Special linear group sections on translation planes

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Abstract

A classification is given of all translation planes of order q^2 that admit a collineation group G admitting a normal subgroup N such that G/N is isomorphic to $PSL(2,q)$.

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1 Introduction.

A theorem that has considerable application for translation planes and related geometries is that of Foulser and Johnson which classifies the translation planes of

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order q^2 that admit $SL(2,q)$ or $PSL(2,q)$ as a collineation group in the translation complement. This theorem invariably arises when considering the action of a group on a finite spread or corresponding geometry. For example, applications arise in the analysis of doubly transitive groups acting non-solvably on a subset of points on a finite spread or acting on a set of spreads forming a parallelism or partial parallelism. Other examples involve the action of groups on flocks of quadratic cones or on partial flocks of elliptic or hyperbolic quadrics. Still other new applications are given in the last section of this article.

Theorem 1. (*Foulser-Johnson [11],[12]*)

Let π denote a translation plane of order q^2 that admits a group G isomorphic to $SL(2,q)$ that induces a non-trivial collineation group acting in the translation complement (G need not act faithfully, but does act non-trivially).

Then π is one of the following planes:

(1) *Desarguesian,*
(2) *Hall,*
(3) *Hering,*
(4) *Ott-Schaeffer,*
(5) *one of three planes of Walker of order* 25*, or*
(6) *the Dempwolff plane of order* 16*.*

The problem is that in certain applications the group $SL(2,q)$ is not presented as directly acting in the translation complement but there is a group G of the translation complement that induces $SL(2,q)$ or $PSL(2,q)$ on a set L of points, or more generally, there is a normal subgroup N of G so that G/N is isomorphic to $PSL(2,q)$. So, in this article, we resolve this technicality and show that even in this case, the complete classification of such translation planes is exactly as in the Foulser-Johnson theorem. Furthermore, we note that it is not necessary to assume that G is in the translation complement.

That is, we prove:

Theorem 2. *Let π denote an affine translation plane of order q^2. Assume that π admits a collineation group G containing a normal subgroup N, such that G/N is isomorphic to $PSL(2,q)$.*

Then π is one of the following planes:

(1) *Desarguesian,*
(2) *Hall,*
(3) *Hering,*
(4) *Ott-Schaeffer,*
(5) *one of three planes of Walker of order* 25*, or*
(6) *the Dempwolff plane of order* 16*.*

2 Background.

In this section, we provide some of the background required to read this article.

2.1 Group Theoretic.

Definition 1. *Let G be a group. The 'Frattini subgroup $\Phi(G)$' is defined to be the intersection of the set of maximal subgroups of G or G itself, if G contains no maximal subgroups. $O_p(G)$ is the largest normal subgroup of G of order a power of p and $O_{p'}(G)$ is the largest normal subgroup of G of order not divisible by p.*

Remark 1. *We shall use the notation $|H|_t$ for t a prime, to denote the order of the Sylow t-subgroup of a finite group H.*

Theorem 3. (*see e.g. [2] or [13]*)
(1) *The Frattini subgroup is a characteristic subgroup.*
(2) *The Frattini subgroup is nilpotent.*
(3) *If H is a p-group then $H/\Phi(H)$ is elementary Abelian.*
(4) *$\Phi(H)$ is the smallest normal subgroup such that $H/\Phi(H)$ is elementary Abelian.*

Theorem 4. (*see e.g. [2] or [13]*) *A finite nilpotent group is a direct product of its Sylow u-subgroups.*

Theorem 5. (*see e.g. Aschbacher [2]* (23.1)) *The Frattini subgroup $\Phi(G)$ of a group G is the set of non-generators; If X is a subset of G such that $\langle X, \Phi(G)\rangle = G$ then $\langle X\rangle = G$.*

We shall be reducing consideration to a finite group G in $GL(4r, p)$, where p is a prime, admitting a normal subgroup N such that $G/N \simeq PSL(2, q)$, where $q > 3$. We shall require aspects of the theory of representations and central extensions of groups, which we list for convenience.

Theorem 6. (*see e.g. Foulser and Johnson [11]*)
Let $q = p^r$, for p a prime. The non-trivial irreducible representations of $SL(2, q)$ on a finite vector space of characteristic p have degree $\geq 2r$.

Theorem 7. (*part of the main theorem of Landazuri and Seitz [23]*)
Let $q = p^r$, for p a prime, and let $l(PSL(2, q), p)$ be the smallest integer $t > 1$ such that $PSL(2, q)$ has a projective irreducible representation over a field of characteristic other than p. Then

$$\begin{aligned} l(PSL(2,q),p) &= (q-1)/d,\ d = (2, q-1) \text{ with exceptions} \\ l(PSL(2,4),2) &= 2,\ l(PSL(2,9),3) = 2. \end{aligned}$$

Theorem 8. (*see e.g. Dickson [9]*)

Assume that $q > 3$. The subgroups of $PSL(2,q)$, $q = p^r$, p a prime, are as follows:

(1) *$PSL(2,p^t)$, where t divides r, or $PGL(2,p^s)$, where $2s$ divides r,*

(2) *subgroups of the semi-direct product of an elementary Abelian subgroup of order p^z, $z \leq t$, and a cyclic group of order dividing $p^{(z,r)} - 1$,*

(3) *subgroups of dihedral groups of orders $2(q \pm 1)/(2, q \pm 1)$,*

(4) *A_4, and r is even if $p = 2$,*

(5) *S_4 and $p^{2r} - 1 \equiv 1 \bmod 16$,*

(6) *A_5 and $p^{2r} - 1 \equiv 0 \bmod 5$.*

Theorem 9. (*see e.g. Dickson [9]*)

(1) *In $PSL(2,q)$, there are $q(q+1)/(2,q+1)$ cyclic subgroups of order $(q \pm 1)$ and exactly $q+1$ Sylow p-subgroups.*

(2) *A subgroup isomorphic to A_4, S_4 or A_5 is its own normalizer in $PSL(2,q)$.*

Definition 2. *A 'central extension of G' is a group H equipped with a surjective homomorphism $\sigma\colon H \longmapsto G$, (H,σ), such that $\ker\sigma$ is contained in the center $Z(H)$ of H.*

Definition 3. *If (H_1,σ_1) and (H_2,σ_2) are central extensions of G, a (homomorphism) 'morphism α' (H_1,σ_1) to (H_2,σ_2) is a group homomorphism $\alpha\colon H_1 \longmapsto H_2$ such that $\sigma_1 = \alpha\sigma_2$.*

A 'universal central extension' is a central extension $(\widetilde{G},\pi)$ such that given any central extension (H,σ) of G, there exists a unique morphism from $(\widetilde{G},\pi)$ to (H,σ).

Definition 4. *A group G is 'perfect' if and only if the commutator subgroup $G' = G$.*

Theorem 10. (*see e.g. Aschbacher [2]* (33.1), (33.2), (33.4))

A finite group G possesses a unique universal central extension $(\widetilde{G},\pi)$ if and only G is perfect. Indeed $\widetilde{G}$ is also perfect. In this case, $\ker\pi$ is called the 'Schur multiplier' of G.

Definition 5. *Let G be a perfect group. A 'covering' of G is a perfect central extension (H,σ) (i.e. H is perfect). The universal central extension is called the 'universal covering group' of G.*

Theorem 11. (*e.g. see Aschbacher [2]* (33.8))

Let G be a finite perfect group and $(\widetilde{G},\pi)$ the universal covering group of G and assume that (H,σ) is a covering group of G.

(1) *Then there is a covering group $(\widetilde{G},\alpha)$ of H with $\pi = \alpha\sigma$.*

(2) *$(\widetilde{G},\alpha)$ is the universal covering group of H.*

(3) *The Schur multiplier $M(H)$ of H is a subgroup of the Schur multiplier $M(G)$ of G.*

(4) *If $Z(G) = 1$ then the Schur multiplier $M(G) = Z(\widetilde{G})$ and $Z(H) \simeq M(G)/M(H)$.*

Theorem 12. (*see e.g. Huppert [15]*).

(1) *If $q > 3$ and $q \notin \{4, 9\}$ then the Schur multiplier $M(PSL(2, q))$ of $PSL(2, q)$ has order $(2, q-1)$ and the universal covering group is isomorphic to $SL(2, q)$.*

(2) *If $q > 3$ and $q \notin \{4, 9\}$ the Schur multiplier $M(SL(2, q))$ is trivial.*

(3) *If $q = 4$ the Schur multiplier of $SL(2, 4)$ is cyclic of order 2 and the universal covering group is isomorphic to $SL(2, 5)$.*

(4) *If $q = 9$, the Schur multiplier of $PSL(2, 9)$ has order 6 and the Schur multiplier of $SL(2, 9)$ has order 3.*

2.2 Translation Planes.

Theorem 13. (*André [1]*) *Let Π denote a finite projective plane. If there are two non-identity homology groups H_1 and H_2 with the same axis L but different centers, then in $\langle H_1, H_2 \rangle$ there is an elation with axis L.*

Theorem 14. (*see e.g. Lüneburg [21]*) *Let π be a finite translation plane and let H denote an affine homology group. Then a Sylow t-subgroup of H is cyclic if $t > 2$ and cyclic of generalized quaternion if $t = 2$.*

Theorem 15. (*see e.g. Foulser [10]*) *Let π be a finite translation plane of order q^2. The group of the translation complememt that fixes a Baer subplane pointwise has order dividing $q(q-1)$, where $q = p^r$, for p a prime. There is a unique Sylow p-subgroup, which is elementary Abelian, which is normalized by a cyclic group of order dividing $p^t - 1$, if the Baer group has order p^t.*

Theorem 16. (*Hering [14], Ostrom [22]*) *Let π be a finite translation plane of order p^z, for p a prime and let G be a collineation group of the translation complement that is generated by affine elation groups.*

(1) *If G is solvable then either G is an elementary Abelian elation group or has order $2t$, where t is odd and $p = 2$.*

(2) *If G is non-solvable and p is odd then G is either isomorphic to $SL(2, 3^a)$ or to $SL(2, 5)$ and $p = 3$. If G is non-solvable and $p = 2$ then G is either isomorphic to $SL(2, 2^a)$ or to $S_z(2^{2b+1})$.*

Theorem 17. (*Jha, Johnson [16]*) *Let π be a finite translation plane of even order q^2. If π admits an affine elation group of order $q^2/2$ then π is a semifield plane (and admits an affine elation group of order q^2).*

Theorem 18. (*Foulser [10]*) *Let π be a finite translation plane of order p^z, where p is prime.*

(1) *Then π cannot admit both non-trivial elation groups and non-trivial Baer p-elements.*

(2) *Let G denote the subgroup of the translation complement that is generated by all Baer p-elements and assume that $p > 2$. Then G is isomorphic to one of the subgroups listed in the Hering-Ostrom theorem on elation groups.*

(3) *If $p > 3$ then all subplanes that are fixed pointwise by Baer p-elements are mutually disjoint or identical and all lie within the same net N of degree $1 + q$.*

Furthermore, if there are at least three distinct Baer subplanes within N then there are exactly $1+k$, where k is the order the kernel of any of the Baer subplanes. In particular, if one Baer subplane is Desarguesian then there are $1+q$ Baer subplanes and N is a derivable net.

(4) *If $p = 3$ and the group generated by the Baer 3-elements contains $SL(2, 9)$ then the results of case (3) apply. Let τ and σ be Baer 3-collineations. Then the following three properties cannot simultaneously hold:*

(a) $Fix\tau \neq Fix\sigma$, (b) $Fix\tau \cap Fix\sigma \neq 0$ and (c) τ leaves $Fix\sigma$ invariant (for example, if (a) and (b) then τ does not commute with σ).

Theorem 19. (*Jha and Johnson [18]*) *Let π be a translation plane of order $q^2 > 4$, where q is even. Let π_o be a Baer subplane which is pointwise fixed by a 2-group B in the translation complement and let E be an an elation group in the translation complement.*

(1) *If E has order q and normalizes B then $|B| \leq 2$.*

(2) *If $|B| > \sqrt{q}$ then $|E| \leq 2$.*

Theorem 20. (*see e.g. Lüneburg [21]*) *Let π be a finite translation plane of order p^z, for p a prime. Let σ be a collineation of order p in the translation complement.*

(1) *The minimal polynomial of σ is $(x - 1)^2$ if and only if σ is a Baer p-element or an affine elation.*

(2) *If the spread for π is in $PG(3, q)$ then every element σ of prime order p has minimal polynomial $(x - 1)^2$ or $(x - 1)^4$. When $p = 3$ only the first alternative is possible.*

(3) *Let V be a finite vector space of dimension s over $GF(p)$, where p is prime. An element σ of order p of $GL(s, p)$, which fixes a subspace W pointwise and fixes V/W pointwise is said to be a 'generalized elation'. The group generated by generalized elations with the same fixed-point space W is Abelian.*

3 Statement of the Main Result in the General Vector Space.

In this section, we shall not assume that G is a collineation group of a translation plane. Rather, we assume only that G is a subgroup of $GL(4r, p)$, where p is a prime integer, containing a subgroup N such that G/N is isomorphic to $PSL(2, q)$, for $q = p^r$.

