Reduction of cocycles and groups of diffeomorphisms of the circle

Andrés Navas

Abstract

We prove two theorems of reduction of cocycles taking values in the group of diffeomorphisms of the circle. They generalize previous results obtained by the author concerning rigidity for smooth actions on the circle of Kazhdan groups and higher rank lattices.

Introduction

As a general principle, it should be possible to extend all results involving groups satisfying Kazhdan's property (T) to the setting of *cocycles* introduced by R. Zimmer (see §1 for definitions). We list below several examples. In all of them, G is a Kazhdan group acting ergodically by measure preserving transformations on a probability space Ω, and $\alpha : G \times \Omega \to H$ is a Borel cocycle taking values in a locally compact topological group H.

(a) If the group H is amenable, then α is cohomologous to a cocycle taking values into a compact subgroup of H [24]. This result has been extended in [14] to the case where H satisfies Haagerup's property, *i.e.* when it is a-(T)-menable (see also [2]).

(b) If H is a Lie group, then α is cohomologous to a cocycle taking values into a Kazhdan subgroup of H [23].

(c) If H is the group of isometries of a real tree, then α is cohomologous to a cocycle taking values into the stabilizer of some point of the tree [1]. This result can be easily extended to the case where H is the group of automorphisms of a measured wall-space [5].

In this article we add to the list above the following result.

Theorem A. *Let $\alpha : G \times \Omega \to \mathrm{Diff}_+^{1+\tau}(\mathrm{S}^1)$ be a Borel cocycle, where $\tau > 1/2$ and G is a compactly generated topological group whose action on Ω is measure preserving and ergodic. Suppose that, for each $g \in G$, the map $x \mapsto \alpha(g,x)$ takes values a.e. into a bounded subset of $\mathrm{Diff}_+^{1+\tau}(\mathrm{S}^1)$. If G has Kazhdan's property (T), then as a cocycle into $\mathrm{Homeo}_+(\mathrm{S}^1)$, α is cohomologous to a cocycle taking values into the group of (euclidean) rotations.*

Recall that, for $\tau > 1/2$, every subgroup of $\mathrm{Diff}_+^{1+\tau}(\mathrm{S}^1)$ that satisfies Kazhdan's property (T) is conjugate to a (compact) group of Euclidean rotations. This result was obtained by the author in [19] (see also [20]). Theorem A above is just a generalisation of this fact into the framework of cocycles. It should be mentioned that a better result is known when G is a Lie group satisfying Kazhdan's property (T). Indeed, using ideas introduced by É. Ghys [11], D. Morris Witte and R. Zimmer proved [17] a similar result in that case for cocycles into $\mathrm{Diff}_+^1(\mathrm{S}^1)$ (and also a partial result for cocycles into $\mathrm{Homeo}_+(\mathrm{S}^1)$).

The reader which is familiar to ideas and techniques from rigidity theory for lattices in semisimple Lie groups could think that the proof of Theorem A is just a translation of the proof given in [19] to the language of cocycles. Nevertheless, there are several (more than technical) problems which appear in that translation, and this is the main reason that motivated the author to write this article. Similar problems appear when we deal with the higher rank case. For that case we follow essentially the approach of [21] (see also [16]): our prototype of higher rank group will be a product $G = G_1 \times \cdots \times G_k$ of $k \geq 2$ compactly generated topological groups G_i. We will also suppose that G acts on Ω *ergodically irreducibly*, that is the action of each subgroup $G_i' = G_1 \times \cdots \hat{G}_i \times \cdots \times G_k$ on Ω is ergodic. A typical example of this situation is given by the action of G on the quotient G/Γ, where Γ is a lattice in G whose projections into each G_i' are dense. The following result is inspired by [18]. The precise meaning of its statement will be clarified in §3.

Theorem B. *Let $\alpha : G \times \Omega \to \mathrm{Diff}_+^{1+\tau}(\mathrm{S}^1)$ be a Borel cocycle, where $G = G_1 \times \cdots \times G_k$ is a product of $k \geq 2$ compactly generated topological groups and $\tau > 1/2$. Suppose that the G-action on Ω is ergodically irreducicle, and that for each $g \in G$ the map $x \mapsto \alpha(g,x)$ takes values a.e. into a bounded subset of $\mathrm{Diff}_+^{1+\tau}(\mathrm{S}^1)$. If the action of G on $\Omega \times \mathrm{S}^1$ does not preserve any probability measure, then up to a topological semiconjugacy and a finite cover, α is cohomologous in $\mathrm{Homeo}_+(\mathrm{S}^1)$ to a cocycle given by a homomorphism from G into the group of direct homeomorphisms of the circle which factors through one of the G_i. Moreover, if each G_i is non discrete and almost topologically simple, then the image of this homomorphism coincides with some finite cover of $\mathrm{PSL}(2,\mathbb{R})$.*

It must be mentioned that Theorem B has a version in class C^1 (and also a partial version in class C^0) when G is a higher rank semisimple Lie group without Kazhdan's property (T). This is stated explicitly in [17], and the proof is based on some of the arguments of [11].

The plan of this paper is the following. In §1 we recall some definitions and the main technical tools used in the proof of theorems A and B, namely the Cocycle

Reduction Lemma (due to R. Zimmer) and the Superrigidity Theorem for Reduced Cohomology (due to Y. Shalom) respectively. In §2 we extend the technique introduced in [19] to the case of cocycles. The main step for the proof of Theorem A is given in §2.5, where we prove that the action of the group of orientation preserving homeomorphisms of the circle on some space of "stable" geodesic currents is *smooth* (that is, the orbits are locally closed). This is a fundamental fact that must be verified in order to apply cocycle reduction and to finish the proof of Theorem A. Finally, the proof of Theorem B is given in §3.

