Bounded sets and dual strong sequences in locally convex spaces

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Abstract

Given a duality $\langle E, F\rangle$, a dual strong sequence is a sequence of bidual enlargements of F in the algebraic dual E^* of E. In this article, we investigate the bounded sets generated by a dual strong sequence and related associated topologies.

1 Introduction

In this article, we continue to investigate the dual strong sequences of a nonbarrelled space, introduced in [9]. Our aim is to enhance the understanding of the associated barrelled topology and its connection to the given nonbarrelled space.

There are plentiful examples of nonbarrelled spaces. The logical equivalence of $(a \to b)$ and (NOT $b \to$ NOT a) converts any necessary condition of barrelledness into a sufficient condition of nonbarrelledness, (for barrelledness conditions see [3], [6], [8]). Important examples of nonbarrelled spaces $C(X)$ of continuous functions can be found in ([8], IV.2.3, IV.2.4; see also Theorem IV.6.2). A concise description of the research of barrelledness in $C(X)$ is given in [7]. New sophisticated examples of nonbarrelled $C(X)$ are developed in [5]. Nonbarrelled $C(X)$ spaces, related to the measure theory, are presented in ([3], Ch. 6).

Dual strong sequences of [9] pave the way to the associated barrelled topology of a nonbarrelled space. The associated barrelled topology emerges as the infimum of topologies finer than the given one and carrying the barrelledness property, ([1],

Received by the editors July 2004.

Communicated by F. Bastin.

1991 *Mathematics Subject Classification :* 46A08.

Key words and phrases : dual strong sequence, transbarrel, locally barrelled, associated topologies, bounded sets.

Definition II.4.8, Proposition II.4.11). The associated barrelled topology can also be approached by a transfinite induction, using the supremum topology for limit ordinals and the strong topology for ordinals with a predecessor, ([6], 4.4., p. 113). The construction of ([6], 4.4., p. 113) embodies the completeness principle of Krein-Smulian-Ptak, ([1], I.2, p. 9, Comment). The idea of the dual strong sequence pointed towards the associated barrelled space, is based on Krein-Smulian-Ptak principle as well. However in developing the concept of dual strong sequence, our target is the process rather than the endpoint of the processs. Dual strong sequences are tools for discovering the intrinsinc properties and crucial ingredients of topologies laying on the way to the associated barrelled topology.

In this article, we focus our attention on general nonbarrelled locally convex spaces, continuing the broad overall approach of [9]. After describing bounded sets affiliated to the dual strong sequence, (Propositions 5.1, 5.2, 5.3), we identify a new (to the best of our knowledge) topology on E between the associated barrelled and ultrabornological, namely the transbarrelled topology, (Definition 6.1). We investigate the connection between the dual strong sequence and the associated topologies of E, (Propositions 6.1, 6.2, 6.3, 6.4).

2 Preliminaries

We follow the notations and definitions of [1], [4], [6], [11]. Let (E, η) be a locally convex topological vector space over the field $\mathbb{K}$ of the real or complex numbers and E', respectively E^*, its topological, respectively algebraic, dual. We denote by $\beta(E, E')$, $\mu(E, E')$, $\sigma(E, E')$ the strong, Mackey, weak topologies on E, respectively, and by $\beta^*(E, E')$ the topology on E of uniform convergence on all strongly bounded subsets of E'. All topologies are locally convex and Hausdorff. A *disk* is an absolutely convex set. Given a disk B, we use E_B for the linear hull of B, equipped with its gauge g_B. If B is bounded for some Hausdorff locally convex topology on E, then E_B is a normed space. A bounded disk B is *barrelled (Banach)*, if E_B is barrelled (a Banach space). A disk is *(weakly) fast compact*, if it is (weakly) compact in E_B for some Banach disk B, ([1], Definition III.1.3). The polars of (weakly) fast compact disks form a 0-neighbourhood base for (infra-)Schwartz topology, ([1], Definition III.3.9; see also [4], t. I, p. 119; see also [11], p. 205). A disk B is (finite-)*infinite-dimensional*, if E_B is (finite-)infinite-dimensional.