Our main general result is as follows:

Theorem 21. *Let V be a $4r$-dimensional vector space over $GF(p)$, for p a prime, and let $q = p^r$. Let G be a subgroup of $GL(4r, p)$ containing a normal subgroup N such that G/N is isomorphic to $PSL(2, q)$.*

Then one of the following occurs.

(1) $q = 2, 3, 4, 9$.

(2) $q = 5, 7, 17, 5^2, 3^3$ *and the following hold:*

(a) $O_{p'}(G)$ is a 2-group S_2, or when $q = 7$, $O_{p'}(G)$ is the product of a 2-group S_2 and a 3-group S_3,

(b) *G/S_2 is isomorphic to $PSL(2,q)$, or when $q=7$, $G/(S_2 \times S_3)$ is isomorphic to $PSL(2,q)$,*

(c) *$\{2^{16}, 2^{20}, 2^{37}\} \geq |S_2| \geq \{2^8, 2^{12}, 2^{13}\}$, respectively as $q = \{17, 5^2, 3^3\}$,*

(d) *If $q=7$ then$\{2^8, 3^4\} \geq \{|S_2|, |S_3|\}$, respectively and either $|S_2| \geq 2^3$ or $|S_3| \geq 3^3$.*

(3) *$|O_p(G)| \geq q^2$.*

(4) *G contains $SL(2,q)$ or $PSL(2,q)$.*

We shall give the proof as a series of lemmas. We shall break up the proof into sections for convenience. So, until we begin our analysis of translation planes, we assume that $G \leq GL(4r,p)$, G contains a normal subgroup N such that G/N is isomorphic to $PSL(2,q)$.

4 N is the Frattini Subgroup.

Lemma 1. *Either there is a p-primitive divisor of q^2-1, or $q=p$ and $p+1=2^b$, for some positive integer b, or $q=8$.*

Proof. Zsigmondy [24]. ∎

We assume that $q > 3$ so that $PSL(2,q)$ is simple.

Lemma 2. *We may assume that N is the Frattini subgroup $\Phi(G)$ of G and $G = G'$. Furthermore, we may assume that no proper subgroup Z of G has the property that ZN/N is isomorphic to $PSL(2,q)$.*

Proof. We note that $G'N/N \simeq G'/G' \cap N$. Hence either G' is contained in N or we may replace G by G'. If G' is contained in N then G/G' is Abelian, a contradiction. Let M be any proper maximal subgroup of G and assume that N is not contained in M. Then $MN = G$ and $M/M \cap N$ is isomorphic to $PSL(2,q)$. So, clearly by induction, we may assume that N is the Frattini subgroup of G. If we assume that G has minimal order for subgroups Z and normal subgroup W of Z such that Z/W is isomorphic to $PSL(2,q)$, the final statement is then valid. ∎

Lemma 3. *N contains all proper normal subgroups of G. Hence,*

$$N = O_p(G) \times O_{p'}(G).$$

Proof. Let T be a proper normal subgroup of G. Then TN/N is a normal subgroup of G/N and since G/N is simple it must be that either $TN = G$ or T contained in N. If $TN = G$, then $T = G$ by minimality. Hence, $O_p(G)$ and $O_{p'}(G)$ are contained in N. Since N is the Frattini subgroup by Lemma 2 and is nilpotent by Theorem 3 then N is the direct product of its Sylow t-subgroups by Theorem 4, so we have the proof of the lemma. ∎

Since N is normal, choose any Sylow p-subgroup S_p^1 of G. Form the normal subgroup $S_p^1 N$ and note that it is possible that there are other Sylow p-subgroups S_p^2 of G that are in $S_p^1 N$. Whenever two Sylow p-subgroups do not have this connective property, they will generate G. More specifically, we record the following lemma.

Lemma 4. (1) *Any two Sylow p-subgroups S_p^1 and S_p^2 such that $S_p^1 \subsetneqq S_p^2 N$ generate G.*

(2) *Assume p-primitive elements $u \neq 3$ or 5 exist. Then any two distinct u-groups T_u^1 and T_u^2 such that $T_u^1 \subsetneqq T_u^2 N$ generate G.*

(3) *If $u = 3$ and is p-primitive then $q = p$ and 3 divides $p + 1$. Any two distinct 3-groups T_3^1 and T_3^2 such that $T_3^1 \subsetneqq T_3^2 N$ either generate A_4, S_4, A_5 or G.*

(4) *If $u = 5$ and is p-primitive then any two distinct 5-groups T_5^1 and T_5^2 such that $T_5^1 \subsetneqq T_5^2 N$ generate A_5 or G.*

(5) *If p-primitive elements do not exist and p is odd > 3 then any two Sylow 2-subgroups S_2^1 and S_2^2 such that $S_2^1 \subsetneqq S_2^1 N$ generate G.*

Proof. We know that $S_p^1 N/N$ and $S_p^2 N/N$ are two distinct groups of $PSL(2, q)$ and hence generate $PSL(2, q)$. Hence, $\left\langle S_p^1, S_p^2, N \right\rangle = G$ but since $N = \Phi(G)$ is the set of non-generators by Theorem 5 then $\left\langle S_p^1, S_p^2 \right\rangle = G$. This proves (1).

Assume that p-primitive elements exist and let u be a prime p-primitive divisor of $q^2 - 1$. Then u divides $q + 1$. Let T_u^i, $i = 1, 2$ be u-subgroups of G. If T_u^1 is in N, and note that T_u^i cannot be in N since the order of $PSL(2, q)$ is $q(q^2 - 1)/(2, q - 1)$. If T_u^1 is not in $T_u^2 N$, then a group $\langle T_u^1, T_u^2 \rangle N/N$ cannot be a subgroup of a dihedral group of order $2(q \pm 1)/(q, q \pm 1)$ so $\langle T_u^1, T_u^2 \rangle N/N$ is either A_4, S_4 or A_5 or isomorphic to $PSL(2, p^t)$, for t dividing r. In the latter case, since u is p-primitive and the order of $PSL(2, p^t)$ is $p^t(p^{2t} - 1)/(2, p^t - 1)$, then u must divide $p^{2t} - 1$ also divides $p^r + 1$. But, if $GF(p^{2t})$ is a proper subfield of $GF(p^{2r})$, then u cannot divide $p^{2t} - 1$, implying that $t = r$ so that $\langle T_u^1, T_u^2, N \rangle = G$, by Lemma 2 and since N is the set of non-generators by Theorem 5, we have $\langle T_u^1, T_u^2 \rangle = G$. This proves (2).

Now assume that $\langle T_u^1, T_u^2 \rangle N/N$ is either A_4, S_4 or A_5. Then u must be either 3 or 5. Assume that the group is A_4 or S_4, forcing $u = 3$ and since 3 divides $p^2 - 1$, it follows that $q = p$, so $r = 1$. Furthermore, 3 divides $p + 1$. More generally, if $u = 3$ and is p-primitive then clearly $q = p$ and 3 divides $p + 1$. This proves part (3). Part (4) follows immediately from Theorem 8.

If there are no p-primitive divisors of $q^2 - 1$, there are two possibilities. If q is odd then $q = p$ and $p + 1 = 2^a$. Using the list of possible subgroups of $PSL(2, q)$ in Theorem 8, then, since $p > 3$, a Sylow 2-subgroup is dihedral of order 2^{a+1}. But then any two Sylow 2-subgroups of $PSL(2, q)$ will clearly generate the full group, proving part (5). ∎

5 Lemmas on Perfect Quotients.

Lemma 5. (1) *The Frattini subgroup of $O_p(G)$, $\Phi(O_p(G))$, is a normal subgroup of G and $G/\Phi(O_p(G))$ is perfect.*

(2) *All proper normal subgroups of $\big(G/\Phi(O_p(G))\big)$ are contained in $N/\Phi(O_p(G))$.*

(3) *Furthermore, $\big(G/\Phi(O_p(G))\big)\Big/\Big(\big(\Phi(O_p(G))\times O_{p'}(G)\big)\big/\Phi(O_p(G))\Big)$ is perfect.*

Proof. We note that $\Phi(O_p(G))$ is characteristic in $O_p(G)$, which is normal in G so $\Phi(O_p(G))$ is normal (see Theorem 3). Let $T/\Phi(O_p(G))$ denote the commutator subgroup of $G/\Phi(O_p(G))$ and assume that this subgroup is proper. Then T is a proper normal subgroup of G, implying that N contains T by Lemma 3. But, then this means that $G/\Phi(O_p(G))\big/\big(N/\Phi(O_p(G))\big)$ isomorphic to $PSL(2,q)$ is Abelian since $N/\Phi(O_p(G))$ contains the commutator subgroup. Hence, $G/\Phi(O_p(G))$ is perfect. This proves part (1). Let $T/\Phi(O_p(G))$ be a normal subgroup of $G/\Phi(O_p(G))$. Then $\big(T/\Phi(O_p(G))\big)N\big/\Phi(O_p(G)) = TN/\Phi(O_p(G))$ is a normal subgroup such that $TN/\Phi(O_p(G))\big/\big(N/\Phi(O_p(G))\big)$ is a normal subgroup isomorphic to TN/N of $PSL(2,q)$. Hence, $TN = G$ if T is contained in N and just as in Lemma 3, by minimality T is contained in N if T is proper. This proves (2). Now consider the commutator subgroup $S/\Phi(O_p(G))\Big/\Big(\big(\Phi(O_p(G))\times O_{p'}(G)\big)\big/\Phi(O_p(G))\Big)$ of $\big(G/\Phi(O_p(G))\big)\Big/\Big(\big(\Phi(O_p(G))\times O_{p'}(G)\big)\big/\Phi(O_p(G))\Big)$ and assume that this subgroup is proper. Then $S/\Phi(O_p(G))$ is a proper normal subgroup of $G/\Phi(O_p(G))$, and applying part (2), we see that S is contained in N. But, then

$$\big(G/\Phi(O_p(G))\big)\Big/\Big(\big(\Phi(O_p(G))\times O_{p'}(G)\big)\big/\Phi(O_p(G))\Big)$$

modulo

$$N/\Phi(O_p(G))\Big/\Big(\big(\Phi(O_p(G))\times O_{p'}(G)\big)\big/\Phi(O_p(G))\Big)$$

is Abelian but this group is clearly isomorphic to $G/N \simeq PSL(2,q)$. So, just as before, the indicated group is perfect. ∎

Lemma 6. *Suppose q is not 4 or 9. If $|O_p(G)| < q^2$ then we may assume that $O_p(G) = \langle 1\rangle$.*

Proof. Recall that $O_p(G)/\Phi(O_p(G))$ is an elementary Abelian p-group of order $< q^2$ by Theorem 3. By Theorem 6, the proper irreducible representations have dimensions at least $2r$. Since $q = p^r$, it follows that $G/\Phi(O_p(G))$ centralizes $O_p(G)/\Phi(O_p(G))$.

Let

$$\overline{N} = \big(O_p(G)/\Phi(O_p(G))\big)\times\Big(\big(O_{p'}(G)\times\Phi(O_p(G))\big)\big/\Phi(O_p(G))\Big),$$
$$\overline{G} = G/\Phi(O_p(G))$$

and

$$\overline{O_{p'}(G)} = \Big(O_{p'}(G) \times \Phi(O_p(G))\Big) \Big/ \Big(\Phi(O_p(G))\Big)$$

Note that $\overline{G}/\overline{O_{p'}(G)}$ is perfect by Lemma 5. It is clear that since $G/\Phi(O_p(G))$ centralizes $O_p(G)/\Phi(O_p(G))$ then $\overline{N}/\overline{O_{p'}(G)}$ is centralized by $\overline{G}/\overline{O_{p'}(G)}$. Furthermore, the quotient

$$\Big(\overline{G}/\overline{O_{p'}(G)}\Big) \Big/ \Big(\overline{N}/\overline{O_{p'}(G)}\Big)$$

is isomorphic to $PSL(2,q)$. Hence, $\overline{G}/\overline{O_{p'}(G)}$ is a perfect central extension of $PSL(2,q)$ by $\overline{N}/\overline{O_{p'}(G)}$. Note that $PSL(2,q)$ is perfect and for $q \neq 4,9$ then $SL(2,q)$ is the universal covering group (universal perfect central extension). If q is odd then the Schur multiplier has order 2 and if q is even the Schur multiplier has order 1. By Theorem 11, the center of $\overline{G}/\overline{O_{p'}(G)}$ has order 1 or 2, respectively as p is even or odd. Furthermore, $\overline{N}/\overline{O_{p'}(G)}$ is a p-group in the center. Hence, $\overline{N}/\overline{O_{p'}(G)}$ is trivial. This group is isomorphic to $O_p(G)/\Phi(O_p(G))$, so that $O_p(G) = \Phi(O_p(G))$. But, recall from Theorem 3 that $\Phi(O_p(G))$ is the smallest normal subgroup of $O_p(G)$ such that the quotient is elementary Abelian. If $O_p(G)$ is elementary Abelian then $\Phi(O_p(G)) = \langle 1 \rangle = O_p(G)$. Assume that $O_p(G)$ is at least p^2. Then, there exists a maximal normal subgroup of index p. But, then $\Phi(O_p(G)) < O_p(G)$. Hence, in either case, we must have $O_p(G) = \langle 1 \rangle$. This completes the proof of the lemma. ■

Lemma 7. *The quotient group G/N' is perfect. More generally, if $N^{(k)}$ is an element in the derived series of N then $G/N^{(k)}$ is perfect.*

Proof. In the derived series for N: $N' \trianglerighteq N'' \trianglerighteq \cdots$, if $N^{(k)}$ is any such element then $N^{(k)}$ is a normal subgroup of G and we may form $G/N^{(k)}$. Let $T/N^{(k)}$ denote the commutator subgroup. If this group is proper then T is a proper normal subgroup of G and hence contained in N. Then $(G/N^{(k)})/(T/N^{(k)}) \simeq G/T$ is Abelian and $T/N^{(k)}$ is a subgroup of $N/N^{(k)}$ so $(G/T)/(N/T)$ is Abelian. But, G/N is isomorphic to $PSL(2,q)$. Hence, $G/N^{(k)}$ is perfect. ■

6 When the Landazuri-Seitz Bound Does Not Hold.

Lemma 8. *Assume q is not $2, 3, 4$ or 9. Assume that $O_p(G)$ is trivial. If p is odd, assume that no v-subgroup of $O_{p'}(G)$ is of order at least $v^{(q-1)/2}$ and if $p = 2$ assume that no v-subgroup of $O_{2'}(G)$ is of order at least v^{q-1}.*

Then G contains a group isomorphic to $PSL(2,q)$ or $SL(2,q)$.