Remark. Theorems A and B (and also the results from [18] and [19]) are still true for cocycles taking values in the group of diffeomorphisms of S^1 of slightly lower differentiability class than $C^{3/2}$. However, we will discuss this point only in the Appendix, since the proofs rely on recent developments on property (T) and the geometry of "almost" Hilbert spaces, and are independent of the rest of the paper.

1 Some preliminary facts

1.1 Zimmer's Cocycle Reduction Lemma

Let Ω be a Borel space endowed with a probability measure μ and let G be a group acting on Ω and preserving the measure class of μ. A Borel map $\alpha : G \times \Omega \to H$ taking values into a topological group H is called a *cocycle* if for all $g_1, g_2 \in G$ and a.e. $x \in \Omega$,

$$\alpha(g_1 g_2, x) = \alpha(g_1, g_2(x))\alpha(g_2, x).$$

When H is identified to a group of automorphisms of some space Ω', the fact that a (Borel) map $\alpha : G \times \Omega \to H$ is a cocycle is equivalent to that the map from G into the group of automorphisms of $\Omega \times \Omega'$ given by $g(x, y) = (g(x), \alpha(g, x)(y))$ is a homomorphism.

Two cocycles α and β are *cohomologous* if there exists a Borel map $\varphi : \Omega \to H$ such that, for a.e. $x \in \Omega$,

$$\alpha(g, x) = \varphi(g(x))^{-1}\beta(g, x)\varphi(x).$$

When H is the group of automorphisms of a space Ω', a Borel function $\psi : \Omega \to \Omega'$ is said to be *equivariant* if for all $g \in G$ one has $\psi(g(x)) = \alpha(g, x)\psi(x)$ for a.e. $x \in \Omega$. In what follows we will always suppose that the Borel structure of H is countably generated ([24], page 10).

Cocycle Reduction Lemma ([24], page 108). *Let $\alpha : G \times \Omega \to H$ be a Borel cocycle, G acting ergodically on Ω. Suppose that Ω' is a continuous H-space on which the action is smooth (i.e. the orbits are locally closed). If there exists an equivariant function from Ω to Ω', then there is a point $y \in \Omega'$ such that α is cohomologous to a cocycle taking values into the stabilizer H_y of y in H.*

The proof of this lemma is very simple: it is based on the principle that functions which are constant along the orbits of an ergodic action are essentially constant. In order to apply this principle, it is important to have a good structure for the space of orbits by H in Ω'. Actually, R. Zimmer states the lemma only for locally compact

groups H, but it is easy to see that this hypothesis is not completely necessary. The essential hypothesis are that the Borel structure of H is countably generated and that the H-action on Ω' is smooth. We clarify all of this because we will deal in the sequel with groups of homeomorphisms of the circle, which in general are not locally compact [10].

1.2 Kazhdan's property (T)

Recall that a locally compact topological group G has Kazhdan's property (T) (or to simplify, is a Kazhdan group) if every (continuous) action of G by isometries of a separable Hilbert space has an invariant vector. Actually, this is not the original definition given by D. Kazhdan, but an equivalent one due to J. P. Serre and denoted property (FH). In this direction, one can naturally define property (FH) for actions, and Theorem A will still be true in this more general setting when G fails to be a Kazhdan group but its action on $\Omega \times S^1$ has property (FH). Let us remark that properties (T) and (FH) are also equivalent for actions, and more generally for equivalence relations [2].

1.3 Shalom's Superrigidity Theorem

A continuous action of a locally compact topological group G by isometries of a metric space (X, d) is called *uniform* if there exist $\varepsilon > 0$ and a compact generating set C of G such that for all $x \in X$ there exists $g \in C$ satisfying $d(g(x), x) \geq \varepsilon$. We will be mainly interested in actions by (affine) isometries of a (real) Hilbert space $\mathcal{H}$. Recall that the group of isometries of such a space is the semidirect product between the unitary group and the group of translations. For an isometry A we will denote by $\theta \in U(\mathcal{H})$ and $c \in \mathcal{H}$ its unitary and translation components respectively.

Among the remarkable results obtained by Y. Shalom in [21], for future reference we state here the Superrigidity Theorem for Reduced Cohomology, which will be essential in §3.

Theorem [21]. Let $G = G_1 \times \cdots \times G_k$ be a topological group which is the product of $k \geq 2$ compactly generated groups G_i. If $A = \theta + c$ is a uniform isometric action of G on a Hilbert space $\mathcal{H}$, then there exists a non zero vector in $\mathcal{H}$ which is fixed by one of the $\theta(G_i')$, where $G_i' = G_1 \times \cdots \times \hat{G}_i \times \cdots \times G_k$.

We will also use the following classical lemma, due to P. Delorme [6].

Lemma. *Let $A = \theta + c$ be an action of a locally compact topological group G by isometries of a Hilbert space $\mathcal{H}$. If A has no global fixed point and is non uniform, then there exists a sequence (K_n) of unitary vectors in $\mathcal{H}$ which is θ-almost invariant, that is for each compact subset C of G the value of $\sup_{g \in C} \|\theta(g)K_n - K_n\|$ tends to zero as n tends to infinity.*

1.4 Stable geodesic currents and convergence groups

Recall that a geodesic current on the circle is a Radon measure defined on the space $S^1 \times S^1 \setminus \Delta$ which is invariant by the flip $(u, v) \mapsto (v, u)$. (Δ denotes the corresponding diagonal.) We say that such a current ν is *stable* if

$$\nu([a, b] \times [c, c]) = 0 \qquad \text{and} \qquad \nu([a, b[\times]b, c]) = \infty \qquad \text{for all} \quad a < b < c < a.$$

The notion of stable geodesic current was introduced in [19], where the following proposition was stated without proof. The author is indebted to J. C. Yoccoz for the appointment of the argument below, which simplifies an original one given in a previous version of this paper.

