Siamese objects, and their relation to color graphs, association schemes and Steiner designs

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1 Introduction

Our motivation stems from the paper [9], in which the authors introduce a class of decompositions of complete graphs into certain strongly regular graphs which share a common spread. Analyzing their construction, we soon came to realize that these strongly regular graphs are $GQ(s,t)$-graphs, and thus, due to a famous result of A.E. Brouwer, deletion of the spread leads in each case to an antipodal distance regular graph which is an $(s+1)$-fold cover of the complete graph K_{st+1}. A relational structure resulting from these distance regular graphs, plus a certain stratification of the spread, yields an association scheme.

We call such objects Siamese association schemes. After recalling some preliminary notions in Section 2, we formulate an axiomatic system for such objects, as well as for some more general ones, and we analyze the resulting combinatorial structures.

We emphasize that the goal of this paper is to introduce the reader to the new area of *"Siamese objects"* in a clear and compelling way. Due to space limitations, we have sought to achieve a desirable balance between general discussion, paying close attention to computational detail, and the presentation of a few nice *ad hoc* models, including elements of proof elaborated on objects of small size. A more comprehensive treatment is thus postponed to [11] and [12]. We refer also to [19] as an important source of information. Finally, we acknowledge the indispensable role played in our investigations by the computer packages COCO [6], [7], GAP [20], Grape [21] and nauty [16].

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2 Preliminaries

2.1 Color graphs

We define a *color graph* Γ to be an ordered pair $(V, \mathcal{R})$, where V is a set of *vertices* and $\mathcal{R}$ a set of (non-empty) disjoint binary relations on V such that $\bigcup_{R \in \mathcal{R}} R = V^2$. We refer to the elements of $\mathcal{R}$ as the *colors* of Γ, and to the number $|\mathcal{R}|$ of its colors as the *rank* of Γ.

Given a color graph $\Gamma = (V, \mathcal{R})$, we define its *adjacency matrix* to be the $v \times v$ matrix $A = (a_{ij})$ for which $a_{ij} = t$ if $(x_i, x_j) \in R_t$, $R_t \in \mathcal{R}$.

Observe that a color graph is nothing more than an edge-coloring of a complete graph. Note further that any function ϕ defined on V^2 defines a color graph. In such case, we can alternatively denote it as (V, ϕ).

Let $\Gamma = (V, \mathcal{R})$ and $\Gamma' = (V', \mathcal{R}')$ be color graphs. An *isomorphism* $\psi : \Gamma \to \Gamma'$ is a bijection of V onto V' which induces a bijection $\Psi : \mathcal{R} \leftrightarrow \mathcal{R}'$ of colors. A *weak* (or *color*) *automorphism* is an isomorphism $\psi : \Gamma \to \Gamma$. If, in addition, the induced map Ψ is the identity on $\mathcal{R}$ we call ψ a (*strong*) *automorphism*. We denote by $CAut(\Gamma)$ and $Aut(\Gamma)$ the groups of all weak and strong automorphisms of Γ, respectively. (For brevity, we will sometimes refer to $Aut(\Gamma)$ as the *group of* Γ, and to $CAut(\Gamma)$ as the *color group of* Γ.) Finally, we denote by $N(\Gamma)$ the normalizer of $Aut(\Gamma)$ in the symmetric group $S(V)$.

2.2 Coherent configurations and association schemes

A coherent configuration $W = (X, \{R_i\})$ is our initial notion which we presume to be known (e.g., see [7]). We shall speak of its basis relations R_i, basis graphs $\Gamma_i = (X, R_i)$, and basis matrices $A_i = A(\Gamma_i)$, thus allowing us to switch freely between the languages of relations, graphs and matrices.

Note that a coherent configuration is defined as a particular case of a color graph. Thus, all definitions in the previous section apply as well to coherent configurations.

If a coherent configuration $W = (X, \{R_i\})$ is homogeneous (i.e., all of its basis graphs Γ_i are regular), then we call it an *association scheme*.