Proof. Consider $G/N' = \overline{G}$. Then, $\overline{G}$ induces an automorphism group on the Abelian group $N/N' = \overline{N}$, with kernel the centralizer $\overline{M}$ of $\overline{N}$ in $\overline{G}$. Since $\overline{N}$ is Abelian, $\overline{N}$ is contained in $\overline{M}$. Hence, $\overline{M}/\overline{N}$ is a normal subgroup of $\overline{G}/\overline{N}$, which is isomorphic to $PSL(2,q)$. So, $\overline{M}$ is either $\overline{G}$ or $\overline{N}$. If $\overline{M} = \overline{G}$, then $\overline{G}$ is a perfect central extension of $PSL(2,q)$ by $\overline{N}$. Now we have $\overline{G}/\overline{N}$ isomorphic to $PSL(2,q)$ and $\overline{N}$ is central in $\overline{G}$, which is a perfect group, by the previous lemma. Hence, $\overline{G}$ is a perfect central extension of $PSL(2,q)$ by $\overline{N}$. Recall that q is not 2, 3, 4 or 9 by

Theorem 12. Since the Schur multiplier of $PSL(2,q)$ has order 1 or 2, respectively as q is even or odd, we see by Theorem 11 that also the order of $\overline{N}$ is 1 or 2. If the order of $\overline{N}$ is 1 then $\overline{G}$ is isomorphic to $PSL(2,q)$. If the order of $\overline{N}$ is 2, recall that then q is odd and the universal covering group of $PSL(2,q)$ is $SL(2,q)$. Hence, the morphism from the universal cover to the perfect central extension $\overline{G}$ is the identity; $\overline{G}$ is isomorphic to $SL(2,q)$.

Hence, we have that $\overline{G}$ is isomorphic to either $PSL(2,q)$ or $SL(2,q)$, when $\overline{G}$ centralizes $\overline{N}$.

Therefore, assume that $\overline{M}$ is $\overline{N}$. Then $PSL(2,q)$ acts on an Abelian subgroup $\overline{N}$. Let v be a prime and let $\overline{N}_v$ denote the Abelian v-subgroup of $\overline{N}$. $\overline{N}_v$ is a direct product of n_v cyclic subgroups. Hence, there is an elementary Abelian normal subgroup $\overline{E}_v$ of order v^{n_v} which contains all groups of order v. Hence, $PSL(2,q)$ further acts on the product of the elementary Abelian subgroups of $\overline{N}$ and, in particular, acts on each elementary Abelian v-subgroup. Thus, $PSL(2,q)$ faithfully acts as a linear group on each $GF(v)$-vector space $\overline{E}_v$ and is therefore a projective representation acting on a vector space of characteristic $v \neq p$. By Theorem 7, the degree of an irreducible component is at least $(q-1)/2$, which means that $v \geq (q-1)/2$, a contradiction to our assumptions.

So, if $\overline{M}$ is $\overline{N}$, then $PSL(2,q)$ acts faithfully as an automorphism group of $\overline{N}$ and commutes with the product of the elementary Abelian subgroups of $\overline{N}$. We claim that this implies that $PSL(2,q)$ commutes with $\overline{N}$, which would contradict our premise. Let $\overline{N}^* = \Pi_v \overline{E}_v$ denote this subgroup, where v is a prime divisor of $O_{p'}(G)$. Then $PSL(2,q)$ acts on $\overline{N}/\overline{N}^* = \overline{N}_2$ as an automorphism group. However, $PSL(2,q)$ then centralizes the product of the maximal elementary v-subgroups of $\overline{N_2}$. Furthermore, if the group contains a cyclic subgroup $\overline{S}_{v^\alpha}$ of order v^α, it follows that $PSL(2,q)$ acts on this cyclic group. However, the automorphism group of a cyclic group is Abelian, implying that $PSL(2,q)$ centralizes all such cyclic subgroups. Since each $\overline{N}_{2,v}$ is a direct product of invariant cyclic subgroups, it follows that $PSL(2,q)$ commutes with $\overline{N}_2$. Hence, $PSL(2,q)$ commutes with $\overline{N}$.

Therefore, we must have that $\overline{G}$ is isomorphic to either $PSL(2,q)$ or $SL(2,q)$ and we may assume that $\overline{G}$ commutes with $\overline{N}$ and $\overline{N}$ has order 1 or 2.

Now assume that G/N' is isomorphic to $PSL(2,q)$, so that $N' = N$. But, then $N^{(k)} = N$, so that N is trivial. In this case, G contains (is isomorphic to) $PSL(2,q)$.

Therefore, assume that G/N' is isomorphic to $SL(2,q)$ and N/N' has order 2. N is nilpotent so is a product of its Sylow v-subgroups. Since then Sylow v-subgroups N_v commute with Sylow v^* Subgroups, for $v \neq v^*$, it follows that N' is a direct product of commutator subgroups N_v' of the Sylow v-subgroups N_v. Since N/N' has order 2, it follows that for v odd then $N_v' = N_v$ and since N is solvable, it follows that $N_v = \langle 1 \rangle$, for $v \neq 2$. Hence, N is a 2-group.

From Lemma 7 G/N'' is perfect so we may repeat the above argument for G/N'' and again N'/N'' is Abelian. By exactly the same argument as for N', we determine that either G contains a group isomorphic to $PSL(2,q)$ or G/N'' is isomorphic to $SL(2,q)$ and N'/N'' has order 2. An obvious induction then shows that either G contains $PSL(2,q)$ or $G/N^{(k)}$ is $SL(2,q)$. If n is the length of the derived series, then $G/N^{(n)} = G$ contains $SL(2,q)$ or G contains $PSL(2,q)$. This completes the proof of the lemma. ∎

7 When the Bound Holds.

Previously, we have considered that any v-subgroup of $O_{p'}(G)$ is of order at least $v^{(q-1)/2}$ and if $p=2$ that no v-subgroup of $O_{2'}(G)$ is of order at least v^{q-1}. Since these are the Landazuri-Seitz bounds for irreducible projective representations of $PSL(2,q)$ of degree other than p, for $p^r=q$, we need to consider when such a bound actually does hold.

To consider the cases when the Landazuri-Seitz bound does hold, we begin by considering the inequality $2^{(q-1)/2} \leq p^{2r(4r+1)}$, when p is odd, and $3^{(q-1)} \leq 2^{2r(4r+1)}$, when $p=2$, where $q=p^r$.

Hence, we need

$$(p^r-1)/4r(4r+1) \leq \log_2 p.$$

When r is at least two, we claim that $(p^r-1)/4r(4r+1) \geq (p^2-1)/8\cdot 9$. Actually, we need this only when $r=3$ or 4, which is easily checked. We first decide what the possible exceptions might be and then use maple to factor $\prod_{i=1}^{4r}(p^i-1)$.

Lemma 9. *Assume that $r=2$. Then $p<73$. Furthermore, the possible exceptions only occur when p is odd and $p=3,5$ so $q=9$ or 25.*

Proof. Now assume that $p-1 \geq 72$. Then we require that $p+1 \leq \log_2 p$, a contradiction. So $p<73$. Notice that when $p=19$, then $20(18)/9\cdot 8=5$ and $\log_2 19 < 5$. Hence, the only possible exceptions are $p=3,5,7,11,13,17$ when $r=2$. Now using the maple factorization program, it is immediate that the only factors of $\prod_{i=1}^{4}(p^i-1)$ whose exponent is at least $(q-1)/2$ is when $p=3$ or 5. ∎

If $r>2$, we consider

$$2^{(p^3-1)/2} \leq p^{78},$$

so

$$(p^3-1)/156 \leq \log_2 p.$$

This fails for $p=11$. Hence, we obtain the proof of the following lemma.

Lemma 10. *If $r>2$ we have that the only possible exception is $(p,r)=(3,3)$, for p odd.*

Now assume that $r=1$ so that $O_{p'}(G)$ is a subgroup of $GL(4,p)$ and its order divides $(p^4-1)(p^3-1)(p^2-1)(p-1)$.

Hence, $2^{(p-1)/2} \leq p^{4+3+2+1=10}$, so $(p-1)/20 \leq \log_2 p$. If $p>201$ then $10<(p-1)/20<\log_2 p$ if and only if $2^{10}<p$, a contradiction since $2^{10}=1024$. Thus, we obtain the following lemma.

Lemma 11. *When $r=1$ and p is odd, the only possible exceptions are $p=3,5,7,11,13,17,19,23$.*

Proof. To verify this, we used the maple factor program to factor the order of $GL(4r,p)$, for $r=1$ and $p<201$. Once factored, the only powers v^α, where α satisfies the Landazuri-Seitz bound are those listed. In fact, v is always 2 unless $p=7$, in which case, v is 2 or 3. ∎

Lemma 12. *The cases $p = 19$ or 23 are not exceptions.*

Proof. In both cases, $(q-1)/2$ is precisely the exponent of 2 in the factorization of $|GL(4,p)|$. But, G/N has a 2-group, pushing the exponent larger than possible. ∎

Lemma 13. *When $p = 2$, and $q > 4$, there is no exception.*

Proof. It is easily seen that $r < 9$, that is, $p = 2$. We have $3^{(q-1)} \leq 2^{2r(4r+1)}$ and we then obtain

$$(2^r - 1) \leq 2r(4r+1)\log_3 2.$$

Let $r > 8$. Then

$$(2^9 - 1) \leq 2(9)(4(9)+1)(\ln 2/\ln 3),$$

does not occur (the left hand side is 511 and the right hand side is approximately 420). Hence, $2 < r < 9$. Hence, we have

$$2^r \leq 2r(4r+1)\log_3 2.$$

Now use the maple factor program to factor the order of $GL(4r,2)$, for these values of r. When $q > 4$, it turns out that there is no exception. ∎

Lemma 14. *The cases $p = 11, 13$ are not exceptions.*

Proof. $p = 11$. We have $(11-1)/2 = 5$, but the only exponent larger than or equal to 5 in the factorization of $|GL(4,11)|$ belongs to 2, and is 9. There we obtain $PSL(2,11)$ acting in $GL(k,2)$. Hence, 11 divides $2^j - 1$, for $j \leq k$. We require that $k = 10$, since 11 divides $2^5 + 1$. This means that a Sylow 2-subgroup of N must have order at least 2^{10}, a contradiction.

$p = 13$. Since 13 divides $2^{12} - 1$, we need a Sylow 2-subgroup to have order at least 2^{12}, and $PSL(2,13)$ has order $13(12)7$, so the Sylow 2-subgroup has order at least 2^{14}. However, the exponent of 2 in $|GL(4,13)|$ is 11. ∎

Lemma 15. (1) *If $q = 9$, the minimal degree is 6.*

(2) *Assume that $q > 3$ and q not 9, then the minimal degree of G is at least q.*

(3) *Furthermore, the minimal degree is at least $q+1$ if p-primitive divisors u exist, where $u \neq 3$ or 5.*

Proof. Let G act as a permutation group on a set X of cardinality $< q$. Take any Sylow p-subgroup S_p and choose an element P of X. Notice that the orbit length for P under S_p is bounded by q/p. There exists an element of $S_p - O_p(G)$ that fixes P. If $p \nmid |X|$ then S_p fixes a point of X. Since this is true for all Sylow p-subgroups, the set of elements fixing P generate a group H that contains an element of each Sylow p-subgroup. Take HN/N. This group contains non-trivial elements from each Sylow p-subgroup of $PSL(2,q)$. By Theorem, 8, HN/N is either A_4, S_4, A_5, $PGL(2,p^t)$, where $2t$ dividing r or $PSL(2,p^s)$, for s dividing r, or $p = 2$ and the group is dihedral D_{q+1} of order $2(q+1)$ (since there are $q+1$ Sylow 2-subgroups in $PGL(2,q)$). First consider that $p=2$ and HN/N is D_{q+1}. This would say that the orbit length for any element P of X is bounded by $q/2$. If the orbit size is strictly smaller for some P, for some Sylow 2-subgroup, then there is a subgroup of that Sylow 2-subgroup of order $4\,|O_2(G)|$ that fixes P. In this case, HN/N cannot be D_{q+1}. Hence, assume that the orbit size of P is $q/2$, for each Sylow 2-subgroup. Furthermore, this argument is independent of the choice of P. So a given Sylow 2-subgroup has only orbits of length $q/2$ in X, a contradiction, since $|X|<q$.