3 The concept of a dual strong sequence and union

Let $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. Let k be a positive integer. Denote by $\mathbf{n} = n_1 n_2 \ldots n_k$ an ordered k-tuple of the cartesian product $\mathbb{N}_0^k$, $k \geq 2$. Define inductively an order relation $\leq^*$ on $\mathbb{N}_0^k$ in the following way :

1) For $k = 1, \leq^*$ is the usual $\leq$ relation.

2) For $k = 2, \mathbf{m} \leq^* \mathbf{n} \Leftrightarrow ((m_1 < n_1) \vee ((m_1 = n_1) \wedge (m_2 \leq n_2))$, for every $\mathbf{n} \in \mathbb{N}_0^2$, $\mathbf{m} \in \mathbb{N}_0^2$.

3) If $k \geq 3$, then $\mathbf{m} \leq^* \mathbf{n} \Leftrightarrow ((m_1 < n_1) \vee ((m_1 = n_1) \wedge (m_2 \dots m_k \leq^* n_2 \dots n_k)))$, for every $\mathbf{n} \in \mathbb{N}_0^k, \mathbf{m} \in \mathbb{N}_0^k$.

Denote by $\mathbf{0}$ the least element of $(\mathbb{N}_0^k, \leq^*)$. Denote by $\mathbf{n}(0) = n_1 n_2 \dots n_k$ an element of $(\mathbb{N}_0^k, \leq^*)$ such that there exists a positive integer k' satisfying :

i) $1 \leq k' \leq k-1$,

ii) $n_{k'} \neq 0$,

iii) $n_p = 0$ for any p such that $k'+1 \leq p \leq k$.

We write $\mathbf{n} <^* \mathbf{m}$, if $\mathbf{n} \leq^* \mathbf{m}$ and $\mathbf{n} \neq \mathbf{m}$. For $\mathbf{n} = n_1 n_2 \dots n_k \in (\mathbb{N}_0^k, \leq^*)$, we denote by $\mathbf{n}+\mathbf{1}$ the least of the elements $\{\mathbf{m} \in (\mathbb{N}_0^k, \leq^*) : \mathbf{n} <^* \mathbf{m}\}$. Speaking informally, $\mathbf{n}+\mathbf{1}$ is obtained by adding 1 to the last coordinate of $\mathbf{n}$.

We extend Definition 3.2 of [9] by defining recursively a map from $(\mathbb{N}_0^k, \leq^*)$ into the set of subspaces of E^* in the following way.

Definition 3.1. Let $\langle E, F \rangle$ be a dual pair and $\mathbf{n} \in \mathbb{N}_0^k$ $(k \geq 2)$. Denote :

1) $F_{\mathbf{0}} = F$,

2) $F_{\mathbf{n+1}} = (E, \beta(E, F_{\mathbf{n}}))'$,

3) $F_{\mathbf{n}(0)} = \cup\{F_{\mathbf{m}} : \mathbf{m} <^* \mathbf{n}(0)\}$.

For any $n_1 \dots n_{k-1} \in \mathbb{N}_0^{k-1}$, the sequence $\{F_{\mathbf{n}} : n_k \in \mathbb{N}_0\}$ is called a *dual strong sequence* of $\langle E, F \rangle$. Any of the subspaces $F_{\mathbf{n}(0)}$ is called a *dual strong union.* The chain $\{F_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}_0^k\}$, (well) ordered by $\leq^*$, is called *the initial dual strong sequence* for $k = 1$ and *the generalized dual strong sequence* for a given $k \geq 2$.

Informally we can think of $\{F_{mn} : mn \in \mathbb{N}_0^2\}$ introduced in [9] as dots of the xy-grid, considering n as the x-coordinate. For $k = 3$, we can visualise F_{kmn} as dots (n, m, k) of xyz-lattice.