Proposition 1.1. *If ν is a stable geodesic current, then the group G_ν of (orientation preserving) homeomorphisms of the circle whose diagonal action on $S^1 \times S^1$ preserves ν is topologically conjugate to a subgroup of $\mathrm{PSL}(2, \mathbb{R})$.*

Proof. We will prove that G_ν has the Convergence Property, that is, each sequence (g_n) in G_ν is either equicontinuous or contains a subsequence (g_{n_k}) such that for some points $a, b \in S^1$ one has that $g_{n_k}(x)$ tends to b for all $x \in S^1 \setminus \{a\}$ and that $g_{n_k}^{-1}(x)$ tends to a for all $x \in S^1 \setminus \{b\}$. By [4, 9, 12] and [22], this implies that G_ν is topologically conjugate to a subgroup of $\mathrm{PSL}(2, \mathbb{R})$.

Let us suppose that (g_n) is a non equicontinuous sequence in G_ν. Up to a subsequence, we may assume that there exists a point $a \in S^1$ and two sequences (x_n) and (y_n) converging to this point, such that $x_n < a < y_n < x_n$ for all $n \in \mathbb{N}$, and such that $g_n(x_n)$ tends to some point $x_\infty \in S^1$ and $g_n(y_n)$ tends to some point $y_\infty \in S^1$, where $x_\infty \neq y_\infty$. Let us fix a point $u \in S^1$ different from a. Up to a new subsequence, one can suppose that $g_n(u)$ tends to some point $b \in S^1$. We will prove that (g_n) tends pointwise to this point $b \in S^1$ on $S^1 \setminus \{a\}$.

Let us fix a point $v \in S^1$ different from u and a. Let us first suppose that v belongs to the interval $]u, a[$. In that case, for $n \in \mathbb{N}$ large enought one has $u < v < x_n < y_n < u$, and since ν is invariant by $g \in G_\nu$,

$$\nu([v, x_n] \times [y_n, u]) = \nu([g_n(v), g_n(x_n)] \times [g_n(y_n), g_n(u)]).$$

The left hand member of this equality converges to $\nu([v, a[\times]a, u]) = \infty$ as n goes to infinity. Thus the right hand member also converges to infinity. But since $g_n(x_n)$, $g_n(y_n)$ and $g_n(u)$ converge to x_∞, y_∞ and b respectively, and since $x_\infty \neq y_\infty$, this implies that $g_n(v)$ converges to b. When v belongs to $]a, u[$, the same argument applied to the product $[u, x_n] \times [y_n, v]$ shows that $g_n(v)$ still converges to the point b.

A similar argument can be given for the sequence (g_n^{-1}), showing that G_ν has the convergence property. This finishes the proof of the proposition. $\quad\square$

The main example of a stable geodesic current is Liouville's measure Lv defined by

$$dLv(u, v) = \frac{du\, dv}{4 \sin^2(\frac{u-v}{2})}.$$

Other examples can be obtained by noting that stability is preserved by small per-turbations. For example, for any function $K \in \mathcal{L}^2(\mathrm{S}^1 \times \mathrm{S}^1, Lv)$ which is invariant by the flip $(u, v) \mapsto (v, u)$, the Radon measure ν_K given by $d\nu_K = [1 + K]^2 dLv$ is stable [19]. This kind of measures will be essential in what follows.

2 Reduction for Kazhdan's groups

2.1 Passing to the 3-fold covering

The study of actions of Kazhdan groups on the circle is simplified by using a beautiful argument pointed to the author by D. Morris Witte: instead of considering the original action, it is sufficient (and easier) to deal with the induced action of a degree 3 central extension of the group on the 3-fold covering of the circle. However, it is not completely evident how to translate this argument to the setting of cocycles, and we will need to be a little bit careful to do this. The idea will consist on looking directly at an affine isometric action on some Hilbert space associated to this 3-fold covering. (Note that this argument does not appear in [18] or [19], and it can be used to simplify some of the proofs therein.)

Let us denote by $\hat{\mathrm{S}}^1$ the 3-fold covering of the original circle S^1. For each $g \in G$ and a.e. $x \in \Omega$, the map $\alpha(g, x) \in \mathrm{Diff}_+^{1+\tau}(\mathrm{S}^1)$ induces 3 diffeomorphisms $\hat{\alpha}_i(g, x)$, $i \in \{1, 2, 3\}$, of $\hat{\mathrm{S}}^1$. Those diffeomorphisms differ each one from the other by an order 3 Euclidean rotation. Remark that in general there is no canonical way to consider $\hat{\alpha}_i$ as a map from $G \times \Omega$ to $\mathrm{Diff}_+^{1+\tau}(\hat{\mathrm{S}}^1)$.

2.2 Construction of the affine isometric action

Let us first consider the Hilbert space $\mathcal{H}' = \mathcal{L}^{2,\Delta}(\hat{\mathrm{S}}^1 \times \hat{\mathrm{S}}^1, Lv)$ of square integrable real valued functions K that satisfy a.e. the equalities $K(u, v) = K(v, u)$ and $K(u + 2\pi/3, v + 2\pi/3) = K(u, v)$. Now let us consider the Hilbert space $\mathcal{H}$ of functions $K \in \mathcal{L}^2(\Omega \times \hat{\mathrm{S}}^1 \times \hat{\mathrm{S}}^1, \mu \times Lv)$ such that, for a.e. $x \in \Omega$, the function K_x sending (u, v) to $K(x, (u, v))$ belongs to $\mathcal{H}'$.