When W is the association scheme of all 2-orbits corresponding to a prescribed transitive permutation group (G, Ω), we often denote its adjacency algebra by $V(G, \Omega)$ and call it the *centralizer algebra* of W.

We refer the reader to [1], [3] and [7] as detailed sources of information about association schemes and distance regular graphs (briefly, *drg*'s). We use the notation $\mathrm{srg}(v, k, \lambda, \mu)$ to denote any strongly regular graph (briefly, *srg*) with parameters (v, k, λ, μ), or when referring to the class of all such graphs.

2.3 Incidence structures

An incidence structure (P, B, I) for which $|P| = v$ is called a *Steiner system*, denoted by $S(t, k, v)$, if each block has size k and every t-element subset of P is contained in exactly one block. When additionally $t = 2$ and $k = 3$, we speak of a *Steiner triple system*, and denote it by $STS(v)$.

Recall that a projective space $PG(3,q)$, when considered as an incidence structure of points and lines, provides a classical example of an $S\,(2, q+1, q^3+q^2+q+1)$, see [8] for more details.

Special attention in this paper is devoted to generalized quadrangles (briefly, GQ's), see [18]. Given a GQ with parameters s and t, we denote it by $GQ(s,t)$, though when $s = t$ we shall simply write $GQ(s)$. We refer to the point graph of a $GQ(s,t)$ as a $GQ(s,t)$-*graph*, and to any srg with the same parameters as a $GQ(s,t)$-graph a *pseudo-geometric* $GQ(s,t)$-*graph*. It is well known that a $GQ(s,t)$-graph is an srg$((s+1)(st+1),\ s(t+1),\ s-1,\ t+1)$.

3 Siamese objects: main definitions

3.1 Siamese color graphs

Let $\Gamma = (V, \{Id_V, S, R_1, R_2, \ldots, R_n\})$ be a color graph for which:

(i) (V,S) is an imprimitive srg, that is a disjoint union of cliques of equal size.

(ii) Each (V,R_i) is an imprimitive drg of diameter 3 with antipodal system S.

(iii) Each $(V, R_i \cup S)$ is an srg.

Then we call Γ a *Siamese color graph*. We refer to S as the *spread* of Γ, and to the number of drg's as the *Siamese rank* of Γ.

Given a Siamese color graph Γ we indicate by (v,k,λ,μ,σ) its parameter set, where (v,k,λ,μ) is the (common) parameter set of each srg $(V, R_i \cup S)$ and σ is the valency of the spread S. There are obvious necessary conditions which must be satisfied by such parameters, and we refer to any set (v,k,λ,μ,σ) which satisfies these conditions as *feasible*.

As previously stated, Siamese color graphs were first studied in [9] by Kharaghani and Torabi. The word "Siamese" comes from the observation that any two of the strongly regular graphs share the spread S, so are like conjoined twins. However, after surgical removal of the spread, both "twins" can live an independent life as distance regular graphs.

3.2 Siamese association schemes

Let $W = (V, \{Id_V, S_1, \ldots, S_n, R_1, \ldots, R_k\})$ be an association scheme. We call W *Siamese* if $\Gamma = (V, \{Id_V, \bigcup S_i, R_1, \ldots, R_k\})$ is a Siamese color graph. Thus, we are allowing the spread of the color graph to be a union of basis relations of the scheme.

Consequently, given a Siamese color graph Γ one may ask whether or not it is coming from a Siamese association scheme W. When it does, we say Γ *admits* W.

A Siamese color graph is said to be *geometric* if each srg $(V, R_i \cup S)$ is the point graph of a suitable generalized quadrangle.

3.3 Siamese Steiner designs

We here explore the relationship between geometric Siamese color graphs and Steiner designs.