Observation 3.1. Definition 3.1 implies that $\mathbf{m} \leq^* \mathbf{n} \Leftrightarrow F_{\mathbf{m}} \subseteq F_{\mathbf{n}}$.

Observation 3.2. For any fixed $\mathbf{n} \in \mathbb{N}_0^k$ $(k \geq 2)$, the tale $\{F_{\mathbf{m}} : \mathbf{m} \,{}^*\!\!\geq \mathbf{n}\}$ is the generalized dual strong sequence of $\langle E, F_{\mathbf{n}} \rangle$.

Observation 3.3. For any fixed $n_1 \in \mathbb{N}_0$, there is an order-preserving isomorphism of $\{F_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}_0^k, n_1 \text{ fixed}\}$ onto $\{F_{\mathbf{m}} : \mathbf{m} \in \mathbb{N}_0^{k-1}\}$ by $\mathbf{n} \to \mathbf{m}$ such that $n_p = m_{p-1}$, for $2 \leq p \leq k$.

Observation 3.4. The following statements are equivalent :

i) There exists $\mathbf{n} \in \mathbb{N}_0^k$ such that $(E, \mu(E, F_{\mathbf{n}}))$ is barrelled.

ii) There exists $\mathbf{n} \in \mathbb{N}_0^k$ such that $F_{\mathbf{m}} = F_{\mathbf{n}}$ for any $\mathbf{m} \,{}^*\!\!\geq \mathbf{n}$.

Given a locally convex nonbarrelled space (E, τ), we say that τ_{bar} is the *associated barrelled topology* for τ if τ_{bar} is the weakest barrelled topology, such that $\tau_{bar} \geq \tau$, ([6], 4.4.10; see also [1], Definition II.4.8).

Proposition 3.1. *Let* $\{F_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}_0^k\}$ $(k \geq 2)$ *be the generalized dual strong sequence of* $\langle E, F\rangle$. *The following statements are equivalent.*

i) τ_{bar} *is the associated barrelled topology for* $\mu(E, F)$.

ii) τ_{bar} *is the associated barrelled topology for any* $\mu(E, F_{\mathbf{n}}), \mathbf{n} \in \mathbb{N}_0^k$.

iii) τ_{bar} *is the associated barrelled topology for any* $\beta(E, F_{\mathbf{n}}), \mathbf{n} \in \mathbb{N}_0^k$.

Proof. For $k = 1$, it is Proposition 6.1 of [9]. For $k = 2$, it is Proposition 6.3 of [9]. For $k > 2$, we use Observations 3.2, 3.3 and the arguments of Proposition 6.1 of [9]. ∎

4 Systematization of bounded disks

Given two families $\mathbf{A}, \mathbf{B}$ of bounded disks in E, we consider the usual order relation : $\mathbf{A} \leq \mathbf{B}$ if and only if for any $A \in \mathbf{A}$, there exists $B \in \mathbf{B}$ such that $A \subseteq B$. We use $\mathbf{A} \subseteq \mathbf{B}$ instead of $\mathbf{A} \leq \mathbf{B}$ if any member of $\mathbf{A}$ belongs to $\mathbf{B}$. We write $\mathbf{A} \sim \mathbf{B}$ if $\mathbf{A} \leq \mathbf{B}$ and $\mathbf{B} \leq \mathbf{A}$. We use $\mathbf{A} = \mathbf{B}$ instead of $\mathbf{A} \sim \mathbf{B}$ if $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{B} \subseteq \mathbf{A}$.

We consider the following families of closed bounded disks of (E, η) :

1) $\mathbf{B}$ – the family of all closed bounded disks,

2) $\mathbf{B}^*$ – the family of all closed strongly bounded disks,

3) **NBAR** – the family of all closed barrelled disks,

4) **BAN** – the family of all closed Banach disks,

5) **WC** – the family of all $\sigma(E, E')$-compact disks,

6) **CO** – the family of all compact disks of (E, η),

7) **WFC** – the family of all weakly fast compact disks,

8) **FC** – the family of all fast compact disks,

9) **FIN** – the family of all finite-dimensional disks.

