ for $r > 0$.

Corollary 2. *Any substantial $(n+r, k+r)$-ruled submanifold $M^{n+k+r} \subset \mathbb{R}^N$ is affinely equivalent to either $\mathcal{C}^{n,k} \times \mathbb{R}^{r-1}$ or $\mathcal{N}_D^{n,k} \times \mathbb{R}^r$, for some $0 \neq D \in \mathbb{R}^{(n+1)\times(k+1)}$. Moreover, $\mathcal{N}_D^{n,k} \times \mathbb{R}^r$ and $\mathcal{N}_{D'}^{n,k} \times \mathbb{R}^r$ are affinely equivalent if and only if rank $D =$ rank D'. In particular, up to affine transformations, there are exactly $k + 1 + \text{sign(r)}$ substantial $(n+r, k+r)$-ruled submanifolds of dimension $n+k+r$ in $\mathbb{R}^N$.*

Both results above also hold for semi-Riemannian space forms, since the properties of being doubly ruled or substantial in Euclidean space have nothing to do with inner products, definite or not.

Several geometric consequences can now be derived. For example, any doubly ruled (substantial) submanifold in a space form $\mathbb{Q}_c^m$ with maximal codimension has sectional curvature less or equal than c. It is also not hard to see that two neighborhoods of $\mathcal{N}_D^{n,k}$ and $\mathcal{N}_{D'}^{n,k}$ are isometric if and only if they are congruent and there are orthogonal endomorphisms $A \in O(n+1)$ and $B \in O(k+1)$ such that $D' = ADB$.

As a simple application, the following corollary implies that the lifting via the Hopf fibration of a totally real and totally geodesic $\mathbb{RP}^n \subset \mathbb{CP}^n$, that is, ${}^1\mathcal{N}_{Id}^{n,1,0} \subset \mathbb{S}^{2n+1}$, is (linearly) the only *complete* doubly ruled submanifold in the sphere, in *any* codimension, for which one of the foliations has minimal rank ($k = 1$). The hypothesis on completeness can be weakened by asking the submanifold to have two complete rulings of the same foliation.

Corollary 3. *Let $M^{n+k+r} \subset \mathbb{S}^{n+k+r+p}$ be a complete $(n+r, k+r)$-ruled submanifold. Then, $p \geq n + r + 1 - k$ and either $r \leq (n+1)(k-1) - 1$ or $r = 0$. In particular, if $k = 1$, then $r = 0$ and $M^{n+1} = {}^1\mathcal{N}_L^{n,1,0} \subset \mathbb{S}^{2n+1}$, with $L \in GL(2n+2, \mathbb{R})$.*

Let us now study the situation in the conformal realm. Since all space forms are conformally equivalent via the standard stereographic projections, there will be no restriction in working on Euclidean space.

It is clear that if we take one of the above examples of doubly ruled submanifolds in space forms in maximal codimension we get, via stereographic projection if necessary, examples of doubly conformally ruled submanifolds in Euclidean space, ${}^c\mathcal{N}_L^{n,k,r} \subset \mathbb{Q}_c^N \cong \mathbb{R}^N$. Particular cases are obtained by choosing the *r-cylinders* over ${}^c\mathcal{N}_L^{n,k,0} \subset \mathbb{Q}_c^{N-r}$, that is, if $L \in GL(N+1-r, \mathbb{R})$, for $c = 0$ we have

$$(L(\mathcal{C}^{n,k}) \times \mathbb{R}^r) \cap (\mathbb{Q}_0^{N-r} \times \mathbb{R}^r) \subset \mathbb{R}^N,$$

and, for $c \neq 0$,

$$(L(\mathcal{C}^{n,k}) \times \mathbb{R}^r) \cap \mathbb{Q}_c^N \subset \mathbb{Q}_c^N \cong \mathbb{R}^N,$$

for the fixed decomposition $\mathbb{E}^{N+1} = \mathbb{E}^{N+1-r} \times \mathbb{R}^r$. Our next result states that, if the codimension is large enough, doubly conformally ruled submanifolds are *conformally equivalent*, i.e., congruent via a conformal diffeomorphism of the ambient space, not only to doubly ruled ones, but in fact to cylinders over doubly ruled submanifolds, provided the two foliations intersect (nontrivially). For the isometric version see Theorem 11.

Theorem 4. *Let $M^{n+k+r} \subset \mathbb{R}^{n+k+r+p}$ be a conformally substantial $(n+r, k+r)$-conformally ruled submanifold, with $n, k, r \geq 1$. Then, $p \leq nk$. If $p > (n-1)k$, then M is conformally equivalent to (the stereographic projection of an open subset of) an r-cylinder over a (n, k)-ruled submanifold of dimension $n+k$ in a space form.*

As a direct consequence of Theorem 1 and Theorem 4 we conclude that the doubly conformally ruled case for maximal codimension is even more restrictive than the doubly ruled case, when the two foliations intersect.

Corollary 5. *Let $M^{n+k+r} \subset \mathbb{R}^N$ be a conformally substantial $(n+r, k+r)$-conformally ruled submanifold, with $n, k, r \geq 1$. Then, M^{n+k+r} is conformally equivalent to an r-cylinder over ${}^c\mathcal{N}_L^{n,k,0} \subset \mathbb{Q}_c^{N-r}$, for some $c \in \mathbb{R}$ and $L \in GL(N+1-r, \mathbb{R})$.*

Observe that the cone $\mathcal{C}^{n,k}$ itself satisfies the hypothesis of the above. In fact, it coincides with the stereographic projection from the point $(0, \ldots, 0, 1)$ of ${}^1\mathcal{N}_{Id}^{n,k,1}$. Similar relation holds for any cone in Euclidean space. A more detailed discussion on r-cylinders will be given in Remark 13.

