Multiple periodic solutions for p-Laplacian problems with convex-concave nonlinearities

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Abstract

We prove the existence of at least three periodic solutions of a nonlinear differential equation involving p-Laplacian and convex-concave nonlinearities.

1 Introduction

In this paper, we deal with the existence of multiple solutions to the periodic boundary conditions problem

$$\begin{cases} -\dfrac{d}{dx}(|\dot{u}|^{p-2}\dot{u}) + |u|^{p-2}u = \lambda\alpha(x)|u|^{q-2}u + \beta(x)|u|^{r-2}u + f(x), \quad u \in \mathbb{R}^N \\ u(0) = u(1), \quad \dot{u}(0) = \dot{u}(1), \end{cases} \tag{1.1}$$

where, f is a nonzero continuous 1-periodic vector-valued function, α and β are positive continuous 1-periodic scalar-valued functions, λ a real positive number and $|\,.\,|$ is the euclidean norm in $\mathbb{R}^N$. Here, we are interested by nonhomogeneous nonlinear equations with concave and convex nonlinearities, so we will assume hereafter

$$1 < q < p < r.$$

The operator $-\frac{d}{dx}(|\dot{u}|^{p-2}\dot{u})$ is the so-called one-dimensional $p-$Laplacian, which reduces to the linear operator "second derivative" for $p = 2$.

Received by the editors August 2005.
Communicated by J. Mawhin.
2000 *Mathematics Subject Classification :* 34C25,34C15,34B15.
Key words and phrases : Ekeland's principle, p-Laplacian operator, Palais-Smale condition, Periodic solutions.

The literature dealing with existence and multiplicity results for two-points boundary value problems involving the $p-$Laplacian operator is very extensive. We can refer the reader to [1, 2, 4, 5, 11, 15, 20, 24] and the references therein. In particular, problems with concave-convex nonlinearities in the homogeneous case ($f = 0$) are considered in [1, 2, 20].

The case of periodic boundary conditions has been studied in [6, 10, 13, 14, 16, 23]. In [16], the authors consider nonhomogeneous nonlinear equations with convex (or concave) nonlinearities. For such kind of problems, various methods are used like topological methods, time map methods and variational methods. In this paper, we use variational tools and follow the ideas developed in the so-called fibering method [17, 18, 19] which was later used and developed by several authors. In particular, in this framework, we use tools like the extraction of Palais-Smale sequences in the Nehari manifold [9] and show existence and multiplicity results. Here, we consider a nonhomogeneous and non autonomous differential equation with convex-concave nonlinearities.

The paper is organized as follow. In the next section, we introduce some notations and definitions and state some technical lemmas. In section 3, we establish ground state constrained minimization problems associated to the problem (1.1) and establish the existence of Palais-Smale sequences for the corresponding functionals. Finally in section 4, we give the proof of our main result.

2 The main result and preliminaries

By solutions of the problem (1.1) we understand critical points of the associated Euler-Lagrange functional

$$E(u) = \frac{1}{p}P(u) - \frac{\lambda}{q}Q(u) - \frac{1}{r}R(u) - L(u)$$

in the Banach space $W := \{u \in W^{1,p}([0,1]) \, : \, u(0) = u(1)\}$ endowed with the norm

$$||u|| = \left(\int_0^1 |\dot{u}|^p \, dx + \int_0^1 |u|^p \, dx\right)^{1/p},$$

where

$$P(u) = \|u\|^p, \;\; Q(u) = \int_0^1 \alpha(x)|u(x)|^q dx,$$

$$R(u) = \int_0^1 \beta(x)|u(x)|^r dx, \;\; L(u) = \int_0^1 f(x) \cdot u(x) dx$$

and "$\cdot$" denotes the inner product in $\mathbb{R}^N$.

Since α and β are positive and continuous functions then the norms defined by $\|u\|_q = Q(u)^{1/q}$ and $\|u\|_r = R(u)^{1/r}$ are equivalent to the respective ones of $L^q([0,1])$ and $L^r([0,1])$. So, in the sequel, we consider these spaces equipped with these norms.