Here is a main point. Even if there is not a well defined action of G on $\Omega \times \hat{\mathrm{S}}^1$, one can naturally define the "regular representation" of G on $\mathcal{H}$ by letting

$$\theta(g^{-1})K(x, (u, v)) = K(g(x), \hat{\alpha}_i(g, x)(u, v)) \, [Jac(\hat{\alpha}_i)(g, x)(u, v)]^{1/2}.$$

Indeed, this definition does not depend on $i \in \{1, 2, 3\}$. (Here and in what follows, we denote also by $\hat{\alpha}_i(g, x)$ the map from $\hat{\mathrm{S}}^1 \times \hat{\mathrm{S}}^1$ to itself induced diagonally by the original one of $\hat{\mathrm{S}}^1$, and we denote by $Jac(\hat{\alpha}_i)(g, x)(u, v)$ the Jacobian at the point (u, v) of this map.)

Lemma 2.1. *For each $g \in G$ the function $c(g)$ given by*

$$(x, (u, v)) \longmapsto 1 - [Jac(\hat{\alpha}_i)(g^{-1}, x)(u, v)]^{1/2}$$

is well defined (i.e. it does not depend on $i \in \{1, 2, 3\}$) and belongs to the Hilbert space $\mathcal{H}$.

Proof. The fact that $c(g)$ is well defined is clear. On the other hand, the proof of Proposition 2.1 in [19] shows that for each $g \in G$ and a.e. $x \in \Omega$, the map

$$(u, v) \longmapsto 1 - [Jac(\hat{\alpha}_i)(g^{-1}, x)(u, v)]^{1/2}$$

belongs to $\mathcal{H}'$. More precisely, for $\tau > 1/2$, there exists a constant $C_\tau < \infty$ such that

$$\left\| 1 - [Jac(\hat{\alpha}_i)(g^{-1}, x)(u, v)]^{1/2} \right\| \leq C_\tau \left\| \alpha_i'(g, x) \right\|_\tau.$$

Since by hypothesis the map $x \mapsto \alpha(g, x)$ takes essentially its values in a bounded subset of $\mathrm{Diff}_+^{1+\tau}(\mathrm{S}^1)$, this inequality proves the lemma.

Now we have a candidate for an isometric affine action of G on the Hilbert space $\mathcal{H}$, namely $A(g) = \theta(g) + c(g)$. The fact that this well defines an isometric action is straightforward to verify.

2.3 The field of equivariant stable geodesic currents

When G has Kazhdan's property (T), the isometric action A above has a fixed point $K \in \mathcal{H}$, that is $\theta(g)K + c(g) = K$ for all $g \in G$. By definition, for all $g \in G$ one has a.e. the equality

$$[1 - K(x, (u, v))]^2 = [1 - K(g^{-1}(x), \hat{\alpha}_i(g^{-1}, x)(u, v))]^2 \, Jac(\hat{\alpha}_i(g^{-1}, x))(u, v). \quad (1)$$

Since K belongs to $\mathcal{H}$, for a.e. $x \in \Omega$ the function K_x that sends (u, v) to $K(x, (u, v))$ belongs to $\mathcal{H}'$. We will denote by ν_x the measure on $\hat{\mathrm{S}}^1 \times \hat{\mathrm{S}}^1$ given by

$$d\nu_x = [1 - K_x]^2 dLv.$$

As a measure on $\hat{\mathrm{S}}^1 \times \hat{\mathrm{S}}^1 \setminus \Delta$, it is an a.e. well defined Radon measure. Moreover, by §1.4, the measure ν_x is an a.e. well defined stable geodesic current.

Let us denote by $\mathcal{SGC}(\hat{\mathrm{S}}^1)$ the space of Radon measures defined on $\hat{\mathrm{S}}^1 \times \hat{\mathrm{S}}^1 \setminus \Delta$ which are stable, invariant by the flip $(u, v) \mapsto (v, u)$, and also invariant by the simultaneous order 3 Euclidean rotation $(u, v) \mapsto (u + 2\pi/3, v + 2\pi/3)$. The group $\mathrm{Homeo}_+(\mathrm{S}^1)$ acts on $\mathcal{SGC}(\hat{\mathrm{S}}^1)$: for $h \in \mathrm{Homeo}_+(\mathrm{S}^1)$ take one of its 3 preimages $\hat{h} \in \mathrm{Homeo}_+(\hat{\mathrm{S}}^1)$ and for $\nu \in \mathcal{SGC}(\hat{\mathrm{S}}^1)$ define $h(\nu)$ as $(\hat{h} \times \hat{h})_*(\nu)$. By the third property above, this definition is independent of the choice of the preimage $\hat{h}$.