So, HN/N is either A_4, S_4, A_5, $PGL(2,p^t)$, where $2t$ dividing r or $PSL(2,p^s)$, for s dividing r. Since this group contains a non-identity element from each Sylow p-subgroup, each of which fixes a unique point of $PG(1,q)$, the group is transitive acting on $PG(1,q)$ and so transitive on the Sylow p-subgroups of $PSL(2,q)$. Since $PGL(2,p^t)$ cannot act transitively on $PG(1,q)$, it follows that HN/N is either A_4, S_4, A_5 or $PSL(2,q)$. Furthermore, if A_4 then r is even if $p=2$, if S_4 then $p^{2r}-1 \equiv 1 \bmod 16$, and if A_5 then $p^{2r}-1 \equiv 0 \bmod 5$.

A_4 and S_4 have normal subgroups of order 4 and 8 respectively and since $q>3$ could not act transitively on $PG(1,q)$, A_4 or S_4 do not occur. If A_5 acts transitively on $PG(1,q)$, since there are 10 Sylow 3-subgroups and 6 Sylow 5-subgroups. Noting that $SL(2,4) \simeq A_5$, there are 5 Sylow 2-subgroups. Hence, it follows that $q=4,5,9$ when A_5 occurs. But, when $q=4$ or 5 we also obtain $PSL(2,q)$ as HN/N.

So HN/N is isomorphic to $PSL(2,q)$ or $q=9$ and is isomorphic to A_5.

If HN/N is isomorphic to $PSL(2,q)$ then $HN = G = \langle H, N\rangle = \langle H\rangle = H$. This means that G fixes P of X and since P was taken as an arbitrary element of X then G fixes X pointwise.

Now assume that $q=9$ and HN/N is A_5 or $PSL(2,9)$. There are six subgroups of $PSL(2,9)$ isomorphic to A_5. Hence, we will not be able to contradict $|X|<9$. However, if $|X|<5$ then all 5-elements fix X pointwise, so that $PSL(2,9)$ fixes X pointwise. If $|X|=5$ then each Sylow 3-subgroup fixes at least two points of X. Since there are at least 10 Sylow 3-subgroup S_i, with the property that any two generate G, and there must be at least two of these groups that share a fixed point on X, it follows that G fixes at least one point of X and then the previous argument shows that G fixes X pointwise. This proves parts (1) and (2).

Now assume that exist p-primitive elements $u \neq 3,5$ and assume that $|X|<q+1$. There are $q(q-1)/(2,q-1)$ cyclic groups of order u. Note that $q(q-1)/(2,q-1)>q$, if $(q-1)/2>1$, or rather if and only if $q>3$. If $|X|<q$ then $q=9$. But, $PSL(2,9)$ has only a p-primitive element equal to 5. Hence, assume that $|X|=q$. Then each cyclic group of order u fixes a point of X and there are at least two distinct cyclic groups of order u that fix the same point. However, the group generated by two distinct u-groups is G, by Lemma 4. This completes the proof of the lemma. ∎

Summary 1. *The bound holds only when*

$$q = 2, 3, 4, 5, 7, 3^2, 17, 5^2, 3^3.$$

8 Proof of the Vector Space Theorem.

We now complete the proof of Theorem 21.

Proof. Assume that we do not have (1), (3) or (4) stated in the theorem; i.e. (1) $q = 2, 3, 4, 9$, (2) $q = 5, 7, 17, 5^2, 3^3$ and the following hold: (a) $O_{p'}(G)$ is a 2-group S_2, or when $q = 7$, $O_{p'}(G)$ is the product of a 2-group S_2 and a 3-group S_3, (b) G/S_2 is isomorphic to $PSL(2, q)$, or when $q = 7$, $G/S_2 \times S_3$ is isomorphic to $PSL(2, q)$ or (3) $O_p(G)$ has order $\geq q^2$ and (4) G contains $SL(2, q)$ or $PSL(2, q)$.

Hence, we need to prove part (2). By Lemmas 6 and 8 and the summary when the Landazuri-Seitz bound can hold, we see that $G/O_{p'}(G)$ is isomorphic to $PSL(2, q)$. Again, using the maple factor program for the factorizations of $\prod_{i=1}^{4r}(p^i - 1)$, we see that the only possible exponent that is at least $(q-1)/2$ is attached to $v = 2$, unless possibly $q = 7$ and $v = 3$.

This proves 2(a),(b). ■

9 The Translation Plane Case.

In the following sections, we prove the following theorem:

Theorem 22. *Let π denote an affine translation plane of order q^2. Assume that π admits a collineation group G containing a normal subgroup N, such that G/N is isomorphic to $PSL(2, q)$.*

Then π is one of the following planes:
(1) *Desarguesian,*
(2) *Hall,*
(3) *Hering,*
(4) *Ott-Schaeffer,*
(5) *one of three planes of Walker of order* 25*, or*
(6) *the Dempwolff plane of order* 16.

We give the proof as a series of lemmas and analysis of special orders. We separate the proof into sections.

Now assume that G leaves invariant a spread for a translation plane π of order q^2, $q = p^r$, and that contains a subgroup N such that G/N is isomorphic to $PSL(2, q)$. We show that G may be considered a subgroup of the translation complement.

Lemma 16. *Assume that $q > 3$. There is a subgroup G^* of the translation complement such that G^*/N^* is isomorphic to $PSL(2, q)$, where N^* is a normal subgroup of G^*.*

Proof. Let T denote the subgroup of translations of π. Then GT/T is isomorphic to a subgroup of the translation complement. Clearly, TN is a normal subgroup of GT and $(GT/N)/(TN/N)$ is isomorphic to $PSL(2, q)$. Since $PSL(2, q)$ is simple, it follows that $PSL(2, q)$ is in the composition series for GT/T. However, this means that $PSL(2, q)$ is in the composition series for a subgroup of the translation complement. Thus, there is a subgroup G^* and a normal subgroup N^* of G^* such that G^*/N^* and G^* is a subgroup of the translation complement. ■

Henceforth, since we may assume that $q > 3$, we may assume that G is a subgroup of the translation complement and hence is a subgroup of $GL(4r, p)$, where $p^r = q$.

We shall utilize our main structure theorem, Theorem 21 given in the previous section, to show that G contains a group isomorphic to $SL(2, q)$ or $PSL(2, q)$ so that the Foulser-Johnson theorem may be applied.

There are essentially two obstacles: First it is possible that $|O_p(G)| \geq q^2$ and second, the Landazuri-Seitz bound holds when $q = 2, 3, 4, 5, 7, 3^2, 17, 5^2, 3^3$.

Remark 2. *All translation planes of orders* $4, 9, 16, 25, 49$ *are known and the only planes admitting a group* G *as in the previous statement are the Desarguesian, Hall planes, Hering, Ott-Schaeffer, Walker of order* 25 *and Dempwolff of order* 16.

Proof. The proof is by checking. For example, see Johnson [19] for the translation planes of order 16 that admit non-solvable groups, see Czerwinski [7] to check the planes of order 25, and see Dempwolff [8] to check that only the planes of order 49 indicated have the given group. ∎

10 $|O_p(G)| < q^2$.

Hence, we may assume that $q > 7$. Let $q = p^r$, where p is a prime and r a positive integer. We begin by showing that the possibility that $|\Phi(O_p(G))| \geq q^2$ cannot hold in translation planes. We actually shall require a stronger version of this when $q = 8$.

Lemma 17. *If* q *is not* 8, *and the order of* $O_p(G)$ *is at least* q^2 *then* $O_p(G)$ *is an elation group of order* q^2 *and the plane is a semifield plane.*

Proof. Assume that there is a prime p-primitive divisor u of $q^2 - 1$ and let g_u be an element of order u. Let X be a $GF(p)$-subspace of the associated vector space that is fixed pointwise by $O_p(G)$. If X does not have order q^2 then g_u will fix X pointwise and since G is generated by its p-primitive elements, then G fixes X pointwise. If L is a component of π which nontrivially intersects X, then L is fixed pointwise by the same argument, implying that G is a central collineation group, a contradiction, by order if nothing else. Hence, X is a component, implying that $O_p(G)$ is an elation group.

Hence the order of X is q^2. By cardinality, X is either a component or a Baer subplane (subplane of order q). Assume that X is Baer. Since $O_p(G)$ fixes X pointwise, it follows that the p-group fixing a Baer subplane pointwise has order at most q and is elementary Abelian by Theorem 15.

Hence, assume that there are no p-primitive elements and that $q \neq 8$. By Lemma 1, we may assume that $p + 1 = 2^a$ and $q = p$. Hence, X must now be a 1- or 2-dimensional $GF(p)$-subspace. Assume first that X is 2-dimensional and X is not a component. Then X is pointwise fixed by $O_p(G)$ and by cardinality, X must be a Baer subplane. The full group fixing X pointwise has order dividing $p(p-1)$ so again $O_p(G)$ is elementary Abelian of order p, contrary to our assumptions.

Now assume that X is a 1-dimensional subspace. The largest p-subgroup fixing a 1-dimensional subspace and acting faithfully on the component L containing X has order q. Hence, $O_p(G)$ contains an elation group of order at least p. Furthermore, we know that there is a Sylow p-subgroup S_p of order at least p^3 which must leave X invariant. The stabilizer in S_p of a second component must have order at least p. But, such a stabilizer then becomes a Baer group since it will have a 1-dimensional $GF(p)$-subspace on two mutually disjoint components. However, this implies that there is an elation group of order p^2 and a Baer group of order p, a contradiction by Theorem 18. Hence, $O_p(G)$ is an elation group, so is elementary Abelian. This proves the lemma if $q \neq 8$. ∎

Lemma 18. *For any order q, $O_p(G)$ cannot contain an elation group of order q^2 or $q^2/2$.*

Proof. If so, let L denote the elation axis. If q is even and $O_p(G)$ contains an elation group of order $q^2/2$, the plane is still a semifield plane by Theorem 17 and there is an elation group E of order q^2. Let M be a component other than L. Then $GE = EG_M$, where G_M does not contain N, since $G_M \cap O_p(G) = \langle 1 \rangle$. Since N is the intersection of all maximal subgroups of G then G_M is not maximal. If q is odd, then $O_p(G)$ is an elation group of order q^2. If q is even then $O_p(G)$ is an elation group of order at least $q^2/2$. So, $G_M O_p(G)$ has index 1 or 2 in G and is therefore normal. $G_M O_p(G) = G$. If the index is 2 then $G_M O_p(G)$ is a proper normal subgroup and hence is contained in N, which implies that G_M is contained in N, a contradiction. Hence, $G_M O_p(G) = G$ so $G_M N/N$ is isomorphic to $PSL(2,q)$, a contradiction to minimality, since $G_M N/N \simeq G_M/G_M \cap N$ and G_M is proper. This completes the proof of the lemma. ∎

Lemma 19. *If $q = 8$ then $|O_2(G)| < 8^2$.*

Proof. Assume that $O_2(G)$ has order $\geq 8^2$. Since $|PSL(2,8)|_2 = 8$, we have a Sylow 2-subgroup S_2 of G of order at least 8^3. Let σ be a central involution in $O_2(G)$. If σ is an elation then G fixes a component L. In this case, $O_2(G)$ will fix pointwise a subspace X_1 of L. If X_1 is proper in L then $G/G_{[X_1]}$ contains $PSL(2,q)$ or $SL(2,q)$ by Lemma 8 or acts trivially on X_1. But then $SL(2,q)$ or $PSL(2,q)$ cannot act non-trivially on X_1 by Theorem 6. Hence, G fixes X_1 pointwise and induces an automorphism group on L/X_1. Consider $O_2(G)$ acting on L/X_1 and assume that $O_2(G)$ fixes X_2/X_1 elementwise. Then G leaves X_2/X_1 invariant, and since G leaves X_1 invariant, G also leaves X_2 invariant. Applying Lemma 8 in combination with Theorem 6, we see that G fixes X_2/X_1 pointwise. Thus, G fixes X_1 and X_2/X_1 pointwise. Hence, G is a generalized elation group (also called a generalized transvection) acting on X_2 therefore is Abelian in its action on X_2, by Theorem 20. If T is the subgroup of G that fixes X_2 elementwise then T is a normal subgroup of G and if T is proper then T is contained in N. But, then G/T would be Abelian, a contradiction. Hence, $T = G$ and G fixes X_2 pointwise. By induction, we see that G fixes L/X_1 pointwise, which again implies a contradiction. Hence, $O_2(G)$ is an elation group, contrary to Lemma 18. ∎

Corollary 1. *$q = 3^2, 17, 5^2, 3^3$ or G contains $SL(2, q)$ or $PSL(2, q)$.*

Proof. By the above three Lemmas 17, 18, 19, $|O_p(G)| < q^2$. Now apply Theorem 21 and Remark 2. ∎

11 The Special Orders $q = 3^2, 17, 5^2, 3^3$.

Lemma 20. *So, we have $q = 7, 17, 5^2, 3^3$ and the following hold:*
(a) $O_{p'}(G)$ is a 2-group S_2,
(b) G/S_2 is isomorphic to $PSL(2, q)$,
(c) $\{2^{16}, 2^{20}, 2^{37}\} \geq |S_2| \geq \{2^8, 2^{12}, 2^{13}\}$, respectively as $q = \{17, 5^2, 3^3\}$.