Notice that :

a) $\mathbf{B} \supseteq \mathbf{B}^* \supseteq \mathbf{NBAR} \supseteq \mathbf{BAN} \supseteq \mathbf{WC} \supseteq \mathbf{CO} \supseteq \mathbf{FC} \supseteq \mathbf{FIN}$,

b) $\mathbf{WC} \supseteq \mathbf{WFC} \supseteq \mathbf{FC}$.

We say that E is *⟨dual⟩ locally (quasi-)barrelled*, respectively *⟨dual⟩ locally (quasi-)complete*, if for any (strongly) bounded set A of E ⟨of E'⟩, there exists $B \in \mathbf{NBAR}$, respectively $B \in \mathbf{BAN}$, in E ⟨in E'⟩, such that $A \subseteq B$. In other words, E is *⟨dual⟩ locally (quasi-)barrelled*, respectively *⟨dual⟩ locally (quasi-)complete*, if $\mathbf{B} \sim \mathbf{BAN}(\mathbf{B}^* \sim \mathbf{BAN})$, in E ⟨in E'⟩. The space $(E, \mu(E, E'))$ is *barrelled (quasi-barrelled)* if $\mathbf{B} = \mathbf{WC}(\mathbf{B}^* = \mathbf{WC})$ in E'.

5 Bounded sets generated by a dual strong sequence

Keeping in mind Observation 3.3, we continue the discussion on dual strong sequences for $k = 2$. Denote by $\mathbf{A}_{kn}$ a family of bounded disks in $(E, \mu(E, F_{kn}))$.

Proposition 5.1. *Let* $\{F_{kn} : kn \in \mathbb{N}_0^2\}$ *be the generalized dual strong sequence of the duality* $\langle E, F \rangle$*. The following properties hold for the bounded families of type* $1, 2$ *of* $(E, \mu(E, F_{kn}))$, $\forall kn \in \mathbb{N}_0^2$.

a) $\mathbf{B}^*{}_{kn} \sim \mathbf{B}_{k,n+1}$,

b) $\mathbf{B}^*{}_{kn} \geq \mathbf{B}_{k+1,0}$.

Proof. a) $F_{k,n+1}$ is the bidual of F_{kn} in the duality $\langle F_{kn}, E \rangle$, therefore $\mathbf{B}^*{}_{kn} \subseteq \mathbf{B}_{k,n+1}$. Let $B \in \mathbf{B}_{k,n+1}$. Since $(E, \beta(E, F_{kn}))$ admits a 0-neighbourhood base consisting of $\mu(E, F_{kn})$-closed disks, the closure of B in $(E, \mu(E, F_{kn}))$ is still $\beta(E, F_{kn})$-bounded, hence $\mu(E, F_{k,n+1}))$-bounded, therefore $\mathbf{B}_{k,n+1} \leq \mathbf{B}^*{}_{kn}$. Hence $\mathbf{B}^*{}_{kn} \sim \mathbf{B}_{k,n+1}$.

b) Let $B \in \mathbf{B}_{k+1,0}$. Then B is bounded in $(E, \mu(E, F_{kn}))$ for every $n \in \mathbb{N}_0$. By the arguments of a), the $\mu(E, F_{kn})$-closure of B belongs to $\mathbf{B}^*{}_{kn}$, therefore $\mathbf{B}^*{}_{kn} \geq \mathbf{B}_{k+1,0}$. ∎

Proposition 5.2. *Let* $\{F_{kn} : kn \in \mathbb{N}_0^2\}$ *be the generalized dual strong sequence of* $\langle E, F \rangle$*. The following statements hold for the bounded families of type* $1 - 9$ *in* $(E, \mu(E, F_{kn}))$ *for any* $kn, mp \in \mathbb{N}_0^2$*, satisfying :* $kn \leq^* mp$.