2.4 The reduction of the cocycle

We proved in §2.3 that there exists an equivariant map $\psi : \Omega \to \mathcal{SGC}(\hat{\mathrm{S}}^1)$, namely $\psi(x) = \nu_x$ (see equality (1)). In §2.5 we will prove that the action of $\mathrm{Homeo}_+(\mathrm{S}^1)$ on $\mathcal{SGC}(\hat{\mathrm{S}}^1)$ is smooth. Assuming this fact for a moment, we can apply the Cocycle Reduction Lemma to conclude that α is cohomologous in $\mathrm{Homeo}_+(\mathrm{S}^1)$ to a cocycle $\beta : G \times \Omega \to H_\nu$, where H_ν is the stabilizer of some element $\nu \in \mathcal{SGC}(\hat{\mathrm{S}}^1)$ by the action of $\mathrm{Homeo}_+(\mathrm{S}^1)$.[1]

---

[1] Is is maybe possible to perform this first reduction of the cocycle α inside the group of orientation preserving *quasisymmetric* homeomorphisms of the circle (compare with [13], [18], and the recent preprint [15]).

Let us denote by $\hat{H}_\nu$ the degree 3 central extension of H_ν. This is a subgroup of $\mathrm{Homeo}_+(\hat{S}^1)$ which preserves ν. So, by Proposition 1.1, $\hat{H}_\nu$ is topologically conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. We claim that the action of H_ν on S^1 is free. Indeed, if $h \in H_\nu$ fixes one point of S^1, then one of its preimages $\hat{h}$ fixes 3 points of $\hat{S}^1$. However, since $\hat{h}$ is topologically conjugate to an element of $\mathrm{PSL}(2,\mathbb{R})$, this is not possible unless $\hat{h}$ (and hence h) is the identity.

We conclude that H_ν is a subgroup of $\mathrm{Homeo}_+(S^1)$ whose action is free and whose degree 3 central extension is topologically conjugate to a subgroup of the Möebius group. This is not possible unless H_ν is topologically conjugate to a group of Euclidean rotations. Modulo the proof of the smoothness of the action of $\mathrm{Homeo}_+(S^1)$ on $\mathcal{SGC}(\hat{S}^1)$, this finishes the proof of Theorem A.

2.5 The smoothness of the action of $\mathrm{Homeo}_+(S^1)$ on $\mathcal{SGC}(\hat{S}^1)$

At first glance, this fact should seem rather surprising. For instance, it is easy to see that the action of $\mathrm{Homeo}_+(S^1)$ on the space of probability measures of the circle is not smooth. However, we will see that the stability property of the measures involved in our case are at the origin of many rigidity phenomena.

For the proof of the smoothness of the action of $\mathrm{Homeo}_+(S^1)$ on $\mathcal{SGC}(\hat{S}^1)$, let us fix a sequence (h_n) in $\mathrm{Homeo}_+(S^1)$ and an element $\nu \in \mathcal{SGC}(\hat{S}^1)$ such that $\nu_n = h_n(\nu)$ tends to some $\nu_\infty \in \mathcal{SGC}(\hat{S}^1)$. We have to prove that there exists $h \in \mathrm{Homeo}_+(S^1)$ such that $h(\nu) = \nu_\infty$. To do this, it suffices to prove that the sequences (h_n) and (h_n^{-1}) are both equicontinuous. Indeed, in this case, any limit h of a subsequence (h_{n_k}) of (h_n) such that $(h_{n_k}^{-1})$ also converges will be an element of $\mathrm{Homeo}_+(S^1)$ satisfying $h(\nu) = \nu_\infty$.

Suppose by contradiction that (h_n) is not equicontinuous and for each $n \in \mathbb{N}$ fix a preimage $\hat{h}_n \in \mathrm{Homeo}_+(\hat{S}^1)$ of h_n. By passing to a subsequence if necessary, we may assume that there exist $\varepsilon > 0$ and two sequences of points (u_n) and (v_n) in $\hat{S}^1$ such that $2\pi/3 > d(u_n, v_n) \geq \varepsilon$ for all $n \in \mathbb{N}$ and such that $d(\hat{h}_n^{-1}(u_n), \hat{h}_n^{-1}(v_n))$ converges to zero as n goes to infinity. Again, by passing to a subsequence if necessary, we can suppose that $x_n = \hat{h}_n^{-1}(u_n)$ and $y_n = \hat{h}_n^{-1}(v_n)$ both converge to the same limit point $z \in \hat{S}^1$, and that u_n (resp. v_n) converges to some point u_∞ (resp. v_∞) in such a way that $2\pi/3 > d(u_\infty, v_\infty) \geq \varepsilon$. Let us denote by $\bar{x}_n, \bar{\bar{x}}_n, \bar{y}_n, \bar{\bar{y}}_n$, etc, the points obtained from x_n, y_n, etc, by Euclidean rotations of order 3 (see Picture 1).
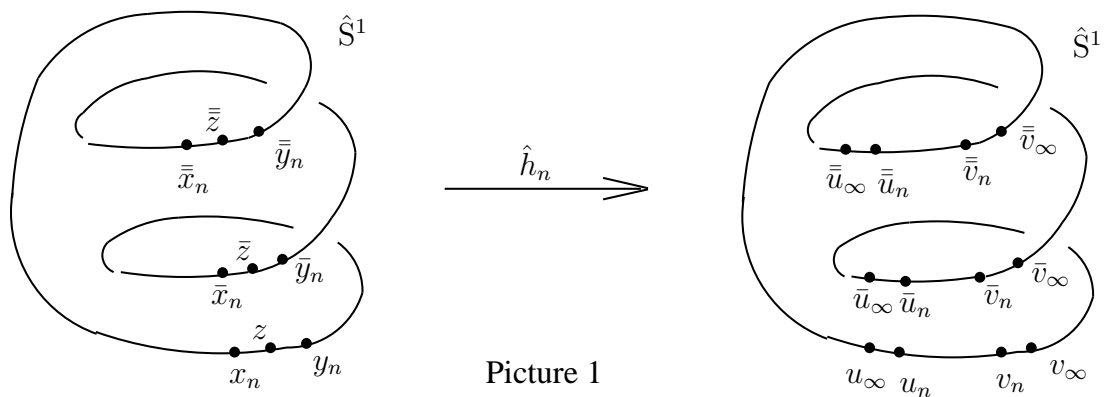
Picture 1