Here we state our main existence result. Let

$$\widehat{\lambda} = \inf_{u \in \mathbb{S}} \widehat{C}_0 \frac{P^{\frac{r-q}{r-p}}(u)}{Q(u)\, R^{\frac{p-q}{r-p}}(u)},$$

where $\mathbb{S}$ is the unit sphere of W and

$$\widehat{C_0} = \frac{r-p}{p-q}\left(\frac{p-q}{r-q}\right)^{\frac{r-q}{r-p}}.$$

If the norm of the nonhomogeneous term f is sufficiently small, then we obtain at least three nontrivial solutions.

Theorem 2.1. *Let $1 < q < p < r$, $\lambda \in]0, \widehat{\lambda}[$ and f a nonzero function in the dual space of W verifying (3.1). Then the problem (1.1) has at least three nontrivial solutions.*

Make precise that our existence and multiplicity result is nonlocal. Indeed, we will see in the next section that $\widehat{\lambda}$ is greater than a positive constant depending only on α, β, p, q and r. Remark also, in [16], the authors consider some nonhomogeneous and non autonomous nonlinear equations with only convex(or concave) nonlinearities.

Let us define the Nehari manifold

$$\mathcal{N} = \{u \in W \setminus \{0\} \,:\, E'(u)(u) = 0\}.$$

It is well known that this manifold contains all critical points of the Euler-Lagrange functional E, which is usually bounded below on it. So we can easily seek critical points of E on $\mathcal{N}$ by minimization. An interesting characterization of the Nehari manifold $\mathcal{N}$, which will be used below, is the following

$$\mathcal{N} = \left\{tu \,:\, u \in W \setminus \{0\} \text{ and } \frac{\partial E}{\partial t}(tu) = 0\right\}$$

Then following the fibering scheme, it is natural to introduce the modified Euler-Lagrange functional [7, 12, 17, 18, 19, 22, 21] $\widetilde{E}$ defined on $\mathbb{R} \times W$ by

$$\widetilde{E}(t,u) := E(tu) = \frac{1}{p}|t|^p P(u) - \frac{\lambda}{q}|t|^q Q(u) - \frac{1}{r}|t|^r R(u) - tL(u).$$

The method consists to find, for a given $u \in W \setminus \{0\}$, critical points $t(u)$ of the real valued function $t \to \widetilde{E}(t,u)$ then substituting t by $t(u)$ into $\widetilde{E}(t,u)$ we obtain a new functional $J(u) := \widetilde{E}(t(u),u)$. If u^* is a critical point of the functional J then $t(u^*)u^*$ is a critical points of E on W.

Let u be an arbitrary element of $W \setminus \{0\}$, we consider

$$\partial_t \widetilde{E}(t,u) = F(t,u) - L(u)$$

where $F(t,u) = t|t|^{p-2}P(u) - \lambda t|t|^{q-2}Q(u) - t|t|^{r-2}R(u)$ and

$$\partial_{tt}\widetilde{E}(t,u) = (p-1)|t|^{p-2}P(u) - \lambda(q-1)|t|^{q-2}Q(u) - (r-1)|t|^{r-2}R(u)$$

respectively the first and the second derivative of the real valued function: $t \mapsto \widetilde{E}(t,u)$.

To determine the zeros of $t \longmapsto \partial_t \widetilde{E}(t,u)$ we analyze the function $t \longmapsto F(t,u)$ which is an odd function so it suffices to consider positive value of t.

Lemma 2.2. *For every $u \in W \setminus \{0\}$, there is a unique $\lambda_0(u) > 0$ such that for all $0 < \lambda < \lambda_0(u)$, the real valued function $t \mapsto F(t,u)$ has exactly one positive global maximum $F(\overline{t}(u), u)$ and one negative local minimum $F(\underline{t}(u), u)$.*

Proof. Let u be an arbitrary element of $W \setminus \{0\}$ and let us show the existence of $\overline{t}(u)$ and $\underline{t}(u)$.