Theorem 4 and Corollary 5 do not hold for $r = 0$. For example, the circular cone in $\mathbb{R}^3$ is a counterexample, since the property that *the normal components of the (non-normalized) mean curvature vectors of both rulings coincide* is a conformal invariant. To see this, observe that if $M^m \subset \mathbb{Q}_c^{m+p}$ is doubly conformally ruled and satisfies this property, for all $x \in M^m$ the two rulings that pass at x are contained in the m-dimensional totally umbilical submanifold of $\mathbb{Q}_c^{m+p}$ that is tangent to M^m at x and which mean curvature vector at x coincides with the ones of the rulings. Our last result shows that this conformally invariant property is, in fact, sufficient to assure the reduction to the doubly ruled case in high codimensions when $r = 0$.

Theorem 6. *Let $M^{n+k} \subset \mathbb{R}^{n+k+p}$ be a conformally substantial (n, k)-conformally ruled submanifold. Assume further that the normal components of the mean curvature vectors of both rulings coincide. Then, $p \leq nk$. If $p > (n-1)k$, then M^{n+k} is conformally equivalent to an (n, k)-ruled submanifold of dimension $n + k$ in a space form. In particular, if $p = nk$, such doubly ruled submanifold must be some ${}^c\mathcal{N}_L^{n,k,0} \subset \mathbb{Q}_c^{n+k+nk}$.*

The analysis of the conformal situation relies on recent developments on generalizations of the Ribaucour transformation; cf. [**4**]. It turns out that the intersection of the rullings gives rise to a Dupin principal normal with integrable conullity, an object which is well locally described by means of this transformation.

1 The conformal realm: cylinders

In this section we study the geometry of the intersection of the rulings of doubly conformally ruled submanifolds, and we show that they are conformally equivalent to cylinders over doubly ruled ones when such intersection is nontrivial or if the normal components of the mean curvature vectors of both rulings coincide. We follow closely the terminology in [**4**].

We need the following basic fact.

Proposition 7. *Any (conformally) substantial doubly ruled submanifold M^m in $\mathbb{Q}_c^{m+p}$ is locally (conformally) substantial.*

Proof: Let us do the doubly ruled case in $\mathbb{R}^{m+p}$, the remaining ones being similar.

We claim that if two points $x_1, x_2 \in M^m$ belong to the same ruling of one of the foliations, say, $\mathcal{F}_1$, and if there is a neighborhood V of x_1 substantially contained in an affine subspace $\mathcal{L}$, then there is a neighborhood $V' \supset V$ of x_2 that is also contained in $\mathcal{L}$. This is so because $\mathcal{L}$ is affinely spanned by the leaves of $\mathcal{F}_1$ that intersect V. Since $x_2 \in \mathcal{F}_1(x_1)$, the union of such leaves gives the desired neighborhood.

Assume M^m is not locally substantial, and let x_0 be an accumulation point of the open subset $U \subset M^m$ consisting of the points $x \in M^m$ for which there exists a nonsubstantial open neighborhood $V_x \subset M^m$ of it. Take $x \in U$ close enough to x_0 such that there is $x' \in \mathcal{F}_1(x_0) \cap \mathcal{F}_2(x)$. By the above claim, there is a neighborhood of x_0, x' and x which is also not substantial. Hence, $x_0 \in U$ and U is closed. Therefore, $U = M^m$ is not substantial, which is a contradiction. ∎

Let $M^m \subset \mathbb{Q}_c^{m+p}$ be a conformally substantial doubly conformally ruled submanifold, with conformal ruled foliations $\mathcal{F}_1$ and $\mathcal{F}_2$. Denote by $\tilde{\nabla}$ the usual Levi-Civita connection of $\mathbb{Q}_c^{m+p}$, and by $\nabla^\perp$ and α the normal connection and second fundamental form of M^m in $\mathbb{Q}_c^{m+p}$, respectively. Assume that $\mathcal{F}_0 := \mathcal{F}_1 \cap \mathcal{F}_2$ is non-trivial. Then, $\mathcal{F}_0$ has constant positive rank and thus it is also a smooth foliation with totally umbilical leaves in $\mathbb{Q}_c^{m+p}$. To simplify the reading, from now on we will also denote by $\mathcal{F}_i$ its associated distribution, i.e., $T\mathcal{F}_i$, $0 \le i \le 2$. Therefore, there are tangent vector fields $Z_i \in TM \cap \mathcal{F}_i^\perp$ and normal vector fields $\eta_i \in T^\perp M$ such that

$$(\tilde{\nabla}_X Y)_{\mathcal{F}_i^\perp} = \langle X, Y\rangle (Z_i + \eta_i), \quad \forall\, X, Y \in \mathcal{F}_i, \quad 0 \le i \le 2. \tag{3}$$

Since $\mathcal{F}_0 \subset \mathcal{F}_j$ is nontrivial, we have that

$$\eta_0 = \eta_1 = \eta_2, \quad \text{and} \quad Z_j = (Z_0)_{TM \cap \mathcal{F}_j^\perp}\,, \quad j = 1, 2, \tag{4}$$

where, throughout this note, for a subbundle E and a vector X, X_E will denote the orthogonal projection of X onto E. In particular, if $r \ge 1$ the normal components

η_j, $j=1,2$, of the (non-normalized) mean curvature vectors of both rulings coincide. The first consequence is that

$$\mathcal{F}_0 \text{ is totally geodesic} \quad \text{if and only if} \quad \mathcal{F}_1 \text{ and } \mathcal{F}_2 \text{ are totally geodesic.} \tag{5}$$