For all $n \in \mathbb{N}$ one has

$$\nu([y_n, \bar{x}_n] \times [\bar{y}_n, x_n]) = \nu_n([v_n, \bar{u}_n] \times [\bar{v}_n, u_n]). \qquad (2)$$

For each $k \in \mathbb{N}$ let us fix 4 points $p_k, q_k, \bar{p}_k, \bar{q}_k$ such that $p_k < z < q_k < \bar{p}_k < \bar{z} < \bar{q}_k$ and such that $d(p_k, z) = d(z, q_k) = d(\bar{p}_k, \bar{z}) = d(\bar{z}, \bar{q}_k) = 1/k$. For fixed k there exists a positive integer $n(k)$ such that for all $n \geq n(k)$ one has $d(x_n, z) < 1/k$ and $d(z, y_n) < 1/k$. So, for $n \geq n(k)$,

$$\nu([y_n, \bar{x}_n] \times [\bar{y}_n, x_n]) \geq \nu([q_k, \bar{p}_k] \times [\bar{q}_k, p_k]).$$

Thus

$$\liminf_{n \to \infty} \nu([y_n, \bar{x}_n] \times [\bar{y}_n, x_n]) \geq \nu([q_k, \bar{p}_k] \times [\bar{q}_k, p_k]).$$

Since this is true for all $k \in \mathbb{N}$, one obtains

$$\liminf_{n \to \infty} \nu([y_n, \bar{x}_n] \times [\bar{y}_n, x_n]) \geq \nu(]z, \bar{z}[\times]\bar{z}, z[) = \infty.$$

However, the right hand term of equality (2) tends to $\nu_\infty([v_\infty, \bar{u}_\infty] \times [\bar{v}_\infty, u_\infty])$, and the value of this expression is finite. (Recall that ν_∞ is a Radon measure on $\hat{S}^1 \times \hat{S}^1 \setminus \Delta$.) This contradiction finishes the proof of the equicontinuity of (h_n). A similar argument allows to show the equicontinuity of (h_n^{-1}), concluding the proof of the smoothness of the action of $\mathrm{Homeo}_+(S^1)$ on $\mathcal{SGC}(\hat{S}^1)$.

3   The higher rank case

The proof of Theorem B is obtained by putting the affine isometric action A of the previous paragraphs into the context of §1.3. The case where this action has a global fixed point was ruled out in §2. If this is not the case but the action is non uniform, then we are in the "degenerate case", and we can apply Delorme's lemma. Finally, in the uniform case, we will use Shalom's Superrigidity Theorem.

3.1   The degenerate case

Associated to the cocycle $\alpha : G \times \Omega \to \mathrm{Diff}_+^{1+\tau}(S^1)$, $\tau > 1/2$, let us consider the affine isometric action A on the Hilbert space $\mathcal{H}$. The arguments in §2 show that if this action has an invariant vector $K \in \mathcal{H}$, then α is cohomologous in $\mathrm{Homeo}_+(S^1)$ to a cocycle taking values in the group of Euclidean rotations.

Let us suppose now that there is no invariant vector for A but this action is not uniform. By Delorme's lemma, there exists a sequence (K_n) of unitary vectors in $\mathcal{H}$ such that for all $g \in G$ one has $\lim_{n \to +\infty} \|\theta(g)K_n - K_n\| = 0$. For each $n \in \mathbb{N}$ let us consider the probability measure $\hat{m}_n$ on $\Omega \times \hat{S}^1$ defined by

$$\frac{d\hat{m}_n}{d(\mu \times Leb)}(x, u) = \int_{\hat{S}^1} \frac{K_n^2(x, (u, v))}{4\sin^2(\frac{u-v}{2})} \, dv.$$

Up to a subsequence, one can suppose that $\hat{m}_n$ tends to some limit $\hat{m}$. This measure $\hat{m}$ induces a probability measure m on $\Omega \times S^1$ by projection $\Omega \times \hat{S}^1 \to \Omega \times S^1$, and a straightforward computation shows that m is invariant by the skew action of G.

3.2 The uniform case

If the affine action A is uniform, then Shalom's Superrigidity Theorem gives the existence of some index $i \in \{1, \ldots, k\}$ such that the space $\mathcal{H}_i$ of $\theta(G_i')$-invariant vectors is non trivial. Due to the commutativity of G_i and G_i', this space $\mathcal{H}_i$ is $\theta(G)$-invariant.

Each non zero function $K \in \mathcal{H}_i$ gives rise to an almost everywhere defined family of finite measures on the fibers over each point $x \in \Omega$. Indeed, denoting by $\pi : \hat{S}^1 \to S^1$ the canonical projection, one can define, for $X \subset S^1$,

$$\mu_{K,x}(X) = \int_{\hat{S}^1} \int_{\pi^{-1}(X)} \frac{K^2(x, (u, v))}{4\sin^2(\frac{u-v}{2})} \, du \, dv.$$

Let us denote by $\mathcal{F}_i$ the collection of all families of finite measures obtained from $\mathcal{H}_i$ by this procedure, and normalized so that

$$\int_\Omega \mu_{K,x}(S^1) \, d\mu(x) = 1.$$

If $\mathcal{F}_i$ contains only one element, then we obtain a normalized equivariant family of a.e. finite measures on S^1, and so an invariant probability measure for the skew action of G on $\Omega \times S^1$. Suppose in what follows that $\mathcal{F}_i$ contains more than one element.