Let

$$\partial_t F(t,u) = t^{q-2} G(t,u),$$

where $G(t,u) = (p-1)t^{p-q}P(u) - \lambda(q-1)Q(u) - (r-1)t^{r-q}R(u)$.

Then

$$\partial_{tt} F(t,u) = (q-2)t^{q-3}G(t,u) + t^{q-2}\partial_t G(t,u),$$

with

$$\partial_t G(t,u) = t^{p-q-1}\left((p-1)(p-q)P(u) - (r-1)(r-q)t^{r-p}R(u)\right).$$

It is clear that the real valued function $t \longmapsto G(t,u)$ is increasing on $(0, t(u))$, decreasing on $(t(u), +\infty)$ and attains its unique maximum for $t = t(u)$, where

$$t(u) = \left(\frac{(p-1)(p-q)}{(r-1)(r-q)} \frac{P(u)}{R(u)}\right)^{\frac{1}{r-p}}. \tag{2.1}$$

Thus, the function $t \longmapsto G(t,u)$ has two positive zeros (resp. one positive zero) if $G(t(u),u) > 0$ (resp. if $G(t(u),u) = 0$) and has no zero if $G(t(u),u) < 0$. On the other hand, a direct computation gives

$$G(t(u),u) = (r-1)\frac{r-p}{p-q}\left(\frac{(p-1)(p-q)}{(r-1)(r-q)} \frac{P(u)}{R(u)}\right)^{\frac{r-q}{r-p}} R(u) - \lambda(q-1)Q(u).$$

Similarly, $G(t(u),u) > 0$ (resp. $G(t(u),u) < 0$) if $\lambda < \lambda(u)$ (resp. $\lambda > \lambda(u)$) and $G(t(u),u) = 0$ if $\lambda = \lambda(u)$, where

$$\lambda(u) = \widehat{C} \frac{P^{\frac{r-q}{r-p}}(u)}{Q(u)\, R^{\frac{p-q}{r-p}}(u)}, \tag{2.2}$$

with

$$\widehat{C} = \frac{(r-1)(r-p)}{(q-1)(p-q)} \left(\frac{(p-1)(p-q)}{(r-1)(r-q)}\right)^{\frac{r-q}{r-p}}.$$

Hence, if $\lambda \in (0, \lambda(u))$, the real valued function $t \longmapsto \partial_t F(t,u)$ has exactly two positive zeros, denoted by $\underline{t}(u)$ and $\overline{t}(u)$, verifying $0 < \underline{t}(u) < t(u) < \overline{t}(u)$.

Since, $G(\underline{t}(u),u) = G(\overline{t}(u),u) = 0$, $\partial_t G(t,u) > 0$ for $t < t(u)$ and $\partial_t G(t,u) < 0$ for $t > t(u)$, it follows that

$$\partial_{tt} F(\underline{t}(u), u) > 0 \text{ and } \partial_{tt} F(\overline{t}(u), u) < 0.$$

This means that the real valued function $t \longmapsto F(t,u)$, $t > 0$, achieves its unique local minimum at $\underline{t}(u)$ and its global maximum at $\overline{t}(u)$.

Now we compute $\lambda_0(u)$ such that $F(\overline{t}(u), u) > 0$ and $F(\underline{t}(u), u) < 0$.

Let $F(t,u) = t^{q-1}H(t,u)$, where $H(t,u) = t^{p-q}P(u) - \lambda Q(u) - t^{r-q}R(u)$. Then similar computations as in the first part of the proof show that there is a constant $\lambda_0(u)$ defined by

$$\lambda_0(u) = \widehat{C_0}\frac{P^{\frac{r-q}{r-p}}(u)}{Q(u)\,R^{\frac{p-q}{r-p}}(u)}, \tag{2.3}$$

with

$$\widehat{C_0} = \frac{r-p}{p-q}\left(\frac{p-q}{r-q}\right)^{\frac{r-q}{r-p}},$$

such that, if $\lambda \in (0, \lambda_0(u))$, the real valued function $t \longmapsto F(t,u)$ has two positive zeros.