Moreover, if we define $\beta = \alpha - \langle\ ,\ \rangle\eta_0$, by (3), the transversality of $\mathcal{F}_1$ and $\mathcal{F}_2$, and $\mathcal{F}_0 \subset \mathcal{F}_j$, $j=1,2$, we have that

$$\beta(\mathcal{F}_0, TM) = 0, \tag{6}$$

$$\beta(\mathcal{F}_1, \mathcal{F}_1) = 0 = \beta(\mathcal{F}_2, \mathcal{F}_2). \tag{7}$$

Equation (6) says that η_0 is a *principal normal* of M^m along the connected components of an open dense subset of M^m where the nullity subspaces of β as a bilinear map,

$$\mathcal{E} = \mathcal{E}_{\eta_0} = \{Z \in TM : \beta(Z, TM) = 0\},$$

have locally constant dimension, and that $\mathcal{F}_0$ is an *umbilical distribution associated to* η_0, i.e, $\mathcal{F}_0 \subset \mathcal{E}$. Since the leaves of $\mathcal{F}_0$ are totally umbilical submanifolds of $\mathbb{Q}_c^{m+p}$, for $v \in \mathcal{F}_0$ we get

$$0 = (\tilde{\nabla}_v(Z_0 + \eta_0))_{T^\perp M} = \beta(Z_0, v) + \nabla_v^\perp \eta_0 = \nabla_v^\perp \eta_0,$$

that is, η_0 is parallel in the normal connection along $\mathcal{F}_0$, a condition which is automatic for umbilical distributions of dimension greater or equal than two (see Proposition 8 in [**3**]). Thus, η_0 is also parallel along $\mathcal{E} \supset \mathcal{F}_0$, i.e., η_0 is a *Dupin principal normal* of M^m, on the open subsets where $\mathcal{E}$ has constant dimension.

Proposition 8. *Let $M^{n+k+r} \subset \mathbb{Q}_c^{n+k+r+p}$ be a conformally substantial (resp. substantial) $(n+r, k+r)$-conformally ruled (resp. $(n+r,k+r)$-ruled) submanifold. In the conformal case, if $r=0$, assume further that the normal components of the mean curvature vectors of both rulings coincide, call them η_0 and define β as above. Then:*

i) It holds that span$\{\beta(\mathcal{F}_1, \mathcal{F}_2)\} = T^\perp M$ almost everywhere. In particular, $p \le nk$.

ii) If $p > (n-1)k$, then $\mathcal{E} = \mathcal{F}_0$ and $\mathcal{E}^\perp$ is integrable and totally umbilical in M.

Proof: In this proof, j will denote an index $j = 1,2$. Set $S(\beta) := \text{span}\{\beta(\mathcal{F}_1, \mathcal{F}_2)\}$.

i). Assume there is an open subset $U \subset M$ where $\dim S(\beta)$ is constant and $S(\beta) \neq T^\perp M$. Hence, $S(\beta)^\perp \subset T^\perp M$ is a nontrivial smooth normal subbundle. From (3) and (4) we see that $Z_j + \eta_0$ is the mean curvature vector of the umbilical leaves of $\mathcal{F}_j$ and then $0 = (\tilde{\nabla}_V(Z_j + \eta_0))_{T^\perp M} = \beta(Z_j, V) + \nabla_V^\perp \eta_0$, for $V \in \mathcal{F}_j$. Thus, $\nabla_V^\perp \eta_0 \in S(\beta)$ for all $V \in TM$. By Codazzi equation, if $X \in \mathcal{F}_1$, $Y, Z \in \mathcal{F}_2$, $\xi \in S(\beta)^\perp$,

$$\langle \beta(X,Y), \nabla_Z^\perp \xi\rangle = -\langle \nabla_Z^\perp \beta(X,Y), \xi\rangle = \langle \langle X, Y\rangle \nabla_Z^\perp \eta_0 - \langle Y, Z\rangle \nabla_X^\perp \eta_0, \xi\rangle = 0. \tag{8}$$

Reversing the roles of $\mathcal{F}_1$ and $\mathcal{F}_2$ we obtain that $S(\beta)^\perp$ is parallel. Since the shape operator A_ξ of M in the direction ξ satisfies $A_\xi = \langle \xi, \eta_0\rangle Id$, it is a standard fact (cf. [**7**]) that U is contained in a proper umbilical (resp. totally geodesic if $\eta_0 = 0$) submanifold of $\mathbb{Q}_c^{n+k+r+p}$, which contradicts the assumption that M is conformally substantial (resp. substantial) in view of Proposition 7. Part $i)$ now follows from (6) and (7).

$ii)$. Set $\mathcal{F}'_j = \mathcal{F}_j \cap \mathcal{F}_0^\perp$. Let $X = X_1 + X_2 \in \mathcal{E} \cap \mathcal{F}_0^\perp$, with $X_j \in \mathcal{F}'_j$. By (7), $X_j \in \mathcal{E}$. Since $n \geq k$ and $S(\beta) = S(\beta|_{\mathcal{F}'_1 \times \mathcal{F}'_2})$, part $i)$ yields that $X = 0$ and then $\mathcal{E} = \mathcal{F}_0$.