Proposition 1. *Let Γ be a Siamese color graph with the parameters*

$$(q^3 + q^2 + q + 1,\ q(q + 1),\ q - 1,\ q + 1,\ q + 1).$$

Suppose further that Γ is geometric. For each point graph $(V, R_i \cup S)$, construct a corresponding generalized quadrangle. Let B denote the union of all lines in all resulting GQ's. Then, defining I to be the usual point-line incidence, the incidence structure (V, B, I) is a Steiner design $S\,(2, q + 1, q^3 + q^2 + q + 1)$. ■

Thus, a geometric Siamese color graph provides a Steiner system with a spread; moreover, it provides a partition of the remaining blocks of the system into sets which together with the spread form generalized quadrangles. We call this a *Siamese partition* of the Steiner system. We further call such a partition *coherent* if the color graph admits a Siamese association scheme.

Conversely, it is easy to see that a Siamese partition of a Steiner system provides a geometric Siamese color graph Γ. The automorphism group of the partition coincides with the color group $CAut(\Gamma)$ of Γ.

3.4 Siamese color graphs as simultaneous antipodal covers

The drg's that occur in Siamese color graphs are a special case of antipodal covers of a complete graph. These are characterized by the following Theorem due to A.E. Brouwer [3].

Theorem 1. *Let Γ be a pseudo-geometric $GQ(s, t)$-graph with a spread. Then removing the spread from Γ gives a distance regular graph which is an $(s + 1)$-fold cover of the complete graph K_{st+1}. Conversely, any drg which is an $(s + 1)$-fold cover of the complete graph K_{st+1} may be obtained by removing a spread from a pseudo-geometric $GQ(s, t)$-graph.* ■

This gives us the following interpretation of Siamese color graphs: The distance regular graphs in a Siamese color graph form simultaneous antipodal covers of K_{st+1} which partition the edges of the complete multipartite graph $\overline{S}$, where S is the spread.

3.5 Pattern of investigation

Given a Siamese color graph Γ there are many interesting questions one may ask: Is Γ geometric? If so, which Steiner design is associated to Γ? Does Γ admit a Siamese association scheme? To answer these questions, as well as to investigate the numerous combinatorial objects one may derive from Γ, it is useful to describe the relevant automorphism groups.

There are quite a few groups to consider: (i) $Aut(\Gamma)$; (ii) $CAut(\Gamma)$; (iii) $N(\Gamma)$; (iv) automorphism groups of the drg's; (v) automorphism groups of the srg's.

If Γ is geometric, we have in addition: (vi) automorphism groups of the GQ's; (vii) automorphism group of the Steiner system; (viii) automorphism group of the Siamese partition.

Finally, if $s = t$ or $s = t + 2$, each srg defines a symmetric design which gives us one more group to consider.

In fact, not all of these groups are distinct. The automorphism group of a GQ coincides with that of its point graph; if Γ admits a Siamese partition, then the automorphism group of the partition coincides with $CAut(\Gamma)$; if Γ admits a Schurian Siamese association scheme, then $CAut(\Gamma)$ coincides with $N(\Gamma)$.

4 Siamese objects on 15 points

4.1 Initial framework

The example we are about to give was our first attempt to develop a general constructive procedure, and as such it played a central role for us. Originally, it was managed with the aid of COCO, although *a posteriori* practically all computational results can be reproduced by hand.

Consider the action of the alternating group $G = A_5$ acting on the coset space G/E, where $E \cong E_4$ is a fixed Sylow 2-subgroup of G. Then we have

Proposition 2. (a) *The rank* r *of* $(G, G/E)$ *is equal to 6.*

(b) *The association scheme* $W = (G/E,\ 2\text{-}orb(G, G/E))$ *has five classes with respective valencies* $1, 1, 4, 4, 4$.

(c) *One such class, say* R_3*, corresponds to a classical antipodal drg, namely the line graph of the Petersen graph* (*see p.2 of [3] for a portrait of this graph*).

(d) *The action of* A_5 *is 2-closed, i.e.,* $Aut(W) = A_5$.