Now it suffices to remark that $\lambda(u) > \lambda_0(u)$, since $\lambda(u) = \frac{r-1}{q-1}\left(\frac{p-1}{r-1}\right)^{\frac{r-q}{r-p}}\lambda_0(u)$ and $\frac{r-1}{q-1}\left(\frac{p-1}{r-1}\right)^{\frac{r-q}{r-p}} > 1$. ■

Let us notice that for every real number $\gamma > 0$, we have

$$\begin{aligned}
\widetilde{E}\left(\gamma t, \frac{u}{\gamma}\right) &= \widetilde{E}(t,u),\\
\partial_t\widetilde{E}\left(\gamma t, \frac{u}{\gamma}\right) &= \frac{1}{\gamma}\partial_t\widetilde{E}(t,u),\\
\partial_{tt}\widetilde{E}\left(\gamma t, \frac{u}{\gamma}\right) &= \frac{1}{\gamma^2}\partial_{tt}\widetilde{E}(t,u).
\end{aligned}$$

It follows that $t\left(\frac{u}{\gamma}\right) = \gamma t(u)$. Consequently, the functionals $u \mapsto F(\underline{t}(u), u)$ and $u \mapsto F(\overline{t}(u), u)$ defined on $W \setminus \{0\}$ are 0-homogeneous, so we can assume hereafter $u \in \mathbb{S}$ where $\mathbb{S}$ is the unit sphere of W.

Let

$$\widehat{\lambda} := \inf_{u\in\mathbb{S}} \lambda_0(u). \tag{2.4}$$

If S_q (resp. S_r) denotes the best Sobolev constant of the embedding $W \subset L^q(0,1)$ (resp. $W \subset L^r(0,1)$), then $\widehat{\lambda} \geq \widetilde{\lambda}$ where

$$\widetilde{\lambda} = \widehat{C}(S_q)^{q/p}(S_r)^{\frac{r(p-q)}{p(r-p)}} > 0.$$

Lemma 2.3. *If $\lambda \in (0, \widehat{\lambda})$ then it holds*

$$\overline{F} = \inf_{u\in\mathbb{S}} F(\overline{t}(u), u) > 0.$$

Proof.
By contradiction suppose that $\overline{F} = 0$, there exists a sequence (u_n) in $\mathbb{S}$ such that $\lim_{n\to+\infty} F(\overline{t}(u_n), u_n) = 0$. This implies, up to a subsequence denoted again by (u_n), that

$$\overline{t}(u_n)^{p-1} - \lambda\overline{t}(u_n)^{q-1}Q(u_n) - \overline{t}(u_n)^{r-1}R(u_n) = o_n(1) \tag{2.5}$$

and by definition of $\overline{t}(u_n)$, $F(\overline{t}(u_n), u_n) > 0$ for all $n \in \mathbb{N}$.
On the other hand for all $n \in \mathbb{N}$, $\partial_t F(\overline{t}(u_n), u_n) = 0$ then

$$(p-1)\overline{t}(u_n)^{p-1} - (q-1)\lambda\overline{t}(u_n)^{q-1}Q(u_n) - (r-1)\overline{t}(u_n)^{r-1}R(u_n) = 0 \tag{2.6}$$

By the embedding $W \subset L^r(0,1)$, there is $C > 0$ such that $R(u_n) \leq C$.
We consider two cases:

- $\lim_{n\to+\infty} \overline{t}(u_n) = 0$.
Since $p < r$ the last term in (2.6) is $o_n(\overline{t}(u_n)^{p-1})$ and we derive from this equation that
$$Q(u_n) = \frac{1}{\lambda}\frac{p-1}{q-1}\overline{t}(u_n)^{p-q} + o_n(\overline{t}(u_n)^{p-q}).$$
It follows
$$F(\overline{t}(u_n), u_n) = \overline{t}(u_n)^{p-1}\left(\frac{q-p}{q-1} + o_n(1)\right).$$
that is strictly negative for sufficiently large n. We get a contradiction.