For $T \in \mathcal{E}$, let $C_T : \mathcal{E}^\perp \to \mathcal{E}^\perp$ be given by

$$C_T X = -(\nabla_X T)_{\mathcal{E}^\perp}.$$

From (3) and $\mathcal{E} \subset \mathcal{F}_j$, it holds that $C_T(\mathcal{F}'_j) \subset \mathcal{F}'_j$. On the other hand, from Codazzi equation we get

$$\beta(C_T X, Y) = \beta(X, C_T Y), \quad \forall\, X \in \mathcal{F}'_1,\ Y \in \mathcal{F}'_2. \tag{9}$$

If $v \in \mathcal{F}'_2$ and $w = C_T v$ are linearly independent, then $\beta(X, w) = \beta(C_T X, v)$ for all $X \in \mathcal{F}'_1$, that contradicts the assumption on p by part $i)$. Hence, $C_T|_{\mathcal{F}'_2} = \mu Id_{\mathcal{F}'_2}$. Now by (7) and (9), $C_T = \mu Id$ and the proof follows. ∎

The paper [**4**] was devoted to the study of the geometry of submanifolds M carrying a Dupin principal normal η whose conullity $\mathcal{E}_\eta^\perp$ is integrable. The techniques developed in that paper allow to give a short proof of the main result in [**3**], where they were studied submanifolds for which $\mathcal{E}_\eta^\perp$ is totally umbilical in M, that is a conformal invariant (in fact, a Lie invariant). Although Theorem 1 in [**3**] will give us a complete description of our doubly conformally ruled submanifolds in view of Proposition 8, we prefer the extrinsic description given in [**4**] as generalized cylinders.

In fact, Theorem 11 and the proof of Theorem 15 of [**4**] say that if $\mathcal{E}_\eta^\perp$ is totally umbilical in $M^m \subset \mathbb{Q}_c^{m+p}$ and $\dim \mathcal{E}_\eta^\perp = q \geq 2$, which is our case since $\dim \mathcal{E}^\perp = n + k$, then M^m is locally conformally congruent to (the stereographic projection, if $c \neq 0$, of an open subset of) an r-cylinder over a submanifold $\mathcal{W}^{m-r} \subset \mathbb{Q}_c^{m-r+p}$ described as follows. Consider the totally geodesic inclusion $\mathbb{Q}_c^{m-r+p} \subset \mathbb{Q}_c^{m+p}$ and call its normal bundle $\mathcal{V} = \mathcal{V}^r$. The *$r$-cylinder over* $\mathcal{W}^{m-r}$ is the m-dimensional submanifold given by

$$S_r(\mathcal{W}^{m-r}) = \{\exp_x(\gamma) : x \in \mathcal{W}^{m-r}, \gamma \in \mathcal{V}(x)\} \subset \mathbb{Q}_c^{m+p}, \tag{10}$$

where exp stands for the exponential map of $\mathbb{Q}_c^{m+p}$. Moreover, it also holds that the cylindrical rulings, that is, the ones obtained by fixing x in (10), correspond to the ones of $\mathcal{E}_\eta$. It is clear that we can also describe $S_r(\mathcal{W}^{m-r})$ as either $\mathcal{W}^{m-r} \times \mathbb{R}^r \subset \mathbb{R}^{m+p}$, if $c = 0$, or $(C\mathcal{W}^{m-r} \times \mathbb{R}^r) \cap \mathbb{Q}_c^{m+p} \subset \mathbb{E}^{m+p+1} = \mathbb{E}^{m-r+p+1} \times \mathbb{R}^r$ if $c \neq 0$, where $C\mathcal{W}^{m-r} \subset \mathbb{E}^{m-r+p+1}$ stands for the cone over $\mathcal{W}^{m-r} \subset \mathbb{Q}_c^{m-r+p}$; see also Remark 13.

Remark 9. Lemma 10 in [**4**] implies that the three classes in Theorem 4 and Corollary 5 given by the sign of $c \in \mathbb{R}$ are conformally disjoint, since $M = S_r(\mathcal{W})$ is conformally substantial if and only if $\mathcal{W}$ is. This implies that submanifolds in different classes cannot be glued together, and then our description as r-cylinders is global. This was also shown by direct methods in more generality at the end of the proof of Theorem 1 in [**3**].

Proof of Theorem 4: By Proposition 8 and the discussion above we already know that M^{n+k+r} is conformally equivalent to an open subset of $S_r(\mathcal{W}^{n+k})$, where the leaves of $\mathcal{E}$ correspond to the cylindrical rulings. Since these rulings are totally geodesic, from (5) we easily conclude that $\mathcal{W}^{n+k}$ should be (n,k)-ruled. ∎

Proof of Theorem 6: We get the inequality on p from Proposition 8. With the notations in (3), by assumption we know that $\eta := \eta_1 = \eta_2$. Let $Z \in TM$ be defined by $Z_{\mathcal{F}_j^\perp} = Z_j$, where $j = 1,2$ throughout this proof. Set

$$\xi = e^{-\frac{1}{2}\|Z\|^2}(Z + \eta).$$

Then, since the leaves of $\mathcal{F}_j$ are totally umbilical, for $X \in \mathcal{F}_j$ we obtain from (3) that

$$e^{\frac{1}{2}\|Z\|^2}\tilde{\nabla}_X\xi = -\|Z_j + \eta\|^2 X + \tilde{\nabla}_X Z_{\mathcal{F}_j} - \langle X, Z\rangle(Z+\eta) \in \mathcal{F}_j. \tag{11}$$

On the other hand, for $X \in \mathcal{F}_1$, $Y \in \mathcal{F}_2$, Codazzi equation yields

$$\nabla^\perp_X\alpha(Y,Z) - \nabla^\perp_Y\alpha(X,Z) - \alpha([X,Y],Z) = \alpha(Y,\nabla_X Z) - \alpha(X,\nabla_Y Z). \tag{12}$$