Lemma 21. *Assume that q is $17, 5^2$ or 3^3. Then either G contains $SL(2, q)$ or $PSL(2, q)$ or G does not fix a component.*

Proof. Assume not and let G fix a component L. In all of the indicated cases, we know that N is a 2-group, or we are finished. Let N^* be the normal subgroup of G that fixes L pointwise. Assume that $N^* = G$. Then there is an elation group E of order exactly q in G. However, then E is normal and proper and so forced into N. Therefore, N^* is a proper normal subgroup and hence is a subgroup of N. If N^* does not fix a second component distinct from L then by Theorem 13, there would be an elation in N^*. Hence, N^* fixes a second component M distinct from L, so that N^* is an affine homology group. From Theorem 14, N^* is either cyclic or generalized quaternion. In any case, we consider G/N^*. Now since G leaves L invariant and the group induced on L by G is G/N^*, note that $|GL(2r, p)| / |PSL(2, q)|_2 = 2^5, 2^7, 2^{10}$, respectively as $q = 17, 5^2, 3^3$. So, G/N^* is a group such that $G/N^*/(N/N^*)$ is isomorphic to $PSL(2, q)$, and the Landazuri-Seitz bound does not hold. By the argument to Lemma 8, G/N^* either contains $PSL(2, q)$, and $N = N^*$ or $SL(2, q)$, and in the latter case N/N^* has order 2. Hence, G induces an automorphism group on N^*. If N^* is cyclic then the induced automorphism group is Abelian. If the normal subgroup T centralizes N^* then G/T is Abelian. If T is not G then T is contained in N so that $(G/T)/(N/T)$ is Abelian. But, this group is isomorphic to $PSL(2, q)$, a contradiction. Hence, when N^* is cyclic then G centralizes N^*. If N^* is generalized quaternion, there is a unique involution σ. Hence, G induces an automorphism group on $N^*/\langle\sigma\rangle$, which is a dihedral group with cyclic stem C of order $|N^*|/4$. Hence, G must centralize the cyclic stem C. Assume that $|N^*| = 2^{n+1}$. If $n > 2$ then there is a unique cyclic subgroup of index 4. The automorpism group G induced on N^* must centralize this cyclic subgroup C and G then centralizes N^*/C. Since G commutes with C and with N^*/C, it must commute with N^*. If $n = 2$, there are three cyclic subgroups of order 4. Since the minimal degree of G is q by Lemma 15, each of these three cyclic subgroups are left invariant by G and hence centralized by G. Clearly, G centralizes N^*. We claim that G/N^* is perfect. If not, there is a proper normal subgroup T of G such that T/N^* is the commutator subgroup of G/N^*. However, T is contained in N. Since $(G/N^*)/(T/N^*)$ is Abelian and isomorphic to G/T then $(G/T)/(N/T) \simeq PSL(2, q)$ is Abelian, a contradiction.

Hence, when N^* is cyclic or generalized quaternion, G centralizes N^* and G/N^* is perfect so G/N^* is a perfect central extension of $PSL(2,q)$ or of $SL(2,q)$. In the first case, G/N^* a perfect central extension of $PSL(2,q)$ then N^* has order 1 or 2, implying that G is $PSL(2,q)$ or $SL(2,q)$. In the second case, since the Schur multiplier of $SL(2,q)$ is trivial, G is $SL(2,q)$. This completes the proof of the lemma. ■

Lemma 22. *Let q be odd and assume that G does not fix a component. Let S_2 be the Sylow 2-subgroup of N. Acting on the plane, S_2 must fix or invert two components $x = 0, y = 0$. Furthermore, S_2 does not contain non-identity affine homology groups with axes and coaxes $x = 0$ and/or $y = 0$.*

Proof. We note that $(q^2 + 1)$ is twice an odd number, when q is odd, since $q^2 - 1$ is divisible by 4 and $(q^2 + 1, q^2 - 1) = 2$. Hence, S_2 must have an orbit of length 2 or two orbits of lengths 1, say $x = 0, y = 0$. Note that S_2 is characteristic in N and hence normal in G. The group generated by the affine homologies with axis $x = 0$ and coaxis $y = 0$ or with axis $y = 0$ and coaxis $x = 0$ is characteristic in S_2 and hence normal in G. If there is an affine homology group of order at least 4, then there is a unique orbit of S_2 of length 2 on the line at infinity, implying that G fixes this orbit and hence fixes two components (by minimal degree), a contradiction to our assumptions on G. Now assume that there are affine involutory homologies σ with axis $x = 0$ and coaxis $y = 0$ and τ with axis $y = 0$ and coaxis $x = 0$. Then, $\langle \sigma, \tau \rangle$ is clearly normal in G and there are three elements permuted by G. By minimal degree, it can only be that G centralizes σ and τ, implying that G fixes a component. Hence, there can be no affine homologies. ■

Lemma 23. *If q is odd and G does not fix a component then S_2 has order dividing $2\,|GL(2r,p)|_2$.*

Proof. There is a subgroup S_2^- of order $|S_2|/2$ that fixes two components $x = 0, y = 0$. The order of S_2^- is at most $|GL(2r,p)|_2$ by Lemma 22. ■

Remark 3. *When considering a collineation group H on a translation plane of order q^2, where H is a subgroup of $GL(4r,p)$, for $q = p^r$, there is always a 'kernel homology' group K of order $p-1$ that commutes with H. If K is not a subgroup of G, when G is considered minimal such that there exists a normal subgroup N and G/N is isomorphic to $PSL(2,q)$, then since K is centralized by any collineation group of the translation plane then GK/NK is still isomorphic to $PSL(2,q)$.*

In any case, any proper normal subgroup of GK is contained in NK.

Also, note that if q is not 9 (since $q > 7$) then we may assume that $O_p(G) = \langle 1 \rangle$.

12 When $q = 5^2$.

When $q = 5^2$, there is a $GF(5)$-kernel homology group K of order 4. Then consider the automorphism group induced on the elementary Abelian 2-group

$$KO_{5'}(G)/\Phi(O_{5'}(G))$$

by G, noting that $\Phi(O_{5'}(G))$ is characteristic in N and hence normal in G and since K centalizes G then $\Phi(O_{5'}(G))$ is normal in G. Form $GK/\Phi(O_{5'}(G))$ and consider the centralizer $\overline{Z}$ of $KO_{5'}(G)/\Phi(O_{5'}(G))$ in $GK/\Phi(O_{5'}(G))$. Assume that $\overline{Z}$ is a proper subgroup. Then there exists a proper normal subgroup T of GK, containing K (since K is centralized by G). Then T is contained in NK by Remark 3 and note that $N = O_{5'}(G)$ by the same remark. But, since

$$KO_{5'}(G)/\Phi(O_{5'}(G))$$

is elementary Abelian, it follows that $T = NK$. Hence, the automorphism group induced is

$$GK/\Phi(KO_{5'}(G))\Big/\Big(NK/\Phi(O_{5'}(G))\Big)$$

isomorphic to $PSL(2,q)$. Otherwise, $\overline{Z} = GK/\Phi(O_{5'}(G))$ and we have a perfect central extension $GK/\Phi(O_{5'}(G))$ of $PSL(2,q)$ by $\Big(NK/\Phi(O_{5'}(G))\Big)$. In this case, we may apply Lemma 8 to show that G contains $PSL(2,q)$ or $SL(2,q)$, for $q = 5^2$. Since $GK/\Phi(O_{5'}(G))$ centralizes $k\Phi(O_{5'}(G))$, for all $k \in K$, we see that NK must have order at least $2^{12}4 = 2^{14}$.

However, $2\,|GL(4,5)|_2 = 2^{11+1} = 2^{12}$, a contradiction. Hence, G must fix a component L, contrary to the above.

Hence, G contains $PSL(2,5)$ or $SL(2,5)$.

13 When $q = 3^3$.

When $q = 3^3$, we have a $GF(3)$-kernel K is order 2. By Lemma 8 and the argument of case 5^2 above, we are finished unless we have that $KO_{3'}(G)$ is a 2-group of order at least $2^{(3^3-1)/2}2 = 2^{14}$. But, $2\,|GL(6,3)|_2 = 2^{14}$, so this is a possibility. Indeed, $O_{3'}(G)$ is a 2-group and so $\Phi(O_3'(G)) < O_{3'}(G)$ and the quotient $O_{3'}(G)/\Phi(O_3'(G))$ has order at least 2^{13}. Note that K fixes all components and is central in the collineation group. So, $O_{3'}(G)$ has order at least 2^{14}. $O_{3'}(G)$ does not fix a component but has an orbit of length 2 of components.

So, $KO_{3'}(G)$ has order at least 2^{14} and so we have a subgroup of order at least 2^{13} that fixes two components, say $x = 0, y = 0$. Furthermore, by Lemma 22, the group induced faithfully on $x = 0$ has order at least 2^{13} and hence exactly 2^{13}, since this is the order of a Sylow 2-subgroup of $GL(6,3)$. Choose a basis so that the Sylow 2-subgroup of order 2^{13} of $GL(6,3)$ containing this induced group also contains the following 2-group A of $GL(6,3)$ acting on $x=0$:

$$A=\left\langle \begin{bmatrix} \alpha_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_6 \end{bmatrix}; \alpha_i \text{ is } 1 \text{ or } -1 \right\rangle.$$

A is a group of order 2^6. Recall that we cannot have involutory affine homologies in $KO_{3'}(G)$.

Note that every element of A has order 2. Recall that any involution acting on a translation plane of odd order either fixes a component pointwise or fixed a Baer subplane pointwise (of order 3^3) (or is the kernel involution). Hence, If a collineation g fixing both $x=0$ and $y=0$ in $KO_{3'}(G)$ such that $(g \mid x=0) \in A$ then $g^2=1$ on $x=0$. By Lemma 22, there is no affine homology with axis $x=0$ or $y=0$. This means that $g^2=1$. Hence, every element of $KO_{3'}(G)$ that induces in A has order 2. But, the only possible involutions in $KO_{3'}(G)$ are the kernel involutions and Baer involutions. If now we assume that g fixes at least 3^4 points of $x=0$ and $g^2=1$ and then g cannot be Baer or the kernel involutory homology. Hence, if g fixes at least 3^4 points of $x=0$ then g fixes all points of $x=0$. Since there are elements of A that fix 3^4 points without fixing all points, we arrive at a contradiction. Hence, this case does not arise. Thus, when $q=3^3$, we obtain that G contains $PSL(2,3^3)$ or $SL(2,3^3)$.

14 When $q=17$.

If G does not fix a component then the order of N cannot be larger than $2\,|GL(2,17)|_2 = 2^7$, by Lemma 22. However, we are finished or the order of N is at least $(17-1)2^{(17-1)/2}$, a contradiction. Hence, G contains $PSL(2,17)$ or $SL(2,17)$.

15 When $q=9$.

If $q=9$, we first show that we may assume that $O_3(G)$ is trivial.

Lemma 24. $O_3(G)=\langle 1\rangle$.

Proof. Assume that $O_3(G)\neq\langle 1\rangle$. Let W be the non-zero subspace pointwise fixed by $O_3(G)$. W is invariant under 5-elements, and 5 is a 3-primitive divisor of 3^4-1. If the cardinality of W is not 3^4 then any 5-element σ will fix W pointwise. If L is any component which nontrivially intersects W, assume that $W\neq L$. Then σ is forced to fix L pointwise. If each element of order 5 fixes L pointwise then the group generated by all elements of order 5, necessarily G, will fix L pointwise, a contradiction. So, the cardinality of W is 3^4 and it then follows immediately that W is a Baer subplane or a component. In either case, $O_3(G)$ is elementary Abelian. Since $O_{3'}(G)$ commutes with $O_3(G)$, it follows that the centralizer of $O_3(G)$ contains N. If the centralizer is exactly N then G/N is induced on $O_3(G)$, forcing $O_3(G)$

to have order at least 9^2. Since Baer p-groups have orders dividing q (i.e. 9), it follows that $O_3(G)$ is an elation group of order 9^2, contrary to Lemma 18. Hence, G centralizes $O_3(G)$, so clearly $G/O_{3'}(G)$ centralizes $N/O_{3'}(G)$. We claim that $G/O_{3'}(G)$ is perfect for if not let $T/O_{3'}(G)$ be the commutator subgroup, so T is a proper normal subgroup and thus contained in N. But, again the quotient is Abelian, contrary to $(G/O_{3'}(G))/(T/O_{3'}(G))/(N/O_{3'}(G))/(T/O_{3'}(G)) \simeq G/N \simeq PSL(2,9)$. Hence, $G/O_{3'}(G)$ is a perfect central extension of $PSL(2,9)$ by $N/O_{3'}(G)$. Since the Schur multiplier of $PSL(2,9)$ has order 6, it follows that $N/O_{3'}(G)$ has order 1 or 3.