- The sequence $\overline{t}(u_n)$ is bounded away from zero.
Combining the equations (2.5) and (2.6), we obtain easily
$$R(u_n) = \frac{p-q}{r-q}\overline{t}(u_n)^{p-r} + o_n(\overline{t}(u_n)^{1-r}) \tag{2.7}$$
and
$$Q(u_n) = \frac{1}{\lambda}\frac{r-q}{r-p}\overline{t}(u_n)^{p-q} + o_n(\overline{t}(u_n)^{1-q}). \tag{2.8}$$
Substituting $R(u_n)$ and $Q(u_n)$ in the equation (2.5), it follows
$$\overline{t}(u_n)^{p-1}\left(\frac{p+q-2r}{(r-p)(r-q)}\right) = o_n(1).$$
Then we get a contradiction. ■

3 Constrained minimization problems

For every $0 < \lambda < \widehat{\lambda}$, if $|F(\underline{t}(u), u)| < |L(u)| < F(\overline{t}(u), u)$ then the real valued function $t \in \mathbb{R} \mapsto \partial_t \widetilde{E}(t, u)$ has exactly five distinct zeros. Hence if $|L(u)| < \overline{F}$ then this function has at least three distinct zeros denoted by $t_i(u)$ $1 \leq i \leq 3$ with $t_1(u) < t_2(u) < t_3(u)$, where $t_2(u)$ corresponds to a local minimum, $t_1(u)$ and $t_3(u)$ correspond to local maxima of $\widetilde{E}(., u)$. In particular, this holds if the inequality
$$||f||_* < \overline{F} \tag{3.1}$$
where $||\ ||_*$ denote the norm on the dual space of W, is satisfied, since for all $u \in \mathbb{S}$, $|L(u)| \leq ||f||_*$. Make precise that if $u \in \mathbb{S}$ is such that $L(u) > 0$ (resp. $L(u) < 0$) then $t_1^+(u) < 0 < t_2^+(u) < t_3^+(u)$ (resp. $t_1^-(u) < t_2^-(u) < 0 < t_3^-(u)$) and we get the following identities
$$t_1^+(u) = -t_3^-(-u),\ \ t_3^+(u) = -t_1^-(-u) \ \text{ and } \ t_2^+(u) = -t_2^-(-u),\ \ \forall\, u \in \mathbb{S}. \tag{3.2}$$
Since we are interested by nontrivial solutions to Problem (1.1), we have to show that $\pm\infty$ and 0 are not accumulation points for $t_i(u)$, $i \in \{1, 2, 3\}$, when u describes

the unit sphere $\mathbb{S}$. To show this, since L is odd and (3.2) holds true, we can confine ourselves to the case where u describes the complete sub-manifold $\mathbb{S}^+ := \{u \in \mathbb{S} : L(u) \geq 0\}$ of $\mathbb{S}$. Indeed, if we define, on the sub-manifolds $\mathbb{S}^{\pm}$, the functionals $J_i^{\pm}(u) := \widetilde{E}(t_i^{\pm}(u), u)$, $i \in \{1,2,3\}$ we have for $u \in \mathbb{S}^+$, $J_i^+(u) = J_i^-(-u)$. So, in the sequel we drop the sign "plus" ; denoting $t_i(u) := t_i^+(u)$ and $J_i(u) := J_i^+(u)$.