In terms of $\beta = \alpha - \langle\ ,\ \rangle\eta$, the normal component of (11) is just

$$\beta(V,Z) + \nabla^\perp_V\eta = 0, \quad \forall\ V \in TM. \tag{13}$$

Hence, (12) becomes $R^\perp(Y,X)\eta = \beta(Y, \nabla_X Z - \langle X,Z\rangle Z) - \beta(X, \nabla_Y Z - \langle Y,Z\rangle Z)$, where $R^\perp$ stands for the curvature tensor of the normal connection of M^{n+k}. Using the Ricci equation in the left hand side we get that

$$\beta(\mathcal{D}X, Y) = \beta(X, \mathcal{D}Y), \quad \forall\ X \in \mathcal{F}_1,\ Y \in \mathcal{F}_2, \tag{14}$$

where $\mathcal{D}V = \nabla_V Z - \langle V,Z\rangle Z - A_\eta V = e^{\frac{1}{2}\|Z\|^2}\tilde{\nabla}_V\xi$, $V \in TM$. Since by (11) we have that $\mathcal{D}\mathcal{F}_j \subset \mathcal{F}_j$, the same argument after (9) applies to (14) and similarly we obtain that $\mathcal{D} = \lambda Id$, for some $\lambda \in C^\infty(M)$. We conclude that $d\xi = aId$, for some $a \in \mathbb{R}$ since the left hand side is a closed 1-form. Equivalently, $\xi = af + v$, $a \in \mathbb{R}$, $v \in \mathbb{R}^{n+k+p}$, where f stands for the immersion $M^{n+k} \subset \mathbb{R}^{n+k+p}$. By means of an inversion and a translation, we can assume without loss of generality that $\xi = af$, for some $a \neq 0$.

For $V \in TM$ we have that

$$V(-e^{-\frac{1}{2}\|Z\|^2}) = e^{-\frac{1}{2}\|Z\|^2}\langle Z, V\rangle = \langle \xi, V\rangle = a\langle f, V\rangle = V((a/2)\|f\|^2).$$

Thus, there is $c \in \mathbb{R}$ such that $\|f\|^2 = -2a^{-1}e^{-\frac{1}{2}\|Z\|^2} - c/4$. Let $\epsilon = \mathrm{sign}(c)$. Consider $\mathbb{R}^{n+k+p} = \{0\} \times \mathbb{R}^{n+k+p} \subset \mathbb{E}^{n+k+p+1}$, where $\mathbb{E}^{n+k+p+1} = \mathbb{R}^{n+k+p+1}$ if $c \geq 0$ or $\mathbb{E}^{n+k+p+1} = \mathbb{L}^{n+k+p+1}$ if $c < 0$ as in the introduction. Let $\mathcal{S} : \mathbb{R}^{n+k+p} \to \mathbb{Q}_c^{n+k+p} \subset \mathbb{E}^{n+k+p+1}$ be map

$$\mathcal{S}(y) = \nu(\sqrt{\epsilon c/4}, y), \qquad \nu = \|(\sqrt{\epsilon c/4}, y)\|^{-2} = (c/4 + \|y\|^2)^{-1}.$$

Observe that this is the stereographic projection for $c \neq 0$ and an inversion followed by a translation for $c = 0$. Consider $\widetilde{M}^{n+k} = \mathcal{S}(M^{n+k}) \subset \mathbb{Q}_c^{n+k+p}$, with its associated rulings $\widetilde{\mathcal{F}}_j = S(\mathcal{F}_j)$ that give rise to the corresponding vectors $\widetilde{Z}$, $\tilde{\eta}$ and map $\tilde{\beta}$. A straightforward computation now shows that the second fundamental form $\tilde{\alpha}$ of $\widetilde{M}^{n+k}$ is given by $\tilde{\alpha} = \nu T(\alpha - \langle\, , \,\rangle\eta)$, where $T(y) = y - 2\nu\langle y, (\sqrt{\epsilon c/4}, y)\rangle(\sqrt{\epsilon c/4}, y)$ (see Theorem 1 in [**4**]). Then, $\tilde{\eta} = 0$ and $\tilde{\alpha} = \tilde{\beta}$. Therefore, the corresponding equation (13) for $\widetilde{M}^{n+k}$ implies that $\tilde{\beta}(\widetilde{Z}, TM) = 0$. By the hypothesis on p and Proposition 8 *i*) we conclude that $\widetilde{Z} = 0$. Thus, $\widetilde{\mathcal{F}}_j$ is totally geodesic, that is precisely what we wanted to show. ∎

2 The maximal cone $\mathcal{C}^{n,k}$

Here we study doubly ruled submanifolds in maximal codimension and prove the remaining results stated in the introduction. We first state a basic result on submanifolds whose relative conullity distribution $\mathcal{E}_0^{\perp}$ is umbilical (cf. Lemma 6 *i*) and $ii - a$) in [**2**]; see also Remark 14 below).

Lemma 10. *Let $M^n \subset \mathbb{R}^{n+p}$ be a submanifold with constant index of relative nullity $n - 2 \geq \dim \mathcal{E}_0 = k > 0$. Assume that $\mathcal{E}_0^{\perp}$ is totally umbilical in M^n. Then, $\mathcal{E}_0^{\perp}$ is totally geodesic if and only if each point has a neighborhood V such that $V \subset \mathcal{L}^{n-k} \times \mathbb{R}^k$, where $\mathcal{L}^{n-k}$ is a submanifold of $\mathbb{R}^{n-k+p}$. On the other hand, $\mathcal{E}_0^{\perp}$ is nowhere totally geodesic if and only if each point has a neighborhood V such that $V \subset W^{n-k+1} \times \mathbb{R}^{k-1}$, where $W^{n-k+1} \subset \mathbb{R}_*^{n-k+1+p}$ is a cone.*

Proof: It is identical to the proof of Lemma 6 *i*) and $ii - a$) in [**2**]. ∎

As a corollary, we get the isometric version of Theorem 4. Similar result holds for any semi-Riemannian space form since Lemma 10 also extends.