Assume that $O_3(G)$ has order 3.

Case 1: W is a Baer subplane.

If so, let N_W denote the net of degree 10 containing W. There are exactly 24 orbits under $O_3(G)$ of components of $\pi - N_W$, which are permuted by G. Let S_3 denote a Sylow 3-subgroup of G, so S_3 has order 3^3. Since 9 does not divide 24, it follows that there is an orbit of $O_3(G)$ orbits of length 1 or 3. This means that the group acting on the $O_3(G)$ orbits has an orbit of length at most 3. In any case, there is a set of 9 components (or three) (three orbits or one orbit of $O_3(G)$) that is left invariant by S_3, implying that there is an element σ of S_3 that fixes a component exterior to N_W. An element σ of S_3 induces a collineation on the Baer subplane W of order 9. Hence, σ acting on W cannot fix W pointwise, so acts as a Baer collineation or an elation on W, as W is either the Hall plane of order 9 or is Desarguesian. In either case, σ fixes 9 points on W. So σ is a planar collineation that fixes a subspace W' of W pointwise and fixes a subspace W'' not in W of dimension at least 1 over $GF(3)$. Hence, σ fixes $W' \oplus W''$ pointwise, a space of dimension at least 3. But, since a subplane has t^2 points, it follows that σ is a Baer collineation. However, since σ commutes with $O_3(G)$, this is a contradiction by Theorem 18.

Case 2: W is a component.

Thus, we may assume that W is a component. Note that any element σ of order 3 has minimal polynomial either $(x-1)^2$ or $(x-1)^3$. Hence, the vector space V is a direct sum of cyclic $\langle\sigma\rangle\, GF(3)$-modules of dimensions $1,2,3$. If the minimal polynomial is $(x-1)^2$ then by Theorem 20, σ is a Baer collineation. If σ is not Baer then the dimension of V is 8 and so V has at least three summands. Hence, σ fixes at least 3^3 points of the translation plane. If σ fixes a component L and is not an elation then L has dimension 4 and thus L has at least two summands so σ fixes at least 3^2 points on any fixed component.

Suppose there exists an element τ of S_3 that fixes a component different from W. Since τ must fix at least 3^2 points of W, it follows that τ is Baer, a contradiction, again by Theorem 18. Hence, S_3 acts semi-regularly on the components not equal to W. Thus, there are three orbits of components under S_3 of lengths 3^3, call these three orbits Γ_i, for $i = 1,2,3$. Note that G cannot be transitive on the components of $\pi - \{W\}$, since the order of S_3 is 3^3.

Suppose that some element of Γ_1 is moved into Γ_2. Then, there is a G-orbit, say Φ_1, of length at least $2 \cdot 3^3$ that contains $\Gamma_1 \cup \Gamma_2$. If Φ_1 is not $\Gamma_1 \cup \Gamma_2$ then G is transitive on $\pi - \{W\}$. Hence, either Γ_1, Γ_2 and Γ_3 are all G-invariant or there is an orbit of length $2 \cdot 3^3$ and one of the S_3 orbits is G-invariant.

So, in any case, there is a G-invariant orbit Γ_1 of length 3^3 and S_3 acts regularly on the components of Γ_1. If L is any component of Γ_1 then G_L has order $|G|/3^3$.

Thus, $G_L N/N$ is a subgroup of $PSL(2,9)$ of order $(9^2-1)/2 = 40$. But, the subgroups of $PSL(2,9)$ that do not contain 3-elements are subgroups of dihedral groups of order (9 ± 1), since S_4, A_4 or A_5, all contain 3-elements. Hence, we have a contradiction. ∎

Using the maple program, we have noted that the only v^α that satisfies the Landazuri-Seitz bound when $q = 9$ is $v = 2$. This means that all other Sylow v-subgroups of N commute with G. Since the Schur multiplier of $PSL(2,9)$ has order 6, and $O_3(G)$ is trivial, this means that we may assume that all Sylow v-subgroups S_v when v is not 2 are trivial. For example, S_v is central so if C is a complement of S_v in N then N/C is central and has order dividing 2 but if v is not 2 then $N = C$. (Note that G/C is perfect, since G is perfect and C is contained in N.)

Thus, we have proved the following lemma.

Lemma 25. *$O_{3'}(G)$ is a 2-group.*

We now show that we may assume that the spread is never in $PG(3,9)$.

Lemma 26. *The spread is not in $PG(3,9)$.*

Proof. Assume that the spread is in $PG(3,9)$ and K, isomorphic to $GF(9)$, is contained in the kernel of the plane π. First consider $G \cap GL(4,9)$ which is normal in G. If this is not the G then $G \cap GL(4,q)$ is contained in N, but $G/G \cap GL(4,9)$ is cyclic of order dividing 2. Hence, G is a subgroup of $GL(4,9)$. Then K^*G, where K^* is the kernel homology group of order 8 of the plane, is a subgroup of $GL(4,9)$. By Theorem 20, in this setting, we know that 3-elements must be either elations or Baer 3-elements, since there cannot be quartic 3-elements. By Theorem 18, there cannot be both non-trivial elations and Baer 3-elements. We have $O_3(G)$ is trivial, so a Sylow 3-subgroup S_3 is elementary Abelian of order 9. If S_3 consists of Baer 3-elements, by Theorem 18, the Baer axes of all the elements must coincide and furthermore there is a net of degree 10 within which all Baer axes lie. It is clear that the 3-elements generate $SL(2,9)$ by Theorem 16. Similarly, if all 3-elements are elations, $SL(2,9)$ is generated by Theorem 16. ∎

Lemma 27. *If G does not contain $SL(2,9)$ then each 3-element is neither an elation nor Baer, so fixes exactly 3^3 points on a unique component.*

Proof. By Theorem 18, if σ is an elation then all elements of the group are elations. Hence, we have an elation group of order 9 with axis L. If L is not invariant then $SL(2,9)$ is generated by Theorem 16. Hence, assume that L is G-invariant. Since G is generated by its Sylow 3-subgroups, then G fixes L pointwise, a contradiction.

A Sylow 3-subgroup of order 9 is elementary Abelian so by Theorem 18 all elements of S_3 fix the same Baer subplane. Again, by Theorem 18, $SL(2,9)$ is generated by the Baer 3-collineations.

Hence, we know that any 3-element fixes exactly 3^3 elements or we are finished.

Assume that a 3-element g in a Sylow 3-subgroup S_3 is planar but fixes 3^3 points, whereas an affine plane has t^2 points, a contradiction. ∎

Lemma 28. *Let S_3 be a Sylow 3-subgroup. Then S_3 fixes a unique component L and fixes at least 9 points of L. If S_3 fixes exactly 9 points of L then the fixed point spaces of the four subgroups of order 3 completely cover L.*

Proof. $81 < 4(27-3)+3$. Hence, S_3 fixes at least 9 points of L. Note that $81 = 4(27-9)+9$. ∎

Lemma 29. *G cannot fix a component.*

Proof. Suppose that G fixes a component L. If two Sylow 3-subgroups that generate G have fixed point spaces on L that non-trivially overlap then G fixes a proper subspace X of L pointwise. However, 5 is a 2-primitive divisor of 3^4-1, implying that all 5-elements fix L pointwise, so that G fixes L pointwise, a contradiction by order of central collineation groups. Hence, if G fixes a component L, then no two distinct Sylow 3-subgroups that generate G have overlapping fixed point subspaces, since these generate G. Indeed, let S_3^i, $i = 1, 2, \ldots, 10$ be Sylow 3-subgroups that pairwise generate G. (That is, it is possible to have two Sylow 3-subgroups in $S_3^i N$.) We know that S_3^i fixes a subspace of at least 9 points of L and S_3^i either fixes 27 points or exactly 9 of L. In the former case, there is space for three Sylow 3-subgroups, with mutually non-overlapping fixed point subspaces. Hence, S_3^i fixes exactly 9 points. If some element $\sigma_j \in S_3^j$ fixes a point of $FixS_3^i$, for $i \neq j$, then $\langle S_3^i, \sigma_j \rangle N/N$ must be $PSL(2,9)$. Hence, $\langle S_3^i, \sigma_j \rangle N = G = \langle S_3^i, \sigma_j, N \rangle = \langle S_3^i, \sigma_j \rangle$, since N is the set of non-generators, a contradiction exactly as before, as G would then fix a point of L. So, $Fix\sigma_j \cap FixS_3^i = 0$. So, we have a direct sum of a subspace of dimension 3 with a subspace of dimension 2, whereas L has dimension 4 over $GF(3)$. This completes the proof of the lemma. ∎

Definition 6. *We call the unique component L_i fixed by S_3^i the 'axis' of the Sylow 3-subgroup S_3^i, $i = 1, 2, \ldots, 10$, where S_3^i, $i = 1, 2, \ldots, 10$, are distinct Sylow 3-subgroups that pairwise generate G.*

Lemma 30. *There are exactly 10 axes for Sylow 3-subgroups S_3^i and $O_{3'}(G)$ fixes each such axis. (An 'axis' for a Sylow 3-subgroup is the unique component fixed by it.)*

Proof. By the previous lemma, G does not fix a component. Thus, each Sylow 3-subgroup fixes a unique component L and there are at least 10 such components. Assume that there are $10 \cdot 2^d$ components, where $d \leq a$. Note that 9 must divide $10 \cdot 2^d - 1 = (9+1)2^d - 1$, so that 9 divides $2^d - 1$. Assume that d is non-zero so that d is at least 6. But, $10 \cdot 2^d \leq 82$, so $d \leq 3$, a contradiction. Hence, $d = 0$. But, now there are exactly 10 axes of Sylow 3-subgroups, each fixed by some Sylow 3-subgroup of $S_3^i O_{3'}(G)$. Now the stabilizer of an axis has the order of a maximal subgroup and hence is a maximal subgroup of G. Therefore, $O_{3'}(G)$ fixes each axis of a Sylow 3-subgroup. Hence, we have the proof of the lemma. ∎

Lemma 31. *Two elements of order* 3 *with the same fixed-point space generate an elementary Abelian* 3*-group.*

Proof. Let σ be a 3-element and let L be the unique component fixed by σ. Since σ fixes set X of 3^3 points on L. Hence, σ induces an automorphism of L/X, a 1-dimensional $GF(3)$-subspace. Hence, σ is a generalized elation. By Theorem 20, the set of elements of order 3 with the same fixed point space generate an Abelian group on L. If an element g of this group fixes L pointwise, then g cannot have order 3. Assume that the order of g is a prime u relatively prime to 3. Since g fixes a 3-dimensional subspace of L pointwise, there is a g-invariant Maschke complement X' such that $X \oplus X' = L$. Hence, either $u = 2$ or g fixes L pointwise (as there are only two non-zero elements in X'). But, if $u = 2$, then g is forced to be an affine homology of order 2 with axis L. But, no 3-element can fix the coaxis of σ, implying that there is an elation in the generated group by Theorem 13. Hence, the group generated is faithful on L and hence is an Abelian so elementary Abelian 3-group, since the Sylow 3-subgroups have order 9 and lie isomorphically in $PSL(2,9)$ ∎

Lemma 32. *If* $O_{3'}(G)$ *is fixed-point-free* 2*-group* (*i.e. no identity element of* $O_{3'}(G)$ *has non-zero fixed points*) *then* $O_{3'}(G)$ *is central and cyclic.*

Proof. Since $O_{3'}(G)$ permutes 80 points, it follows that the order of $O_{3'}(G)$ is at most 16 when it is fixed-point-free. Let σ be an involution in $O_{3'}(G)$. Since σ fixes at least 10 components, σ is either a kernel involution (and hence unique) or is Baer. But, by assumption σ does not fix non-zero points. Hence, there is a unique involution in $O_{3'}(G)$, assuming it is non-trivial. Hence, $O_{3'}(G)$ is cyclic or quaternion. If $O_{3'}(G)$ is cyclic then its automorphism group is Abelian. But, similarly, if $O_{3'}(G)$ is quaternion then G induces an automorphism group on $O_{3'}(G)/Z(O_{3'}(G))$, of order $\leq 2^3$. Since there are no 5-elements in $GL(3,2)$ or $GL(2,2)$, this implies that every 5-element centralizes $O_{3'}(G)/Z(O_{3'}(G))$ in $G/Z(O_{3'}(G))$, so that $G/Z(O_{3'}(G))$ centralizes $O_{3'}(G)/Z(O_{3'}(G))$. Hence, $G/Z(O_{3'}(G))$ is a perfect central extension of $PSL(2,9)$ by $O_{3'}(G)/Z(O_{3'}(G))$, so that $O_{3'}(G)/Z(O_{3'}(G))$ has order dividing 6, as 6 is the order of the Schur multiplier of $PSL(2,9)$. Hence, $O_{3'}(G)/Z(O_{3'}(G))$ has order 1 or 2. If the order is 1 then $O_{3'}(G)$ has order 2 so is cyclic. If the order of $O_{3'}(G)/Z(O_{3'}(G))$ is 2, then $O_{3'}(G)$ is of order 4 and has a unique involution so that $O_{3'}(G)$ is cyclic of order 4, contrary to our assumption. ∎

Lemma 33. *If* $O_{3'}(G)$ *is central then* G *contains* $PSL(2,9)$ *or* $SL(2,9)$.