Lemma 3.1. *Let $\lambda \in (0, \widehat{\lambda})$ and f in the dual space of W verifying (3.1). Then the functionals $J_i(u)$ defined on $\mathbb{S}^+$ are bounded below.*

Proof. For every $i \in \{1,2,3\}$, using the fact that $\partial_t \widetilde{E}(t_i(u), u) = 0$ in the expression $\widetilde{E}(t_i(u), u)$, we get

$$J_i(u) = \left(\frac{1}{p} - \frac{1}{r}\right) |t_i(u)|^p + \lambda \left(\frac{1}{r} - \frac{1}{q}\right) |t_i(u)|^q Q(u) + \left(\frac{1}{r} - 1\right) t_i(u) L(u) \qquad (3.3)$$

Then,

$$J_i(u) \geq \left(\frac{1}{p} - \frac{1}{r}\right) |t_i(u)|^p + \lambda \left(\frac{1}{r} - \frac{1}{q}\right) |t_i(u)|^q S_q^q + \left(\frac{1}{r} - 1\right) |t_i(u)| \|f\|_*,$$

where S_q denotes the best Sobolev constant in the embedding $W \subset L^q(0,1)$. The hypothesis $1 < q < p < r$ achieves the claim. ∎

At this stage, we can define

$$\alpha_i = \inf_{u \in \mathbb{S}^+} J_i(u), \quad i \in \{1,2,3\}. \qquad (3.4)$$

Under the assumptions of the previous lemma, we have

Lemma 3.2. *If $(u_n^i) \subset \mathbb{S}^+$, $i \in \{1,2,3\}$, are minimizing sequences of α_i then*

$$\text{(i)} \quad \liminf_{n \to +\infty} t_1(u_n^1) > -\infty \quad \textit{and} \quad \limsup_{n \to +\infty} t_1(u_n^1) < 0,$$

$$\text{(ii)} \quad \liminf_{n \to +\infty} t_2(u_n^2) > -\infty \quad \textit{and} \quad \limsup_{n \to +\infty} t_2(u_n^2) < 0,$$

$$\text{(iii)} \quad \liminf_{n \to +\infty} t_3(u_n^3) > 0 \quad \textit{and} \quad \limsup_{n \to +\infty} t_3(u_n^3) < +\infty,$$

Proof. First of all we show that, for $i \in \{1,2,3\}$, the sequence $(t_i(u_n^i))$ is bounded. Since $\partial_t \widetilde{E}(t_i(u_n^i), u_n^i) = 0$, it follows that

$$\begin{aligned} |t_i(u_n^i)|^p &= \lambda |t_i(u_n^i)|^q Q(u_n^i) + |t_i(u_n^i)|^r R(u_n^i) + t_i(u_n^i) L(u_n^i) \\ &\leq \lambda |t_i(u_n^i)|^q C(R(u_n^i))^{\frac{q}{r}} + |t_i(u_n^i)|^r R(u_n^i) + \varepsilon |t_i(u_n^i)|^p + C_\varepsilon \|f\|_*^{p'}, \end{aligned}$$

for some $0 < \varepsilon < 1$ and some positive constant C_ε. Hence

$$(1-\varepsilon)|t_i(u_n^i)|^p \leq \lambda C(|t_i(u_n^i)|^r R(u_n^i))^{\frac{q}{r}} + |t_i(u_n^i)|^r R(u_n^i) + C_\varepsilon \|f\|_*^{p'}.$$

Suppose that $|t_i(u_n^i)|^p \to +\infty$ as $n \to +\infty$. Then $|t_i(u_n^i)|^r R(u_n^i) \to +\infty$ and it follows, since $q/r < 1 < p$, that $(|t_i(u_n^i)^r| R(u_n^i))^{\frac{q}{r}}$ and $t_i(u_n^i) L(u_n^i)$ are negligible with respect to $|t_i(u_n^i)|^r R(u_n^i)$. Thus, $|t_i(u_n^i)|^p = |t_i(u_n^i)|^r R(u_n^i)(1 + o_n(1))$ and consequently

$$J_i(u_n^i) = |t_i(u_n^i)|^r R(u_n^i) \left(\frac{1}{p} - \frac{1}{r} \right) (1 + o_n(1)),$$

which tends to $+\infty$ with n, and this is impossible.