a) $\mathbf{B}_{kn} \geq \mathbf{B}_{mp}$,

b) $\mathbf{B}^*{}_{kn} \geq \mathbf{B}^*{}_{mp}$,

c) $\mathbf{NBAR}_{kn} \subseteq \mathbf{NBAR}_{mp}$,

d) $\mathbf{BAN}_{kn} \subseteq \mathbf{BAN}_{mp}$,

e) $\mathbf{WC}_{kn} \supseteq \mathbf{WC}_{mp}$,

f) $\mathbf{CO}_{kn} \supseteq \mathbf{CO}_{mp}$,

g) $\mathbf{WFC}_{kn} = \mathbf{WFC}_{mp}$,

h) $\mathbf{FC}_{kn} = \mathbf{FC}_{mp}$,

i) $\mathbf{FIN}_{kn} = \mathbf{FIN}_{mp}$.

Proof. a) $\{(E, \sigma(E, F_{kn})) : kn \in \mathbb{N}_0^2\}$ is a projective spectrum with continuous identity maps $(E, \sigma(E, F_{mp})) \to (E, \sigma(E, F_{kn}))$ for $mp \;{}^* \geq kn$. Therefore each bounded disk of $(E, \mu(E, F_{mp}))$ is bounded in $(E, \mu(E, F_{kn}))$.

b) By Proposition 5.1, $\mathbf{B}^*{}_{kn} \sim \mathbf{B}_{k,n+1} \supseteq \mathbf{B}^*{}_{k,n+1} \geq \mathbf{B}_{k+1,0} \supseteq \mathbf{B}^*{}_{k+1,0} \sim \mathbf{B}_{k+1,1} \supseteq \mathbf{B}^*{}_{k+1,1}$. Therefore $\mathbf{B}^*{}_{kn} \geq \mathbf{B}^*{}_{mp}$ for any $kn \leq^* mp$.

c) Let $B \in \mathbf{NBAR}_{kn}$. Then B is closed in $(E, \sigma(E, F_{mp}))$ for any $mp \;^*\!\geq kn$. Since $\mathbf{NBAR}_{kn} \subseteq \mathbf{B}^*{}_{kn} \sim \mathbf{B}_{k,n+1}$, B is a barrelled disk in $(E, \sigma(E, F_{k,n+1}))$, hence $B \in \mathbf{NBAR}_{k,n+1}$, and we conclude that $B \in \mathbf{NBAR}_{kp}$ for any $p \geq n$. Since $F_{k+1,0} = \cup\{F_{kn} : n \in \mathbb{N}_0\}$, B is bounded in $(E, \sigma(E, F_{k+1,0})$, hence $B \in \mathbf{NBAR}_{k+1,0}$. Therefore $\mathbf{NBAR}_{kn} \subseteq \mathbf{NBAR}_{mp}$ for any $mp \;^*\!\geq kn$.

d) Using the same arguments as in c), we conclude that $\mathbf{BAN}_{kn} \subseteq \mathbf{BAN}_{mp}$.

e) The identity map $(E, \sigma(E, F_{mp})) \to (E, \sigma(E, F_{kn}))$ is continuous for $kn \leq^* mp$. Therefore $\mathbf{WC}_{kn} \supseteq \mathbf{WC}_{mp}$.

f) By the arguments of e), $\mathbf{CO}_{kn} \supseteq \mathbf{CO}_{mp}$.

g) Observing that any closed Banach disk of $(E, \sigma(E, F_{mp}))$ is still a Banach disk but is not necessarily closed in $(E, \sigma(E, F_{kn}))$ for $kn \leq^* mp$, we conclude that $(E, \sigma(E, F_{kn}))$ and $(E, \sigma(E, F_{mp}))$ have the same Banach disks. Hence $\mathbf{WFC}_{kn} = \mathbf{WFC}_{mp}$.

h) By the arguments of g), $\mathbf{FC}_{kn} = \mathbf{FC}_{mp}$.

i) It is trivial.