Proof. G is a perfect central extension of $O_{3'}(G)$. So $O_{3'}(G)$ is a subgroup of the Schur multiplier of $PSL(2,9)$. Hence, $O_{3'}(G)$ has order 1 or 2 and the proof of the lemma is immediate. ∎

Hence, we may assume that there is a non-identity element involution g in $O_{3'}(G)$ that fixes a non-zero point. Since g fixes 10 axes of Sylow 3-subgroups, it follows that g is Baer, and the pointwise fixed subplane lies in the net N_{10} defined by the 10 axes of Sylow 3-subgroups.

Suppose that $Fixg = \pi_o$ then there is a unique Baer subplane π_1 of N_{10} also fixed by g, called the its coaxis subplane $\pi_1$of the same net N_{10}. We wish to determine the number of Baer subplanes of N_{10}. So, assume that S_3 fixes or interchanges π_o and π_1 and hence fixes both. Each Sylow 3-subgroup S_3 fixes a unique component of N_{10} and hence must induce elation groups on both π_o and π_1 because, since either is of order 9, each is either Hall or Desarguesian and there cannot be planar 3-groups acting on either subplane. Hence, both subplanes are Desarguesian. Since both π_o and π_1 are invariant under then there are two fixed-point-subspaces of S_3 of order 9, $L \cap \pi_o$ and $L \cap \pi_1$, implying that the direct sum of these two fixed point subspaces intersecting the common component is fixed pointwise by S_3. Thus, S_3 is an elation group, a contradiction. Hence, there are at least three Baer subplanes in N_{10}. By Theorem 18 there are 1+ k subplanes, where k is the order of the kernel of π_o. The kernel is $GF(3)$ if π_o is Hall and $GF(9)$ if π_o is Desarguesian. Hence, there are either 4 or 10 subplanes. But, since the minimum degree of G is 6, by Lemma 15, it follows that there must be 10 subplanes. Hence, we have shown the following lemma.

Lemma 34. *N_{10} is a derivable net.*

Thus, G acts on the set of Baer subplanes of N_{10}. Now since an element τ of order 3 fixes 3^3 points, τ must fix at least three Baer subplanes, implying a contradiction as above, as any two intersections with L, generate L and τ induces elation groups on all three.

Hence, we have shown that $O_{3'}(G)$ is central so that G contains $SL(2, 9)$ or $PSL(2, 9)$.

Thus, in general, we have shown that G must contain either $SL(2, q)$ or $PSL(2, q)$ and we then may apply the Foulser-Johnson theorem to obtain the extension theorem.

16 Applications.

In this section, we mention a few new applications of the extension of the Foulser-Johnson Theorem.

The complete details of these results are given in the individual papers. In the first three results on 'quartic groups,' there are both group theoretic and translation plane theoretic results that are intertwined to show that under the existing conditions, $SL(2, q)$ is always generated.

In the application on parallelisms of $PG(3, q)$, there is a wide interplay of group theoretic, translation plane theoretic and combinatorial results that produce the result mentioned.

When we consider general quartic groups below, for odd order, we assume that there is a translation plane π of order q^2 admitting a group Q of order $> \sqrt{q}$ that fixes a component L and induces a generalized elation group on L; Q fixes a proper subspace W elementwise and also fixes L/W elementwise. Furthermore, assume that

Q is the maximal generalized elation group of the translation complement that fixes W pointwise. If we assume that W is of cardinality q, we claim that the stabilizer of L in the full translation complement group G normalizes Q or we may classify π. If Q is not normal in G_L then let Q^x for $x \in G_L$ be distinct from Q. If $Q \cap Q^x$ is non-trivial, then there is an element g of G_L of order p that fixes both W and Wx pointwise. But g commutes with both Q and Q^x, implying that g is a generalized elation with axis W and with axis Wx, so that $Wx = W$. Hence, Q and Q^x fix W pointwise, a contradiction. Hence, Q and Q^x are disjoint. Assume also that W and Wx are then disjoint or equal. If W and Wx are disjoint, then on L there is an induced partial spread consisting of images of W under $\langle Q, Q^x \rangle$ and there are two elation groups of orders $> \sqrt{q}$. If there is a partial spread on L, an 'elation' group is defined as a generalized elation group with axis of order q (as opposed to simply a group that fixes a set of q points). Under these conditions, the theorem of Hering and Ostrom (see Theorem 16) for elation groups on translation planes actually applies. Hence, if we assume that $p > 3$, we obtain a group induced on L isomorphic to $SL(2, p^t)$, and $p^t > \sqrt{q}$. Furthermore, this partial spread of $1 + p^t$ r-dimensional $GF(p)$-subspaces, when $p^r = q$, is a Desarguesian partial spread coordinatizable by a field isomorphic to $GF(p^t)$. Indeed, the corresponding net is a subplane covered net, covered by subplanes of order p^t. This implies that $p^t - 1$ divides $q - 1$. Thus, $SL(2, q)$ is induced on L. By the extension theorem, $SL(2, q)$ is a subgroup of G and the plane is classified (actually as a Hering plane).

Theorem 23. *Let π be a translation plane of odd order q^2, $q = p^r$, $p > 3$. If π admits two generalized elation groups of orders $> \sqrt{q}$ with disjoint fixed point spaces that fix the same component L. Then the group generated by the generalized elation groups induces $SL(2, q)$ on L and one of the following situations occurs.*

(1) *the generalized elation groups are Baer groups (fix Baer subplanes pointwise) and the plane is a Hall plane,*

(2) *the generalized elation groups contain no Baer p-element and the plane is a Walker plane of order* 25.

Proof. ¿From the above comments, we know that the generalized elation groups induce elation groups on L, and there is a partial Desarguesian spread generated from the fixed point spaces of the generalized elation groups. This implies that $SL(2, q)$ is generated on L. Hence, it follows that there is a collineation isomorphic to $SL(2, q)$ acting on the translation plane. But, if G is the group generated and N is the group fixing L pointwise, then G/N is isomorphic to $SL(2, q)$. Hence, by our extension result this implies that the plane is classified. The situations (1) and (2) are simply the possible actions when the p-groups induce generalized elation groups on components. ∎

In the above result, we assumed that the fixed point spaces are disjoint. If the spread is in $PG(3, q)$, so the associated vector space is 4-dimensional, then fixed point spaces are normally 1-dimensional $GF(q)$-subspaces and hence are identical or are disjoint.

When L is not invariant, and the spread is in $PG(3, q)$, there are similar analyses that produce $SL(2, q)$ acting as a quotient. Without such assumptions, and in terms of applications, what we can say about planes of even order is stronger than what we know about odd order planes.

17 Even Order.

When q is even, there is a different definition for quartic groups, more in keeping with our treatment of generalized elation groups mentioned above.

Definition 7. *Let π be a translation plane of even order q^2 and let Q be an elementary Abelian 2-group consisting of Baer involutions whose axes share a common line (subline). For example, Q could be a Baer group, otherwise, there is a unique common line, called the 'center' of the group and the unique component containing the center is called the axis.*

Any such group Q shall be called a 'quartic group'. Any quartic group that is not Baer shall be called a 'proper quartic group'.

Definition 8. *Let π be a translation plane of even order q^2 and let Q be an elementary Abelian 2-group consisting of Baer involutions whose axes share a common line (subline). If there is a partition of Q into Baer groups of order 2^i (i.e. the subgroups of order 2^i fix Baer subplanes pointwise) we shall say that Q is a quartic group of index i.*

The common subline is called the 'center' of the group and the component containing the center is called the 'axis'.

Remark 4. *The Ott-Schaeffer planes admit quartic groups of index 1 and the Hall planes admit quartic groups of index r, where $q = 2^r$.*

Theorem 24. (*Biliotti, Jha, Johnson [4]*) *Let Q be a quartic group in a collineation group G of the translation complement of a translation plane π of even order q^2. Then one of the following occurs.*

(1) *Q contains a Baer group of index 1 or 2.*

(2) *Q is contained in an elementary Abelian normal 2-subgroup of G.*

(3) *Q contains a Baer group Q^- of index 2 and there exists an element x of $G - N_G(Q)$, such that $\langle Q, Q^x \rangle$ fixes a Baer subplane π_o that is fixed pointwise by Q^- and induces a solvable group of order $2d$, for d odd, on π_o.*

(4) *$|Q| = 8$ and there is a conjugate Q^x, for $x \in G - N_G(Q)$ such that $\langle Q, Q^x \rangle$ is the universal covering group of $SL(2, 4)$.*

(5) *There is an elementary Abelian 2-subgroup $\overline{Q}$ containing Q and fixing a Baer subline pointwise and a solvable subgroup W of order $2d$, $d > 1$, generated by elations and a subgroup S of G such that S/W is isomorphic to $SL(2, |\overline{Q}|)$, $S_z(|\overline{Q}|)$, $SU(3, |\overline{Q}|)$ or $PSU(3, |\overline{Q}|)$. In all cases, $|\overline{Q}W/W|$ is the order of the center of a Sylow 2-subgroup of S/W.*

Theorem 25. (*Biliotti, Jha, Johnson [4]*) *Let π be a translation plane of even order q^2 and let Q be a quartic group of order at least* $\max(16, 4\sqrt{q})$ *of a collineation group G of a translation plane π of order q^2. Then one of the following occurs:*

(1) *Q is contained in a normal elementary Abelian 2-group of G,*
(2) *π is Hall, or*
(3) *π is Ott-Schaeffer.*

When the spread is contained in $PG(3, q)$, we have the following result:

Theorem 26. (*Biliotti, Jha, Johnson [4]*) *Let π be a translation plane of even order q^2 with spread in $PG(3, q)$ that contains an elementary Abelian 2-group Q in the linear translation complement G and which contains no elations.*

(1) *If the order of Q is at least* $\max(16, 4\sqrt{q})$*, then one of the following occurs:*
  (a) *Q is contained in an elementary Abelian normal subgroup of G,*
  (b) *π is Hall, or*
  (c) *π is Ott-Schaeffer.*

(2) *Assume the order of Q is at least* $\max(16, 2\sqrt{q})$*. Assume that Q is Baer, or is not Baer and does not contain a Baer group of index 2. Then one of the following occurs:*
  (a) *Q is contained in an elementary Abelian normal subgroup of G,*
  (b) *π is Hall, or*
  (c) *π is Ott-Schaeffer.*

In fact, we may go slightly further in the dimension 2 case, as is shown in Biliotti, Jha, Johnson [5], when the groups are very large.

Theorem 27. (*Biliotti, Jha, Johnson [5]*) *Let π be a translation plane of even order q^2 with spread in $PG(3, q)$ admitting two disjoint linear quartic groups of order q. Then the plane is one of the following planes:*

(1) *Desarguesian of order 2,*
(2) *Hall,*
(3) *Ott-Schaeffer.*

18 When the Spread Is in $PG(3, q)$, q Odd.

Since the principal planes admitting $SL(2, q)$ have spreads in $PG(3, q)$, we now consider this situation. In some sense, the term 'quartic group' is most accurately portrayed when q is odd and when the spread is in $PG(3, q)$.

Definition 9. *Let π be a translation plane of order $q^2, q = p^r$, p a prime, with spread in $PG(3, q)$. A 'quartic group' T is an elementary Abelian p-group all of whose non-identity elements are quartic (i.e. have minimal polynomials $(x - 1)^4$) and which fix the same 1-dimensional $GF(q)$-subspace pointwise. The fixed-point space is called the 'quartic center' of the group and the unique component of π containing the center is called the 'quartic axis'.*

Theorem 28. (*Biliotti, Jha, Johnson [3]*) *Let π be a translation plane with spread in $PG(3,q)$, where $q = p^r$, p a prime. If π admits at least two mutually disjoint quartic p-groups of orders $> \sqrt{q}$ all with distinct centers then*

(1) *the group generated by the quartic p-elements is isomorphic to $SL(2,q)$, and*

(2) *the plane is one of the following planes:*

(a) *a Hering plane or*

(b) *one of the three exceptional Walker planes of order* 25.

19 Quartic Groups with Equal Centers.

The results mentioned in the previous applications require that distinct quartic groups have distinct centers. Hence, this raises the question on what happens when the centers of two disjoint quartic groups are equal? This is considered as follows. The reader is directed to [20] for the definition of a 'desirable plane'.

Theorem 29. (*Biliotti, Jha, Johnson [5]*)

(1) *Let π be a translation plane of odd order q^2 and spread in $PG(3,q)$ that admits two disjoint quartic groups with orders q that share their quartic centers. Then π is a desirable translation plane.*

Conversely,

(2) *If π is a desirable translation plane then there exists a group G of order q^2 that contains two disjoint quartic groups T and S with the same quartic center such that $\langle T, S\rangle = G$.*

All of the results mentioned above use the extension of the Foulser-Johnson theorem more-or-less directly. The authors have recently classified deficiency one transitive partial parallelisms in $PG(3,q)$, where it is necessary at several places to use the extension more indirectly, to show that such a group does not act.

Theorem 30. (*Biliotti, Jha, Johnson [6]*) *Let $\mathcal{P}$ be a parallelism in $PG(3,q)$ admitting an automorphism group G that fixes one spread (the socle) and acts transitively on the remaining spreads.*

Let $q = p^r$, for p a prime. If $q \neq 8$ assume that the Sylow p-subgroups of G are in $PGL(4,q)$. If $q = 8$, assume that G is a subgroup of $PGL(4,q)$.