Now we show that the sequence $(t_2(u_n^2))$ is bounded away from zero. Since (u_n^2) is bounded in W then, there exists a subsequence of (u_n^2), still denoted by (u_n^2), which converges to some $\overline{u}$, weakly in W and strongly in $L^s([0,1])$, for every $s \geq 1$. Recall that for every $u \in \mathbb{S}^+$ and $\lambda \in (0, \widehat{\lambda})$, we have $\partial_t \widetilde{E}(t_2(u), u) = 0$ and $\partial_{tt} \widetilde{E}(t_2(u), u) \neq 0$. Then the implicit function theorem implies that $t_2(u)$ is C^1 with respect to u since E is. Then $J_2(u_n)$ tends to zero when n tends to $+\infty$. Since α_2 is strictly negative we get a contradiction.

We show, in the same manner for $i = 1$ and $i = 3$, that the sequences $(t_i(u_n^i))$ are bounded away from zero.
Using $\partial_t \widetilde{E}(t_i(u_n^i), u_n^i) = 0$ in the inequality $\partial_{tt} \widetilde{E}(t_i(u_n^i), u_n^i) < 0$, we get

$$(p - q)t_i(u_n^i)^{p-1} - (r - q)t_i(u_n^i)^{r-1} R(u_n^i) + (q - 1)L(u) < 0$$

Since, $(q - 1)L(u) \geq 0$ and $R(u) \leq S_r^r$, it follows

$$p - q < (r - q)S_r^r t_i(u_n^i)^{r-p}$$

Then if $t_i(u_n^i)$ tends to zero when n tends to $+\infty$, we get a contradiction. ■

Under the assumptions of Lemma 3.1, we have

Proposition 3.3. *Let* $(u_n^i) \subset \mathbb{S}$ *be a minimizing sequence of* α_i, $i \in \{1, 2, 3\}$. *Then,* $(U_n^i) := (t_i(u_n^i)u_n^i)$ *is a Palais-Smale sequence for the functional* E.

The main idea of this proof is in [9], we thank A. El Hamidi for bringing to our attention this issue.

Proof. For every $i \in \{1, 2, 3\}$, we prove the statement in the same way so we drop the index and consider $(u_n) \subset \mathbb{S}^+$ be a minimizing sequence of α of the functional J. First, according to the previous lemma, it is clear that (U_n) is bounded in W. Using the Ekeland variational principle on the complete manifold $(\mathbb{S}^+, ||\,.\,||)$ to the functional J, we conclude that

$$|J'(u_n)(\varphi_n)| \leq \frac{1}{n} ||\varphi_n||, \text{ for every } \varphi_n \in \mathrm{T}_{u_n}\mathbb{S}^+,$$

where $\mathrm{T}_{u_n}\mathbb{S}^+$ is the tangent space to $\mathbb{S}^+$ at the point u_n. Moreover, for every $\varphi_n \in \mathrm{T}_{u_n}\mathbb{S}^+$, one has

$$\begin{aligned} J'(u_n)(\varphi_n) &= \partial_t \widetilde{E}(t(u_n), u_n) t'(u_n)(\varphi_n) + \partial_u \widetilde{E}(t(u_n), u_n)(\varphi_n), \\ &= \partial_u \widetilde{E}(t(u_n), u_n)(\varphi_n), \end{aligned}$$

since $\partial_t \widetilde{E}(\underline{t}(u_n, \lambda), u_n) = 0$.
Furthermore, let

$$\begin{array}{rccl} \pi : & W \setminus \{0\} & \longrightarrow & \mathbb{R} \times \mathbb{S} \\ & u & \longmapsto & \left(||u||, \frac{u}{||u||} \right) := (\pi_1(u), \pi_2(u)). \end{array}$$

Applying Hölder's inequality, we get for every $(u,\varphi)\in(W\setminus\{0\})\times W$:

$$\begin{cases} |\pi_1'(u)(\varphi)| & \leq & ||\varphi||, \\ ||\pi_2'(u)(\varphi)|| & \leq & 2\frac{||\varphi||}{||u||}. \end{cases}$$