Proof: Applying the classical orbit-counting lemma to the group E (e.g., see [10]) we obtain $r = \frac{1}{4}(15+3\cdot 3) = 6$, proving (a). Identifying the given action $(G, G/E)$ with that of A_5 on the 15 edges of the Petersen graph, we get an evident description of 2-orbits of $(G, G/E)$ from which both (b) and (c) readily follow. Lastly, one proves (d) by using arguments similar to those found in Section 2.5.2 of [7]. ∎

4.2 A Siamese association scheme on 15 points

In what follows, denote $\Omega = G/E$, so that $W = (\Omega,\ 2\text{-orb}(G, \Omega))$.

Proposition 3. (a) *The color group* $CAut(W)$ *acts transitively on the non-reflexive relations* R_1, R_2 *of valency 1, and on the relations* R_3, R_4, R_5 *of valency 4.*

(b) *The graph* $(\Omega, R_1 \cup R_2)$ *is an imprimitive srg of valency 2, i.e., a spread with 5 connected components.*

(c) *The automorphism group of graph* (Ω, R_3) (*i.e., line graph of the Petersen graph*) *is isomorphic to* S_5.

(d) *The graph* $(\Omega, R_1 \cup R_2 \cup R_3)$ *is a strongly regular graph of valency 6; it is isomorphic to the complement* $\overline{T(6)}$ *of the triangular graph* $T(6) = L(K_6)$. *Its automorphism group is* S_6. ∎

Under the action of $CAut(W)$, we obtain two more strongly regular graphs each isomorphic to the graph in part (d) of Proposition 3. Thus we get

Corollary 4. W *is a Siamese association scheme.* ∎

We may now investigate the Siamese triple system which corresponds to W. It turns out that the automorphism group of this system is a large overgroup of $Aut(W)$, a fact which is quite advantageous for our purposes.

4.3 A Siamese partition of $PG(3,2)$

Proposition 5. *$W = (\Omega,\ 2\text{-}orb(G,\Omega))$ gives rise to a Siamese $STS(15)$.*

Proof: It is well known that $\overline{T(6)}$ is the point graph of the unique $GQ(2)$. Now apply Propositions 3 and 1 to complete the proof. ∎

Proposition 6. *The $STS(15)$ of Proposition 5 is isomorphic to $PG(3,2)$. Moreover, the embedded GQ's of order 2 correspond bijectively to the 2-subsets of an 8-element set.*

Proof: We begin by imitating the proof of a classical exceptional isomorphism between $PSL(4,2)$ and A_8, see [22]. Clearly, $PG(3,2)$ is a classical $STS(15)$ which we denote by $\mathcal{S}$. We now derive a combinatorial model for $\mathcal{S}$ as follows.

Consider an affine design $S(3,4,8)$ with base set $\Omega_0 = \{1,\dots,8\}$ and its resulting orbit Ω_1 of length 15 under the action induced from (A_8,Ω_0). Now take Ω_1 to be the point set of $\mathcal{S}$, and define the block set B to consist of all partitions of Ω_0 having shape $4+4$. We define $D \in \Omega_1$ to be incident to $\pi \in B$ if the cells of π are blocks in D.

We now construct a subset of blocks which forms a generalized quadrangle. Recall that $S(3,4,8)$ is also a 2-design with $\lambda = 3$. Let us fix a certain 2-element subset of Ω_0, say $\{1,2\}$. Then there are exactly three blocks in $S(3,4,8)$ containing $\{1,2\}$. It is clear that in a concrete copy $S(3,4,8)$, the remainders of these three blocks form a partition of the set $\Omega_0 \setminus \{1,2\}$ of the form $2+2+2$. Moreover, taking into account that A_8 acts 6-transitively on Ω_0, we easily get that each such partition arises exactly once from a suitable design in the orbit Ω_1 defined above. This establishes a bijection between Ω_1 and the set P of partitions of $\Omega_0 \setminus \{1,2\}$ of shape $2+2+2$.

On the other hand, let us consider the set B_0 of all partitions from B which do not separate set $\{1,2\}$. We have exactly 15 such partitions which bijectively correspond to the 2-element subsets of $\Omega_0 \setminus \{1,2\}$. Thus we are coming to a famous model of $W(2)$ for $GQ(2)$ (cf. [3], Example (ii), page 30). Indeed, we have a natural incidence structure (B_0,P) in which a 2-subset from B_0 is incident to a partition from P if it is one of the cells of this partition.