Theorem 11. *Let $M^{n+k+r} \subset \mathbb{R}^{n+k+r+p}$ be a substantial $(n+r, k+r)$-ruled submanifold, with $n, k, r \geq 1$. Then, $p \leq nk$. If $p > (n-1)k$, then M^{n+k+r} is congruent to an open subset of either $N^{n+k} \times \mathbb{R}^r$ or $CN^{n+k} \times \mathbb{R}^{r-1}$, where N^{n+k} is a (n, k)-ruled submanifold in $\mathbb{R}^{n+k+p}$ or $\mathbb{S}^{n+k+p}$, respectively.*

Proof: It is a consequence of Proposition 8 and Lemma 10. Again, the description is global in view of Remark 9. ∎

We now show the linear uniqueness of $\mathcal{C}^{n,k}$.

Proposition 12. *Any substantial $(n+1, k+1)$-ruled cone $\mathcal{W}^{n+k+1} \subset \mathbb{R}^{n+k+1+nk}$ is linearly equivalent to (an open subset of) $\mathcal{C}^{n,k}$.*

Proof: Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the rulings of $\mathcal{W}^{n+k+1}$. Then, $\mathcal{F}_0(x) = \mathrm{span}\{x\}$ is 1-dimensional, for all $x \in \mathcal{W}^{n+k+1}$. Fix $y \in \mathcal{W}^{n+k+1}$ and let $\mathcal{F}_1^0 = \mathcal{F}_1(y)$ (resp. $\mathcal{F}_2^0 = \mathcal{F}_2(y)$) be the leaf of $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) that contains y. Fix also the following notations for indices:

$$1 \leq i \leq n, \quad 0 \leq j, l \leq k.$$

Since $\mathcal{F}_1^0$ and $\mathcal{F}_2^0$ are both (open subsets of) linear subspaces, there are bases $\{e_0, \ldots, e_n\}$ of $\mathcal{F}_1^0$ and $\{v_{00}, \ldots, v_{0k}\}$ of $\mathcal{F}_2^0$, with $e_0 = v_{00} = y$, and such that each one of the vectors is close enough to y. Thus, there is a leaf $\mathcal{F}_2^i$ of $\mathcal{F}_2$ such that

$e_i \in \mathcal{F}_2^i$. Moreover, there are a neighborhood $y \in U \subset \mathcal{F}_2^0$ and well defined smooth maps $\varphi_i : U \subset \mathcal{F}_2^0 \to \mathcal{F}_2^i$,

$$\varphi_i(x) = \mathcal{F}_1(x) \cap \mathcal{F}_2^i \cap \{z \in \mathbb{R}^{n+k+1+nk} : \|z\| = \|x\|, \langle z, x\rangle > 0\}.$$

It holds that $\|\varphi_i(x)\| = \|x\|$, $\varphi_i(y) = \|y\|\|e_i\|^{-1}e_i$ and $\varphi_0 = Id$. Since we have that $\{\varphi_0(y), \dots, \varphi_n(y)\}$ is a basis of $\mathcal{F}_1^0$, for $x \in \mathcal{F}_2^0$ close enough to y we have that $\{\varphi_0(x), \dots, \varphi_n(x)\}$ is a basis of $\mathcal{F}_1(x)$. Therefore, $h : \mathbb{R}^n \times U \to \mathbb{R}^{n+k+1+nk}$ given by

$$h(t, x) = x + \sum_i t_i \varphi_i(x) \tag{15}$$

is a smooth parametrization of $\mathcal{W}^{n+k+1}$ around $y = h(0, y)$ since $h(t, \lambda x) = \lambda h(t, x)$, for $\lambda \in \mathbb{R}_*$. Clearly, by fixing the line span$\{x\}$ and moving t we parametrize the leaves of $\mathcal{F}_1$. However, this is the general form of a cone with *one* ruled foliation $\mathcal{F}_1$ of rank $n+1$, but only $n+1$ transversal rulings $\mathcal{F}_2^0, \dots, \mathcal{F}_2^n$ in its closure, since there is no *a priori* linearity on the φ_i's. We search for it as follows.

Take a basis $\{v_{i0}, \dots, v_{ik}\}$ of $\mathcal{F}_2^i$ and write $\varphi_i = \sum_j \varphi_{ij} v_{ij}$. A key point is that, if $\mathcal{B} = \{v_{mj} : \ 0 \le m \le n, \ 0 \le j \le k\}$, from (15) and Proposition 7 we get that

$$\mathcal{W}^{n+k+1} \text{ is substantial} \quad \text{if and only if} \quad \mathcal{B} \text{ is linearly independent.} \tag{16}$$