∎

Observation 5.1. Using the arguments of Proposition 5.2, we conclude that the members of the chain $\{(E, \mu(E, F_{kn})) : kn \in \mathbb{N}_0^2\}$ have the same Banach (not necessarily closed) disks. Generally, the family of nonclosed Banach disks in a locally convex space is wider than the family of the closed Banach disks. A non-trivial example of a nonclosed Banach disk, based on James Condition for Reflexivity, can be found in ([6], 3.2.21).

Observation 5.2. The arguments of Proposition 5.2 assure us that the members of the chain $\{(E, \mu(E, F_{kn})) : kn \in \mathbb{N}_0^2\}$ have the same barrelled (not necessarily closed) disks. Generally, the family of barrelled disks is wider than the family of Banach disks. Each infinite-dimensional Banach disk has dense barrelled subspaces with barrelled non-Banach disks, ([11], Ch. 1, 3.2.9). For every infinite-dimensional Banach disk B, there exists a barrelled disk A such that $SpA = SpB$ and $A \subseteq B$, ([2], Proposition 3.7).

Proposition 5.3. *Let $\{F_{kn} : kn \in \mathbb{N}_0^2\}$ be the generalized dual strong sequence of $\langle E, F\rangle$. Suppose there exists $kn \in \mathbb{N}_0^2$ such that $(E, \mu(E, F_{kn}))$ is locally quasi-barrelled (resp. locally quasi-complete). Then $(E, \mu(E, F_{mp}))$ is locally barrelled (resp. locally complete) and $\mathbf{B}_{mp} \sim \mathbf{NBAR}_{mp} \sim \mathbf{NBAR}_{kn}$ (resp. $\mathbf{B}_{mp} = \mathbf{BAN}_{mp} \sim \mathbf{BAN}_{kn}$) for any $mp \;^*\!> kn$.*

Proof. Suppose $\mathbf{B}^*{}_{kn} \sim \mathbf{NBAR}_{kn}$ for some $kn \in \mathbb{N}_0^2$. By Propositions 5.1 and 5.2, $\mathbf{B}_{k,n+1} \sim \mathbf{B}^*{}_{kn} \sim \mathbf{NBAR}_{kn} \subseteq \mathbf{NBAR}_{mp} \subseteq \mathbf{B}^*{}_{mp} \subseteq \mathbf{B}_{mp} \leq \mathbf{B}_{k,n+1}$ for $mp \;^*\!\geq kn$. Therefore $\mathbf{B}_{mp} \sim \mathbf{NBAR}_{mp} \sim \mathbf{NBAR}_{kn}$, for any $mp \;^*\!\geq k, n+1$. Hence $(E, \mu(E, F_{mp}))$ is locally barrelled for any $mp \;^*\!\geq k, n+1$. In a similar way, if $\mathbf{B}^*{}_{kn} = \mathbf{BAN}_{kn}$ for some $kn \in \mathbb{N}_0^2$, then $\mathbf{B}_{mp} = \mathbf{BAN}_{mp} \sim \mathbf{BAN}_{kn}$ and $(E, \mu(E, F_{mp}))$ is locally complete for any $mp \;^*\!\geq k, n+1$. ∎

6 The associated topologies on E

A disk of E is absorbing, bornivorous or ultrabornivorous if it absorbs every element, every bounded disk or every Banach disk of E, ([1], Definition III.2.1). A disk of E is a *transbarrel* if it absorbs every barrelled disk of E. Any bornivorous disk of E is a transbarrel and any transbarrel is ultrabornivorous. Since any barrelled disk is strongly bounded, any barrel of E is a transbarrel. The gauge of a transbarrel is a seminorm, ([4], t. I, p. 21), and finite intersections of transbarrels are transbarrels.