An extra important message is that $W(2)$ is self-dual, thus we have now complete freedom to embed the dual of $W(2)$ into our combinatorial model of $PG(3,2)$. This embedding is uniquely determined by the selected subset $\{1,2\}$. ∎

Corollary 7. (a) *There are exactly 28 embedded GQ's in $PG(3,2)$.*

(b) *The generalized quadrangles defined by the pairs $\{x,y\}$, $\{x,z\}$, $\{y,z\}$ intersect pairwise in a fixed spread which depends only on the subset $\{x,y,z\} \subset \Omega_0$.*

(c) *This construction provides a Siamese partition of the classical $STS(15)$.*

(d) *The automorphism group of the Siamese partition in part (c) is isomorphic to $(S_5 \times S_3)^+$, where G^+ denotes the subgroup of all even permutations in G.*

(e) *There are exactly 56 different Siamese partitions of the classical $STS(15)$.* ∎

Coming back from $\mathcal{S}$ to W, we again obtain all desired information about W, only this time in a much more natural and clear manner.

Corollary 8. (a) $Aut(W) \cong A_5$.
(b) $CAut(W) \cong (S_5 \times S_3)^+$. ∎

4.4 Another Siamese partition of an $STS(15)$

Using GAP and COCO we construct a Siamese color graph Γ with $Aut(\Gamma) \cong A_4$, which does not admit a Siamese association scheme despite the fact that Γ is geometric. Namely, we consider the $STS(15)$ designated as #7 in the enumeration of Mathon-Rosa-Phelps [15]. This Steiner system is invariant under $(S_4 \times S_4)^+$ in its natural action on the two orbits $O_1 = \{1, 2, 3, 4\}$ and $O_2 = \{5, 6, 7, 8\}$ (each copy of S_4 acts on one orbit). There are two orbits of points of lengths 3 and 12 and three orbits of blocks of lengths 18, 15 and 1. Descriptions of points, blocks and incidence may be easily realized in terms of the initial action of $(S_4 \times S_4)^+$ on $O_1 \cup O_2$ (see [11], [19], [12]). In these terms we also get a clear, computer-free description of the three embedded GQ's which form the desired Siamese partition of $STS\#7$.

4.5 All Siamese color graphs on 15 vertices are obtained

Proposition 9. *Every Siamese color graph on 15 vertices is necessarily geometric. Thus, there are exactly two non-isomorphic Siamese color graphs on 15 vertices, one of which admits a Siamese association scheme.* ∎

The proof available to us relies strongly on a computer inspection of all 80 $STS(15)$'s, see [15]. We also use the fact that $\overline{T(6)}$ is the only srg$(15, 6, 1, 3)$, and it is the point graph of the unique $GQ(2)$.

5 A classical Siamese association scheme on 40 points

5.1 Initial framework

We here describe a Siamese association scheme corresponding to the classical generalized quadrangle $W(3)$ of order 3, such that its associated Siamese Steiner design is the classical $PG(3, 3)$. (We shall refer to the resulting Siamese scheme as "classical" for exactly this reason, despite the fact that it has never been before presented.)

We begin with the alternating group $G = A_6$ in its natural action on six letters, and consider the action of G on the coset space $\Omega = G/E$, where E is the Sylow 3-subgroup of G given by $E = \langle(1, 2, 3), (4, 5, 6)\rangle$. Clearly then, $|\Omega| = 360/9 = 40$. Set $W = V(G, \Omega)$. By way of computer we obtain the following initial information.

Proposition 10. (a) *W has rank 8, with valencies* $1^4, 9^4$.
(b) $Aut(W) \cong S_2 \times A_6$.
(c) *$CAut(W)$ is a non-split extension of $S_2 \times A_6$ by the dihedral group D_4 of order 8. It acts transitively on the set of directed relations of valency 1, as well as on the*

set of relations of valency 9.