Since $\mathcal{W}^{n+k+1}$ is doubly ruled, there is a smooth function $s : \mathbb{R}^n \times U \to \mathbb{R}^n$, $s = s^t(x) = (s_1^t(x), \dots, s_n^t(x))$, such that, if we write $x = \sum_j x_j v_{0j}$, the map h_t given by

$$h_t(x) = h(s^t(x), x) = \sum_j \left(x_j v_{0j} + \sum_i s_i^t(x)\varphi_{ij}(x) v_{ij} \right) \tag{17}$$

parametrizes around y the leaf of $\mathcal{F}_2$ that passes through $h(t, y)$. Observe that from the substantiallity of $\mathcal{W}^{n+k+1}$, no s_i^t nor φ_{ij} can vanish identically. Because these leaves are linear subspaces, taking partial derivatives in (17) yields, for each l, that

$$v_{0l} + \sum_{ij} \frac{\partial(s_i^t \varphi_{ij})}{\partial x_l}\, v_{ij} \ \in \ \text{span}\{h_t(v_{00}), \dots, h_t(v_{0k})\}.$$

In view of (16), we obtain that $\frac{\partial}{\partial x_l}(s_i^t\varphi_{ij}) = s_i^t(v_{0l})\varphi_{ij}(v_{0l})$. Therefore,

$$s_i^t(x)\varphi_{ij}(x) = \sum_l x_l s_i^t(v_{0l})\varphi_{ij}(v_{0l}). \tag{18}$$

Dividing the above by s_i^t we see that the functions $s_i^t(v_{0l})/s_i^t$ do not depend on t. Then, there are functions b_{il} such that $s_i^t(x) = b_{il}(x)s_i^t(v_{0l})$. Since the left-hand side does not depend on l, there are constants $a_{il} \in \mathbb{R}_*$ such that $b_{il}(x) = a_{il}b_i(x)$ and $s_i^t(v_{0l}) = a_{il}^{-1}r_i(t)$, where $b_i = b_{i0}$ and $r_i(t) = s_i^t(v_{00})$. Thus, (18) becomes $b_i(x)\varphi_{ij}(x) = \sum_l x_l c_{ij}^l$, with $c_{ij}^l = a_{il}^{-1}\varphi_{ij}(v_{0l}) \in \mathbb{R}$, and hence (17) is $h_t(x) = \sum_j x_j \left(v_{0j} + \sum_i r_i(t) w_{ij}\right)$, where $w_{ij} = \sum_l c_{il}^j v_{il}$. From the regularity of h, the substitution $y_i = r_i(t)$ is just a change of coordinates in $\mathbb{R}^n$ and we get $h(y, x) = \sum_j x_j \left(v_{0j} + \sum_i y_i w_{ij}\right)$, which parametrizes an open dense subset of a cone that is linearly equivalent to $\mathcal{C}^{n,k}$. The result now follows easily. ∎

Proof of Theorem 1: The inequality on p follows from Proposition 8. Assume that $p = nk$, and consider $\mathbb{Q}_c^N \subset \mathbb{E}^{N+1}$ as in the introduction. Therefore, the cone over M^{n+k+r}, $CM = \{tx : t \in \mathbb{R}_*, x \in M\} \subset \mathbb{E}^{N+1}$ is a substantial $(n+k+r+1)$-dimensional $(n+r+1, k+r+1)$-ruled submanifold in maximal codimension nk. Since these properties have nothing to do with the definite (or not) inner product in $\mathbb{E}^{N+1}$, we can argue for such cones in $\mathbb{R}^{N+1}$ with the usual inner product that, by Proposition 8 *ii*), have totally umbilical relative conullity $\mathcal{E}^\perp$. Since CM is a cone, its conullity is nowhere totally geodesic and then, by Lemma 10, it has the structure

$$CM = \mathcal{W}^{n+k+1} \times \mathbb{R}^r \subset \mathbb{R}^{N+1}, \tag{19}$$

where $\mathcal{W}^{n+k+1} \subset \mathbb{R}^{n+k+1+nk}$ should then be a $(n+1, k+1)$-ruled cone in maximal codimension nk. The proof follows from Proposition 12. ∎

Remark 13. Using (19) we can say a little more in Theorem 1 in terms of cylinders. With the notations of the introduction, according to the position of the $\mathbb{R}^r$ factor in (19) relative to $\mathbb{Q}_c^N$, there exists $T \in GL(N - r + 1, \mathbb{R})$ such that $M^{n+k+r} \subset \mathbb{Q}_c^N$ is (an open subset of) one of the following:

1. $c > 0$. $(T(\mathcal{C}^{n,k}) \times \mathbb{R}^r) \cap \mathbb{S}_c^N \subset \mathbb{S}_c^N \subset \mathbb{R}^{N+1}$. Complete only for $r = 0$. Otherwise its singular set is $\mathbb{S}_c^{r-1} = (\{0\} \times \mathbb{R}^r) \cap \mathbb{S}_c^N$.

2. $c = 0$. $T(\mathcal{C}^{n,k}) \times \mathbb{R}^{r-1} \subset \mathbb{R}^N$ with $r \geq 1$, if the factor $\mathbb{R}^r$ is *not* parallel to $\mathbb{R}^N = \mathbb{Q}_0^N \subset \mathbb{R}_*^{N+1}$. Always singular with singular set $\mathbb{R}^{r-1} = \{0\} \times \mathbb{R}^{r-1}$.

3. $c = 0$. $(T(\mathcal{C}^{n,k}) \cap \mathbb{R}_D^{N-r}) \times \mathbb{R}^r \subset \mathbb{R}_D^{N-r} \times \mathbb{R}^r = \mathbb{R}^N$, if the factor $\mathbb{R}^r$ is parallel $\mathbb{Q}_0^N$. All are complete. Observe that $T(\mathcal{C}^{n,k}) \cap \mathbb{R}_D^{N-r} = T(\mathcal{N}_{T^*D}^{n,k})$.