Let us fix an orthonormal basis $\{K_1, K_2, \ldots\}$ of $\mathcal{H}_i$, and let us define

$$\bar{K} = \sum_i \frac{|K_i|}{2^i}, \qquad K = \frac{\bar{K}}{\|\bar{K}\|}.$$

This function $K \in \mathcal{H}_i$ gives an element of $\mathcal{F}_i$ such that the support of $\mu_{K,x}$ is "maximal" on the fiber of a.e. $x \in \Omega$. Denote by S_K^1 the equivariant topological circle obtained by cutting up the connected components of the complementary of the support of measures $\mu_{x,K}$. (Note that equivariance follows from the fact that $\mathcal{H}_i$ is a $\theta(G)$-invariant subspace.) Denote also by μ_K the (G_i'-equivariant) probability measure on the (G-equivariant topological) circle S_K^1 induced by the family $\mu_{x,K}$, with $x \in \Omega$. The procedure of cutting induces a new cocycle

$$\bar{\alpha} : G \times \Omega \to \mathrm{Homeo}_+(S_K^1),$$

which can be considered as a "semiconjugate" of the cocycle α.

By hypothesis, there exists a unitary vector $K' \in \mathcal{H}_i$ inducing a different element of $\mathcal{F}_i$ from that induced by K. Denote by $\mathcal{FM}(S_K^1)$ the space of finite measures on S_K^1, and denote by $\mathcal{LM}(S_K^1)$ the subspace of $\mathcal{FM}(S_K^1)$ whose elements have no atoms and total support (i.e. the measure of each non empty open set is positive). Let $\mathcal{M}^\Delta(S_K^1)$ be the space $\mathcal{LM}(S_K^1) \times \mathcal{FM}(S_K^1) \setminus \Delta$, where Δ denotes the corresponding diagonal. The map $\psi : G_i' \to \mathcal{M}^\Delta(S_K^1)$ given by $\psi(g_i') = (\mu_K, \mu_{K'})$ is G_i'-equivariant, and it is easy to verify that the action of $\mathrm{Homeo}_+(S_K^1)$ on $\mathcal{M}^\Delta(S_K^1)$ is smooth. So, applying the Cocycle Reduction Lemma to the restriction $\bar{\alpha}|_{G_i'}$, we obtain that $\bar{\alpha}$ is cohomologous to some cocycle $\bar{\beta} : G \times \Omega \to \mathrm{Homeo}_+(S_K^1)$ which takes values in the stabilizer of some point (μ, μ') in $\mathcal{LM}(S_K^1) \times \mathcal{FM}(S_K^1) \setminus \Delta$. On

the other hand, the intersection H of the stabilizers of μ and μ' is a finite subgroup of $\mathrm{Homeo}_+(\mathrm{S}^1)$. Indeed, the stabilizer of μ is topologically conjugate to the group of Euclidean rotations, and the only closed and strict subgroups of the group of rotations are the finite ones.

By identifying (equivariantly) the points of the orbits by H, one obtains another topological equivariant circle $\mathrm{S}^1_K/\sim$, and $\bar\beta$ induces a cocycle

$$\beta : G \times \Omega \to \mathrm{Homeo}_+(\mathrm{S}^1_K/\sim)$$

such that the image $\beta(G'_i \times \Omega)$ is trivial. (Thus $\bar\beta$ can be seen as a "finite cover" of the cocycle β.) Since G_i and G'_i commute, for $g \in G_i$, $g' \in G'_i$ and a.e. $x \in \Omega$, we have

$$\beta(g, g'(x)) = \beta(g, g'(x))\beta(g', x) = \beta(gg', x) = \beta(g'g, x) = \beta(g', g(x))\beta(g, x) = \beta(g, x).$$

Since G'_i acts ergodically on Ω, we conclude that β does not depend on the point $x \in \Omega$, and thus it is indeed a Borel (hence continuous) homomorphism from G to $\mathrm{Homeo}_+(\mathrm{S}^1_K/\sim)$ factoring through G_i. This finishes the proof of the first part of Theorem B.

Suppose now that each G_i is non discrete and almost topologically simple (*i.e.* the closed normal subgroups are compact or cocompact). Recall that each locally compact subgroup of $\mathrm{Homeo}_+(\mathrm{S}^1)$ is a (real) Lie group [10]. Moreover, if such a group is connected, then it is topologically conjugate to a subgroup of a direct product of some groups of the following list: $\mathbb{R}$, $\mathrm{Aff}_+(\mathbb{R})$, $\mathrm{SO}(2,\mathbb{R})$, $\mathrm{PSL}_k(2,\mathbb{R})$ for some $k \geq 1$ or $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$. The cocycle β then induces naturally homomorphisms from G_i to some of these groups, and G_i being non discrete and almost topologically simple, these homomorphisms have compact image or are surjective with compact kernel. It is easy to see that the only possibility for which there is no invariant probability measure for the G-action on $\Omega \times \mathrm{S}^1$ is that $\beta(G_i)$ is topologically conjugate to some finite cover $\mathrm{PSL}_k(2,\mathbb{R})$ of $\mathrm{PSL}(2,\mathbb{R})$, and this finishes the proof of Theorem B.

Remark 3.1. When each G_i is a simple Lie group, it has been proven in [17] that the invariant probability measure on $\Omega \times \mathrm{S}^1$ can be taken so that its projection into Ω coincides with the original measure μ. We ignore if this is still true in our setting.