(d) *Up to isomorphism, W has exactly 10 proper fusion schemes.* ■

One fusion scheme of W defines a spread of the form $10 \circ K_4$, another an antipodal drg Γ of valency 9 and diameter 3, and still another an srg (in fact the point graph of $W(3)$). The metric scheme defined by Γ is non-Schurian.

5.2 New model of the classical generalized quadrangle $W(3)$.

The drg Γ implicitly mentioned in the previous section is antipodal distance regular, with intersection array $\{1, 2, 9; 9, 6, 1\}$. Thus, if we add a spread to it we get a pseudo-geometric $GQ(3)$-graph. In fact, this graph is geometric.

To prove this, is suffices to find 40 4-cliques in this graph. However, we can also establish this directly. Indeed, let $\Omega_0 = \{1, 2, 3, 4, 5, 6\}$, and let P be the set of directed cycles of length 3 in Ω_0. Let B_1 be the set of all partitions of Ω_0 into two triples, and B_2 the set of directed arcs from Ω_0. Set $B = B_1 \cup B_2$. Define incidence as follows. Let a cycle be incident to a partition if its set of vertices is one of the partition cells; let it be incident to an arc if it contains this arc. We are now prepared to prove the following proposition.

Proposition 11. (a) *The incidence structure (P, B) is a GQ of order 3 which is isomorphic to $W(3)$.*

(b) *B_1 corresponds to a spread of $W(3)$ (associate a partition of two triples with the set of four directed cycles defined on these triples).*

(c) *Deletion of B_1 from the point graph of $W(3)$ results in the drg Γ.*

(d) *$Aut(\Gamma) \cong S_2 \times S_6$.*

Proof: We provide an outline of basic facts which must be confirmed to elaborate a proof. Most of these are straightforward to verify.

(1) $|P| = 40$, $|B_1| = 10$, and $|B_2| = 30$, hence $|B| = 40$.

(2) Each directed cycle is contained in one partition, and contains 3 arcs. Hence, each point is incident to 4 blocks.

(3) Each partition contains two parts; for each of them, there are two possible orientations. Thus each block in B_1 is incident to 4 points.

(4) Given an arc (x, y), there are 4 points remaining to complete the arc to a cycle of length 3. Hence, each block in B_2 is incident to 4 points.

(5) Two distinct partitions cannot contain the same 3-set. In particular, two blocks in B_1 cannot be incident with the same point. Thus, B_1 forms a spread of the incidence structure.

(6) Two arcs are incident with the same triangle if the endpoint of one is the starting point of the other. In this case, the triangle is uniquely determined; thus, two points in B_2 intersect in at most one point.

(7) An arc and a partition are incident with the same triangle if the arc is contained in one of the classes of the partition. Also here the triangle is unique; thus two blocks, one from B_1 and one from B_2, share at most one point.

(8) Altogether we have that two blocks intersect in at most one point; thus, (P, B)

is a partial linear space of order $(3,3)$.

(9) (P,B) is a generalized quadrangle.

As fact (9) is quite more subtle than the previous facts, we provide a detailed proof. Let p be a point, and b be a block not containing p. If $b \in B_1$, then since $p \notin b$, not all three points of the triangle lie in the same class of b. Thus there is exactly one arc in p lying in one class; this is the unique block incident with p and intersecting b. If, on the other hand, $b \in B_2$, then the arc b is not contained in the triangle p. If b shares a point with p, then there is exactly one arc in p such that its endpoint is the starting point of b, or vice versa. If it doesn't share a point with p, then it intersect the unique block from B_1 containing p.

(10) $Aut(\Gamma) = S_6 \times S_2$, where Γ is the drg obtained by deleting a spread from the point graph of the resulting $GQ(3)$.