We say that (E, t) is *transbarrelled* if every transbarrel of (E, t) is a 0-neighbourhood in (E, t). Notice that a transbarrelled space is barrelled.

Given a locally convex space (E, τ), we say that $\tau_b(\tau_{ub})$ is *the associated bornological (ultrabornological) topology* for τ if $\tau_b(\tau_{ub})$ is the weakest bornological (ultrabornological) topology of E such that $\tau_b \geq \tau(\tau_{ub} \geq \tau)$, ([1], Definition III.2.3, see also [6], 6.2.4). Clearly, $\tau_{ub} \geq \sup(\tau_b, \tau_{bar})$. For two locally convex topologies τ, η on E, we define $\sup\{\tau, \eta\}$ as the weakest locally convex topology on E, finer that τ and η. Remind that τ_{bar} is the associated barrelled topology for $(E, \mu(E, E'))$.

Proposition 6.1. *Let E a locally convex space. Let τ_{trb} be a locally convex topology on E such that a disk is a 0-neighbourhood in (E, τ_{trb}) if and only if it is a transbarrel of $(E, \mu(E, E'))$. The space (E, τ_{trb}) is transbarrelled and $\sup(\tau_b, \tau_{bar}) \leq \tau_{trb} \leq \tau_{ub}$.*

Proof. Clearly, τ_{trb} is a locally convex Hausdorff topology on E, satisfying : $\tau_{trb} \geq \beta(E, E')$ and $\tau_b \leq \tau_{trb} \leq \tau_{ub}$. Since a 0-neighbourhood in (E, τ_{trb}) is a transbarrel of $(E, \mu(E, E'))$, any barrelled disk of $(E, \mu(E, E'))$ is bounded in (E, τ_{trb}), implying that $(E, \mu(E, E'))$ and (E, τ_{trb}) have the same barrelled disks. Hence any transbarrel of (E, τ_{trb}) is also a transbarrel in $(E, \mu(E, E'))$, therefore a 0-neighbourhood in (E, τ_{trb}), and we proved that (E, τ_{trb}) is transbarrelled. Since (E, τ_{trb}) is barrelled, $\tau_{trb} \geq \tau_{bar}$. Therefore $\sup(\tau_b, \tau_{bar}) \leq \tau_{trb} \leq \tau_{ub}$. ∎

Definition 6.1. Let (E, τ) be a locally convex space. The topology τ_{trb} such that a disk is a 0-neighbourhood in (E, τ_{trb}) if and only if it is a transbarrel of (E, τ) is called the *associated transbarrelled topology* for (E, τ).

Proposition 6.2. *Let $\{F_{kn} : kn \in \mathbb{N}_0^2\}$ be the generalized dual strong sequence of $\langle E, F\rangle$ and τ_{bar}, (resp. τ_{trb}, τ_{ub}), the associated barrelled, (resp. transbarrelled, ultrabornological), topology for $\mu(E, F)$. The following statements are true for each $kn \in \mathbb{N}_0^2$.*

a) $(E, \mu(E, F_{kn})), (E, \tau_{bar}), (E, \tau_{trb})$ *have the same barrelled disks.*

b) $(E, \mu(E, F_{kn})), (E, \tau_{bar}), (E, \tau_{trb}), (E, \tau_{ub})$ *have the same Banach disks.*

c) $(E, \mu(E, F_{kn})), (E, \tau_{bar}), (E, \tau_{trb}), (E, \tau_{ub})$ *have the same fast compact and weakly fast compact disks.*

d) τ_{trb}*, (resp.* τ_{ub}*), is the associated transbarrelled, (resp. ultrabornological), topology for* $\mu(E, F_{kn})$*).*

Proof. a) follows from Observation 5.2 and Proposition 6.1.

b) follows from Observation 5.1, Proposition 6.1 and ([1], Proposition III.2.4).

c) follows from b).

d) follows from a) and b).