4. $c < 0$. $(T(\mathcal{C}^{n,k}) \times \mathbb{L}^r) \cap \mathbb{H}_c^N \subset \mathbb{H}_c^N \subset \mathbb{R}^{N-r+1} \times \mathbb{L}^r = \mathbb{L}^{N+1}$ with $r \geq 1$, if the factor $\mathbb{R}^r$ is Lorentzian. Always singular, with singular set $\mathbb{H}_c^{r-1} = (\{0\} \times \mathbb{L}^r) \cap \mathbb{H}_c^N$.

5. $c < 0$. $(T(\mathcal{C}^{n,k}) \times \mathbb{R}^r) \cap \mathbb{H}_c^N \subset \mathbb{H}_c^N \subset \mathbb{L}^{N-r+1} \times \mathbb{R}^r = \mathbb{L}^{N+1}$, if the factor $\mathbb{R}^r$ is Riemannian. All complete.

6. $c < 0$. $(T(\mathcal{C}^{n,k}) \times \mathbb{O}^r) \cap \mathbb{H}_c^N \subset \mathbb{H}_c^N \subset \mathbb{O}^{N-r+1} \times \mathbb{O}^r = \mathbb{L}^{N+1}$, if the factor $\mathbb{R}^r = \mathbb{O}^r$ is degenerate: here we write $\mathbb{L}^{N+1}$ as the sum of degenerate subspaces. Complete.

The r-cylinders correspond precisely to cases 1, 3 and 5, according to the sign of c. Hence, Corollary 5 says that we can forget about cases 2, 4 and 6 when working in the conformal setting.

Remark 14. We take this opportunity to observe that Lemma 6 in [**2**] does not hold for $\dim \mathcal{E}_0 = n-1$ since flat hypersurfaces are generically noncylindrical. Moreover, its part $ii-b)$ is incomplete, since it is missing the product case that corresponds to a degenerate factor, like case 6 in Remark 13. The corresponding case in Theorem 3 part (I) $ii) - b)$ in [**2**] is thus also missing. However, the description using warped product representations that follows that lemma is correct and is an alternative and more intrinsic way to describe all cases. Observe also that, similarly as Remark 13, for $c = 0$ the lemma is correct, for $c > 0$ can be improved by taking $\tilde{c} = c$ (i.e., totally geodesic), and for $c < 0$ there are only three cases, according with the sign of $\tilde{c}$.

Remark 15. Let us argue that $\mathcal{C}^{n,k}$, and hence ${}^c\mathcal{N}_L^{n,k,r}$, is embedded. Given $0 \leq i' \leq n$, $0 \leq j' \leq k$, let $\mathcal{U}_{i'j'}$ be the open dense subset $\mathcal{U}_{i'j'} = \{S \in \mathbb{R}^{(n+1)\times(k+1)} : S_{i'j'} \neq 0\}$. Let $f = f_{i'j'} : \mathcal{U}_{i'j'} \to \mathbb{R}^{n\times k}$ be the map given by $f(S) = (S_{i'j'}S_{ij} - S_{i'j}S_{ij'})_{i\neq i', j\neq j'}$. Then, $f^{-1}(0) = \mathcal{C}^{n,k} \cap \mathcal{U}_{i'j'}$ are all regular points of f because $d/dt|_{t=0}(f(S + tE_{ij})) = S_{i'j'}E_{ij}$, for all $i \neq i'$, $j \neq j'$ and $S \in \mathcal{C}^{n,k} \cap \mathcal{U}_{i'j'}$, where E_{ij} stands for the matrix in $\mathbb{R}^{(n+1)\times(k+1)}$ that has zeroes in all entries except a one at (i, j).

Proof of Corollary 2: The first part follows from Remark 13 cases 2 and 3. For the second, assume we have an affine isomorphism that carries $\mathcal{N}_D^{n,k}$ to $\mathcal{N}_{D'}^{n,k}$. This is equivalent to have $L \in GL(N - r + 1, \mathbb{R})$ that takes one into the other extrinsically. Since L is linear and $\mathcal{C}^{n,k}$ is a cone, we have that $\mathcal{C}^{n,k}$ is invariant under L and hence under L^{-1}. Because any rank s matrix is the sum of s rank one matrices, we easily see that both L and L^{-1} must preserve ranks. Therefore, from $L(\mathcal{N}_D^{n,k}) = \mathcal{N}_{L^{-1*}D}^{n,k}$, all we have to see is that L^* must also preserves ranks, or, equivalently, $\mathcal{C}^{n,k}$.

If L preserve ranks, then we have a bilinear map $(x, y) \mapsto L(x^t y) \in \mathcal{C}^{n,k} \cup \{0\}$, $x \in \mathbb{R}^{n+1}$, $y \in \mathbb{R}^{k+1}$. Thus, by fixing y and moving x, it is easy to see that there are well defined (up to factors) $A \in GL(n + 1, \mathbb{R})$, $B \in GL(k + 1, \mathbb{R})$ such that $L(x^t y) = (Ax)^t(By)$. Therefore, $L^*(x^t y) = (A^*x)^t(B^*y)$ also preserves $\mathcal{C}^{n,k}$. ■

Proof of Corollary 3: Take two complete leaves of $\mathcal{F}_1$. Since they cannot intersect and they are obtained as the intersection between the sphere and linear $(n+r+1)$-dimensional (complete) subspaces, we have that $n + k + r + p + 1 \geq 2(n + r + 1)$. Hence, by Theorem 1, $n + r + 1 - k \leq p \leq nk$. The proof follows from the fact that no ${}^1\mathcal{N}_L^{n,k,r}$ is complete if $r > 0$; see Remark 13 case 1. ■

Proof of Corollary 5: It follows immediately from Theorems 1 and 4. ■

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