4   Appendix: improving the differentiability class

Recently, some new rigidity phenomena have been discovered for Kazhdan groups. More precisely, the fixed point property is not only valid for isometric actions of such groups on Hilbert spaces, but also on any L^p-space for $p \in [1,2]$ and for "almost" isometric actions: see [3] and [8] respectively. By combining the methods of those works, one can easily prove that for any group G satisfying Kazhdan's property (T), there exists a constant $\delta(G) > 0$ such that G satisfies the fixed point property for isometric actions on L^p-spaces for all $p \in [1, 2 + \delta(G)]$.

By considering an L^p-version of Liouville's cocycle and then using the same arguments of [19] and this work, the preceding result allows to prove a sharper version of theorem A for cocycles into $\mathrm{Diff}^{1+\tau}_+(\mathrm{S}^1)$ for $\tau > \tau(G)$, where $\tau(G) < 1/2$

depends on G (one can take $\tau(G) = 1/\delta(G)$).[2] A similar (and stronger) remark can be done for the higher rank case. Indeed, it has been recently remarked [3] that a weak version of the Superrigidity Theorem for Reduced Cohomology holds for representations in L^p-spaces for any $p \in]1, \infty[$. Using this new result, it is not very difficult to obtain an improved version of Theorem B (and the related results from [18]) for cocycles (resp. for representations) into $\mathrm{Diff}_+^{1+\tau}(S^1)$ for any $\tau > 0$.

Acknowledgments. The author would like to thank A. Valette for his invitation to the University of Neuchâtel (where this work was carried out), and D. Gaboriau for many interesting discussions on the subject.

References

[1] ADAMS, S. & SPATZIER, R. Kazhdan groups, cocycles and trees. *Am. Journal of Maths* **112** (1990), 271-287.

[2] ANANTHARAMAN-DELAROCHE, C. Cohomology of property T groupoids and applications. Preprint (2003).

[3] BADER, U., FURMAN, A., GELANDER, T. & MONOD, N. Property (T) and actions on L_p. Preprint (2004).

[4] CASSON, A. & JUNGREIS, D. Convergence groups and Seifert fibered 3-manifolds. *Invent. Math.* **118** (1994), 441-456.

[5] CHÉRIX, P., COWLING, M., JOLISSAINT, P., JULG, P. & VALETTE, A. *Groups with the Haagerup property. Gromov's a-(T)-menability.* Progress in Mathema- tics **197**, Birkhäuser (2001).

[6] DE LA HARPE, P. & VALETTE, A. *La propriété* (T) *de Kazhdan pour les groupes localement compacts.* Astérisque **175** (1989).

[7] FARLEY, D. A proper isometric action of Thompson group V on Hilbert space. *Internat. Math. Res. Notices* **45** (2003), 2409-2414.

[8] FISHER, D. & MARGULIS, G. Almost isometric actions, property (T), and local rigidity. Preprint (2003).

[9] GABAI, D. Convergence groups are Fuchsian groups. *Annals of Maths* **136** (1992), 447-510.

[10] GHYS, É. Groups acting on the circle. *L'Enseignement Mathématique* **47** (2001), 329-407.

[11] GHYS, É. Actions de réseaux sur le cercle. *Invent. Math.* **137** (1999), 199-231.

---

[2]It is unknown but very plausible that Theorem A (or at least the main result of [19]) still holds for C^1-diffeomorphisms. A very interesting problem is to study the case of piecewise affine homeomorphisms of the circle. For instance, it is not known if some generalized Thompson's groups satisfy Kazhdan's property (T) (see however [7]).

[12] HINKKANEN, A. Abelian and non discrete convergence groups of the circle. *Trans. of the AMS* **318** (1990), 87-121.

[13] HINKKANEN, A. The structure of some quasisymmetric groups. *Mem. of the AMS* **83** (1990), 1-87.

[14] JOLISSAINT, P. Borel cocycles, approximation properties and relative property (T). *Erg. Theory and Dyn. Systems* **20** (2000), 483-499.

[15] MARKOVIC, V. Quasisymmetric groups. Preprint, Univ. of Warwick (2004).

[16] MONOD, N. & SHALOM, Y. Cocycle superrigidity and bounded cohomology for negatively curved spaces. *Journal of Differential Geometry* **67** (2004), 1-61.

[17] MORRIS WITTE, D. & ZIMMER, R. Actions of semisimple Lie groups on circle bundles. *Geom. Dedicata* **87** (2001), 91-121.

[18] NAVAS, A. Quelques nouveaux phénomènes de rang 1 pour les groupes de difféomorphismes du cercle. *Comment. Math. Helvetici* **80** (2005), 355-375.

[19] NAVAS, A. Actions de groupes de Kazhdan sur le cercle. *Ann. Sci. de l'ENS* **35** (2002), 749-758.

[20] REZNIKOV, A. *Analytic topology of groups, actions, strings and varietes.* Preprint (1999).

[21] SHALOM, Y. Rigidity of commensurators and irreducible lattices. *Invent. Math.* **141** (2000), 1-54.

[22] TUKIA, P. Homeomorphic conjugates of Fuchsian groups. *J. für Reine und Angew. Math.* **391** (1989), 1-54.

[23] ZIMMER, R. Kazhdan groups acting on compact manifolds. *Invent. Math.* **75** (1984), 425-236.

[24] ZIMMER, R. *Ergodic theory of semisimple groups.* Monographs in Mathematics, Birkhäuser (1984).

Dpto. de matemáticas, Univ. de Chile,
Las Palmeras 3425, Ñuñoa, Santiago, Chile
email : andnavas@uchile.cl