Then

(1) *the socle is Desarguesian,*

(2) *the associated group G contains an elation group of order q^2 acting on the socle, and*

(3) *the remaining spreads of the parallelism are isomorphic derived conical flock spreads.*

Theorem 31. (*Biliotti, Jha, Johnson [6]*) *Let $\mathcal{P}$ be a parallelism in $PG(3,q)$ admitting an automorphism group G that fixes one spread (the socle) and acts transitively on the remaining spreads. If $q = p^r \neq 8$, for p a prime and $(r,q) = 1$, then the following occur:*

(1) *the socle is Desarguesian,*

(2) *the associated group G contains an elation group of order q^2 acting on the socle, and*

(3) *the remaining spreads of the parallelism are isomorphic derived conical flock spreads.*

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Appendix: Computer Data

The Bound When $r = 1$.

In this section, we list the prime decompositions of $|GL(4,p)|_{p'}$ If S_v is a Sylow v-subgroup of $GL(4,p)$, $v \neq p$, then the Landazuri-Seitz bound for a group $PSL(2,q)$, $p^r = q$ acting on an elementary Abelian v-sgroup requires that the order of S_v is at least $v^{(q-1)/(2,q-1)}$.

In the following table we list (p,N), where p is an odd prime < 201 and $N = |GL(4,p)|_{p'}$. If the bound holds, we indicate this.

p	(p,N)
3	$2^9\,5^1\,13^1$, holds
5	$2^{11}\,3^2\,13^1\,31^1$, holds
7	$2^{11}\,3^5\,5^2\,19^1$, holds
11	$2^9\,3^2\,5^4\,7^1\,19^1\,61^1$, holds
13	$2^{11}\,3^5\,5^1\,7^2\,17^1\,61^1$, holds
17	$2^{19}\,3^4\,5^1\,29^1\,307^1$, holds
19	$2^9\,3^9\,5^2\,127^1\,181^1$, holds
23	$2^{11}\,3^2\,5^1\,7^1\,11^4\,53^1\,79^1$, holds
29	$2^{11}\,3^2\,5^2\,7^4\,13^1\,67^1\,421^1$
31	$2^{15}\,3^5\,5^4\,13^1\,37^1\,331^1$
37	$2^{11}\,3^9\,5^1\,7^1\,19^2\,67^1\,137^1$
41	$2^{15}\,3^2\,5^4\,7^2\,29^2\,1723^1$
43	$2^9\,3^5\,5^2\,7^4\,11^2\,37^1\,631^1$
47	$2^{13}\,3^2\,5^1\,13^1\,17^1\,23^4\,37^1\,61^1$
53	$2^{11}\,3^6\,5^1\,7^1\,13^4\,281^1\,409^1$
59	$2^9\,3^2\,5^2\,29^4\,1741^1\,3541^1$
61	$2^{11}\,3^5\,5^4\,13^1\,31^2\,97^1\,1861^1$
67	$2^9\,3^5\,5^1\,7^2\,11^4\,17^2\,31^1\,449^1$
71	$2^{11}\,3^4\,5^4\,7^4\,2521^1\,5113^1$
73	$2^{15}\,3^9\,5^1\,13^1\,37^2\,41^1\,1801^1$
79	$2^{13}\,3^5\,5^2\,7^2\,13^4\,43^1\,3121^1$
83	$2^9\,3^2\,5^1\,7^2\,13^1\,19^1\,41^4\,53^1\,367^1$
89	$2^{15}\,3^4\,5^2\,11^4\,17^1\,233^1\,8011^1$

p	(p,N)
97	$2^{23}\,3^5\,5^1\,7^4\,941^1\,3169^1$
101	$2^{11}\,3^2\,5^8\,17^2\,5101^1\,10303^1$
103	$2^{11}\,3^5\,5^1\,13^2\,17^4\,1061^1\,3571^1$
107	$2^9\,3^6\,5^2\,7^1\,13^1\,53^4\,127^1\,229^1$
109	$2^{11}\,3^{13}\,5^2\,7^1\,11^2\,13^1\,457^1\,571^1$
113	$2^{19}\,3^2\,5^1\,7^4\,13^1\,19^2\,991^1\,1277^1$
127	$2^{19}\,3^9\,5^1\,7^4\,1613^1\,5419^1$
131	$2^9\,3^2\,5^4\,11^2\,13^4\,8581^1\,17293^1$
137	$2^{15}\,3^2\,5^1\,7^1\,17^4\,23^2\,37^1\,73^1\,1877^1$
139	$2^9\,3^5\,5^2\,7^2\,13^1\,23^4\,499^1\,9661^1$
149	$2^{11}\,3^2\,5^4\,7^1\,17^1\,31^1\,37^4\,103^1\,653^1$
151	$2^{11}\,3^5\,5^8\,7^1\,13^1\,19^2\,877^1\,1093^1$
157	$2^{11}\,3^5\,5^2\,13^4\,17^1\,29^1\,79^2\,8269^1$
163	$2^9\,3^{17}\,5^1\,7^1\,19^1\,41^2\,67^1\,2657^1$
167	$2^{11}\,3^2\,5^1\,7^2\,83^4\,2789^1\,28057^1$
173	$2^{11}\,3^2\,5^1\,29^2\,41^1\,43^4\,73^1\,30103^1$
179	$2^9\,3^4\,5^2\,7^1\,37^1\,89^4\,433^1\,4603^1$
181	$2^{11}\,3^9\,5^4\,7^2\,13^2\,79^1\,139^1\,16381^1$
191	$2^{17}\,3^2\,5^4\,7^1\,13^2\,17^1\,19^4\,29^1\,31^1\,37^1$
193	$2^{27}\,3^5\,5^3\,7^1\,97^2\,149^1\,1783^1$
197	$2^{11}\,3^4\,5^1\,7^8\,11^2\,19^1\,2053^1\,3881^1$
199	$2^{11}\,3^9\,5^4\,11^4\,13267^1\,19801^1$

Prime power factorization of the p'-subgroups of $GL(4r,p)$, for $r > 1$.

We determine the prime power factorization for $GL(4r,p)$ for relevant values of (p,r), $r > 1$, that we require.

p	prime power factorization
	The Cases $GL(8,p)$
17	$2^{39}\, 3^{9}\, 5^{2}\, 7^{1}\, 13^{1}\, 29^{2}\, 307^{2}\, 41761^{1} \cdot 88741^{1}\, 25646167^{1}$, holds
13	$2^{23}\, 3^{10}\, 5^{2}\, 7^{4}\, 17^{2}\, 61^{2}\, 157^{1}\, 14281^{1} \cdot 30941^{1}\, 5229043^{1}$
11	$2^{19}\, 3^{5}\, 5^{9}\, 7^{2}\, 19^{2}\, 37^{1}\, 43^{1}\, 61^{2}\, 3221^{1} \cdot 7321^{1}\, 45319^{1}$
7	$2^{23}\, 3^{10}\, 5^{4}\, 19^{2}\, 29^{1}\, 43^{1}\, 1201^{1}\, 2801^{1} \cdot 4733^{1}$
5	$2^{23}\, 3^{5}\, 7^{1}\, 11^{1}\, 13^{2}\, 31^{2}\, 71^{1}\, 313^{1} \cdot 19531$, holds
3	$2^{19}\, 5^{2}\, 7^{1}\, 11^{2}\, 13^{2}\, 41^{1}\, 1093^{1}$, holds
2	$3^{5}\, 5^{2}\, 7^{2}\, 17^{1}\, 31^{1}\, 127^{1}$, holds
	The Cases $GL(12,p)$
7	$2^{34}\, 3^{17}\, 5^{6}\, 11^{1}\, 13^{1}\, 19^{4}\, 29^{1}\, 37^{1}\, 43^{2} \cdot 181^{1}\, 191^{1}\, 1063^{1}\, 1123^{1}\, 1201^{1} \cdot 2801^{2}\, 4733^{1}\, 293459^{1}$
5	$2^{34}\, 3^{8}\, 7^{2}\, 11^{2}\, 13^{3}\, 19^{1}\, 31^{4}\, 71^{2}\, 313^{1} \cdot 521^{1}\, 601^{1}\, 829^{1}\, 19531^{1} \cdot 12207031^{1}$
3	$2^{28}\, 5^{3}\, 7^{2}\, 11^{4}\, 13^{4}\, 23^{1}\, 41^{1}\, 61^{1}\, 73^{1} \cdot 757^{1}\, 1093^{1}\, 3851^{1}$, holds
2	$3^{8}\, 5^{3}\, 7^{4}\, 11^{1}\, 13^{1}\, 17^{1}\, 23^{1}\, 31^{2}\, 73^{1} \cdot 89^{1}\, 127^{1}$
	Case $GL(16,p)$
5	$2^{47}\, 3^{10}\, 7^{2}\, 11^{3}\, 13^{4}\, 17^{1}\, 19^{1}\, 29^{1}\, 31^{5} \cdot 71^{3}\, 181^{1}\, 313^{2}\, 449^{1}\, 521^{1}\, 601^{1} \cdot 829^{1}\, 1741^{1}\, 11489^{1}\, 19531^{2} \cdot 12207031^{1}\, 305175781^{1}$
3	$2^{39}\, 5^{4}\, 7^{2}\, 11^{6}\, 13^{5}\, 17^{1}\, 23^{1}\, 41^{2}\, 61^{1} \cdot 73^{1}\, 193^{1}\, 547^{1}\, 757^{1}\, 1093^{2}\, 3851^{1} \cdot 4561^{1}\, 797161^{1}$
2	$3^{10}\, 5^{4}\, 7^{5}\, 11^{1}\, 13^{1}\, 17^{2}\, 23^{1}\, 31^{3}\, 43^{1} \cdot 73^{1}\, 89^{1}\, 127^{2}\, 151^{1}\, 257^{1}\, 8191^{1}$

2	prime power factorization
	Cases $GL(4r,2)$, for $r \geq 1$
1:	$GL(4,2)$ $3^{2}\, 5^{1}\, 7^{1}$
2:	$GL(8,2)$ $3^{5}\, 5^{2}\, 7^{2}\, 17^{1}\, 31^{1}\, 127^{1}$
3:	$GL(12,2)$ $3^{8}\, 5^{3}\, 7^{4}\, 11^{1}\, 13^{1}\, 17^{1}\, 23^{1}\, 31^{2}\, 73^{1} \cdot 89^{1}\, 127^{1}$
4:	$GL(16,2)$ $3^{10}\, 5^{4}\, 7^{5}\, 11^{1}\, 13^{1}\, 17^{2}\, 23^{1}\, 31^{3}\, 43^{1} \cdot 73^{1}\, 89^{1}\, 127^{2}\, 151^{1}\, 257^{1}\, 8191^{1}$
5:	$GL(20,2)$ $3^{14}\, 5^{6}\, 7^{6}\, 11^{2}\, 13^{1}\, 17^{2}\, 19^{1}\, 23^{1}\, 31^{4} \cdot 41^{1}\, 43^{1}\, 73^{2}\, 89^{1}\, 127^{2}\, 151^{1}\, 257^{1} \cdot 8191^{1}\, 131071^{1}\, 524287^{1}$
6:	$GL(24,2)$ $3^{17}\, 5^{7}\, 7^{9}\, 11^{2}\, 13^{2}\, 17^{3}\, 19^{1}\, 23^{2}\, 31^{4} \cdot 41^{1}\, 43^{1}\, 47^{1}\, 73^{2}\, 89^{2}\, 127^{3}\, 151^{1} \cdot 241^{1}\, 257^{1}\, 337^{1}\, 683^{1}\, 8191^{1} \cdot 131071^{1}\, 178481^{1}\, 524287^{1}$
7:	$GL(28,2)$ $3^{19}\, 5^{8}\, 7^{10}\, 11^{2}\, 13^{2}\, 17^{3}\, 19^{1}\, 23^{2}\, 29^{1} \cdot 31^{5}\, 41^{1}\, 43^{2}\, 47^{1}\, 73^{3}\, 89^{2}\, 113^{1} \cdot 127^{4}\, 151^{1}\, 241^{1}\, 257^{1}\, 337^{1}\, 601^{1} \cdot 683^{1}\, 1801^{1}\, 2731^{1}\, 8191^{2} \cdot 131071^{1}\, 178481^{1}\, 262657^{1} \cdot 524287^{1}$
8:	$GL(32,2)$ $3^{22}\, 5^{9}\, 7^{11}\, 11^{3}\, 13^{2}\, 17^{4}\, 19^{1}\, 23^{2}\, 29^{1} \cdot 31^{6}\, 41^{1}\, 43^{2}\, 47^{1}\, 73^{3}\, 89^{2}\, 113^{1} \cdot 127^{4}\, 151^{2}\, 233^{1}\, 241^{1}\, 257^{2}\, 331^{1} \cdot 337^{1}\, 601^{1}\, 683^{1}\, 1103^{1}\, 1801^{1} \cdot 2089^{1}\, 2731^{1}\, 8191^{2}\, 65537^{1} \cdot 131071^{1}\, 178481^{1}\, 262657^{1} \cdot 524287^{1}\, 2147483647^{1}$