From Lemma 3.2, there is a positive constant C such that

$$|t(u_n)|\geq C,\ \ \forall\, n\in\mathbb{N}.$$

Then for every $\varphi\in W$, there are $\varphi_n^1\in\mathbb{R}$ and $\varphi_n^2\in \mathrm{T}_{u_n}\mathbb{S}^+$ such that $|\varphi_n^1|\leq||\varphi||$, $||\varphi_n^2||\leq\frac{2}{C}||\varphi||$ and

$$\begin{array}{rcl} E'(t(u_n)u_n)(\varphi) & = & \partial_t\widetilde{E}(t(u_n),u_n)(\varphi_n^1)+\partial_u\widetilde{E}(t(u_n),u_n)(\varphi_n^2), \\ & = & \partial_u\widetilde{E}(t(u_n),u_n)(\varphi_n^2), \\ & = & J'(u_n)(\varphi_n^2). \end{array}$$

Therefore,

$$\begin{array}{rcl} E'(t(u_n)u_n)(\varphi) & \leq & \dfrac{1}{n}||\varphi_n^2|| \\ & \leq & \dfrac{2}{nC}||\varphi||. \end{array}$$

We easily conclude that

$$\lim_{n\to\infty}||E'(U_n)||_*=0.$$

■

4 Proof of the main theorem

Proof of Theorem 2.1

We will use the notations of the previous theorem. According to proposition 3.3, we know that $E(U_n^i)$ converges to α_i, $||E'(U_n^i)||_*$ converges to 0 as n tends to $+\infty$ and that (U_n^i) is bounded in W. Passing if necessary to a subsequence, we have

$$\begin{array}{lcll} U_n^i & \rightharpoonup & U^i & \text{in}\ \ W, \\ U_n^i & \to & U^i & \text{in}\ \ L^\delta(0,1),\ \ \text{for all}\ \ \delta\geq 1, \\ U_n^i & \to & U^i & \text{a.e}\ \ (0,1). \end{array}$$

Let $v_n^i=U_n^i-U^i$, then using Theorem 3.3 and a lemma due to Brézis-Lieb [3], we get

$$\begin{array}{rcl} ||v_n^i||^p & = & ||U_n^i||^p-||U^i||^p+\mathrm{o}_n(1), \\ ||v_n^i||_q^q & = & ||U_n^i||_q^q-||U^i||_q^q+\mathrm{o}_n(1), \\ ||v_n^i||_r^r & = & ||U_n^i||_r^r-||U^i||_r^r+\mathrm{o}_n(1). \end{array}$$

It follows that

$$E(v_n^i) = E(U_n^i) - E(U^i) + \mathrm{o}_n(1),$$
$$E'(v_n^i) = E'(U_n^i) - E'(U^i) + \mathrm{o}_n(1),$$

and consequently $E'(v_n^i)v_n^i \to 0$ as $n \to +\infty$, which implies that

$$||v_n^i||^p = \lambda ||v_n^i||_q^q + ||v_n^i||_r^r + \mathrm{o}_n(1).$$

Therefore, $||v_n^i||^p \to 0$ as $n \to +\infty$. We conclude that U_n^i converges strongly to U^i in W, and consequently U^i, $i \in \{1,2,3\}$, are nontrivial solutions to Problem 1.1. On the other hand, it is clear that

$$U^i = t_i(u^i)u^i, \quad i \in \{1,2,3\}.$$

hold true. Moreover, using the fact that

$$\widetilde{E}_{tt}(t_2(u^2), u^2) > 0 \text{ and } \widetilde{E}_{tt}(t_i(u^i), u^i) < 0, \quad i \in \{1,3\},$$

we get

$$U^2 \neq U^1 \text{ and } U^2 \neq U^3.$$

Suppose that $U^1 = U^3$, then we get necessarily $t_1(u^1) = -t_3(u^3)$. Therefore $u^1 = -u^3$, $L(u_1) = L(u_3) = 0$ and $L(U_1) = L(U_3) = 0$. Recall that if $L(u) = 0$ then $E(u) = E(-u)$, so in this case, if U^1 is a solution then $-U^1$ is too. This achieves the proof. ■

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