∎

Proposition 6.3. *Let (E,τ) be a locally convex space, $F_{00} = E', \{F_{kn} : kn \in \mathbb{N}_0^2\}$ the generalized dual strong sequence of $\langle E, E'\rangle$ and τ_b, (resp. $\tau_{bar}, \tau_{trb}, \tau_{ub}$) the associated bornological, (resp. barrelled, transbarrelled, ultrabornological), topology for τ. The following statements are true for each $kn \in \mathbb{N}_0^2$.*

a) If (E,τ) is locally barrelled, then $\tau \le \mu(E, F_{kn}) \le \tau_{bar} \le \tau_b = \tau_{trb} \le \tau_{ub}$.

b) If (E,τ) is locally complete, then $\tau \le \mu(E, F_{kn}) \le \tau_{bar} \le \tau_b = \tau_{trb} = \tau_{ub}$.

Proof. a) By ([1], Proposition III.2.4), (E,τ_b) is the inductive limit of $\{E_B : B \in \mathbf{B}\}$. Therefore (E,τ_b) is barrelled, hence $\tau_b \ge \tau_{bar}$. Since $(E,\tau), (E,\tau_b)$ and (E,τ_{trb}) have the same bounded subsets, τ_b is the associated bornological topology for $\tau, \mu(E, F_{kn})$, τ_{bar} and τ_{trb}. Applying Propositions 6.1 and 6.2, we obtain the conclusion.

b) Follows from ([1], Proposition III.2.4) and Proposition 6.2.

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For a duality $\langle E, F\rangle$, we denote by $(F, fc(F,E))$ the Schwartz topology on F of uniform convergence on the fast compact disks of E.

Proposition 6.4. *Let $\{F_{kn} : kn \in \mathbb{N}_0^2\}$ be the generalized dual strong sequence of $\langle E, F\rangle$. Let G be the completion of $(F, fc(F,E))$. Then F_{kn} is a dense subspace of $(G, fc(G,E))$, for each $kn \in \mathbb{N}_0^2$.*

Proof. By ([1], Proposition III.2.8), G is the dual of (E,τ_{ub}). Since by Proposition 6.2, τ_{ub} is the associated ultrabornological topology for $\mu(E, F_{kn})$, we apply ([1], Proposition III.2.8). ∎

We conclude this article with a simple illustration of the associated transbarrelled topology τ_{trb} on a locally convex space, such that $\tau_{trb} \ne \tau_{bar}$ and $\tau_{trb} \ne \tau_{ub}$.

Example 6.1. Take a barrelled, nonbornological, locally complete space X, such that X admits an infinite-dimensional bounded disk, ([12], Theorem 1, see also [6], 6.2.16). Let E be a "remarkable hyperplane" in X equipped with the induced topology, ([10]; see also [6], 6.3.11; for discussion on cardinality issues and requirements see [3], 2.2). Then E is barrelled, locally barrelled, admits an infinite-dimensional bounded disk, and any Banach disk of E is finite-dimensional (i.e. $\mathbf{B} = \mathbf{B}^* \sim$ **NBAR** and **BAN** = **WC** = **CO** = **WFC** = **FC** = **FIN** in E). Since E is locally dense in X, E is nonbornological ([6], Proposition 6.2.7). Then $\mu(E, E') = \tau_{bar} < \tau_b$. By Observation 3.4 and Proposition 6.3 a), $\tau_b = \tau_{trb}$. Therefore the space (E, τ_{trb}) is the inductive limit of $\{E_B : B \in \mathbf{B}\}$, ([1], III.2; see also [6], 6.1; see also [11], Ch. 1, 3.2). Noticing that the associated ultrabornological topology τ_{ub} is the finest locally convex topology on E, we conclude that $\mu(E, E') = \tau_{bar} < \tau_b = \tau_{trb} < \tau_{ub}$.

The author is very grateful to Professor J. Schmets of the University of Liege for improving this article by valuable remarks and suggestions after carefully reading the preliminary version.

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