As with fact (9), we provide all essential details needed to verify (10). By construction, (P,B) is invariant under S_6, as well as under reversal of each cycle. Clearly such reversal is not induced from an element of S_6, but commutes with each such element. As it is involutory, we have that (P,B) is invariant under $S_6 \times S_2$. By Theorem 1, deleting a spread from the obtained $GQ(3)$ results in a drg Γ for which we clearly have $Aut(\Gamma) \geq S_6 \times S_2$. It is well known (e.g., see [18]) that $W(3)$ is the unique $GQ(3)$ containing a spread, and that $Aut(W(3)) \cong PSU(4,3).\mathbb{Z}_2$. As this group contains a unique conjugacy class of maximal subgroups of order 1440 (see Section 3.5 of [7]), we conclude that $Aut(\Gamma) = S_6 \times S_2$. ∎

Corollary 12. *W is a Siamese association scheme.* ∎

Remark. Our model of $W(3)$ is presented in much the same spirit as the one in [17]. We believe, however, that ours has a certain advantage, providing in a more evident form the "essence" of the group $Aut(\Gamma)$.

6 Computational approach to Siamese objects

6.1 More Siamese objects on 40 points

We wish to describe Steiner designs $S(2,4,40)$ which admit a Siamese partition; one of these is the classical design $PG(3,3)$. In fact, at present 475 such designs have been found with the aid of a computer.

Recall that there are exactly two geometric srg$(40,12,2,4)$ up to isomorphism; only one of these has a spread, namely the point graph of $W(3)$. Moreover, this spread is unique up to isomorphism. Therefore, in a geometric Siamese color graph on 40 points there is only one option for the drg and the srg. Note that all Siamese color graphs found by us are geometric. It turns out that exactly three of the corresponding Siamese designs have a point-transitive automorphism group, while only the classical one has a block-transitive automorphism group.

6.2 On 85 points

Several approaches were used to search for Siamese color graphs related to $W(4)$ but only one example was found; it is an association scheme, and will be discussed in the next section in a more general context. The existence of other examples remains a challenging open problem.

6.3 Toward an infinite series of Siamese association schemes

Recall that we used A_5 and A_6 as starting groups to manufacture classical Siamese schemes on 15 and 40 points, respectively. A quite realistic prospect for an infinite series of such schemes now emerges from two classical exceptional isomorphisms, namely $A_5 \cong PSL(2,4)$ and $A_6 \cong PSL(2,9)$.

Consider the group $G = PSL(2,q^2)$. It has a subgroup H of the form $E_{q^2}.S$, where S is the set of all non-zero squares in $\mathbb{F}_q$. (In fact, H is a subgroup of the affine group $AGL(1,q^2)$.) The index of H in G is $(q^2+1)(q+1)$.

We consider the action of G on the coset space G/H. The corresponding centralizer algebras have been investigated for $q = 2,3,4,5,7$; in all cases we get a Siamese association scheme. Theoretical consequences of this computational experiment are discussed in the Section 7.

6.4 More about methodology

The Steiner designs and their Siamese partitions were obtained using one of three computational approaches. First, we considered the stabilizer of a point in the natural action of A_5 and A_6 on $PG(1,4)$ and $PG(1,9)$, respectively, and constructed all Siamese color graphs invariant under this action. A second approach was to start from a Steiner system and check to see if it admitted a Siamese partition. In order to do this, a great number of $S(2,4,40)$ had to be generated using the Kramer-Mesner method. Using the first and second approaches in tandem, we were able to construct all Siamese objects on 15 points (here "all" equaled just two), and nine such objects on 40 points each having a rather rich automorphism group.

A third approach had been arranged by us in order to enumerate all Siamese objects on 40 points. It is based on the use of double cosets and is still being developed. Early experimental results using this approach seem to be very encouraging.

7 Classical Siamese objects

7.1 Initial imprimitive group

We start with the group $G = PSL(2,q^2)$, and consider its action on the coset space $\Omega = G/H$ where H is as described in Section 6.3. Our main object of interest is the association scheme corresponding to the centralizer algebra $W = V(G,\Omega)$.

7.2 Complete projective model

Consider the projective line $PG(1, q^2)$ of order q^2. We label its points by $\mathbb{F}_{q^2} \cup \{\infty\}$ and note that G acts naturally on this point set.

We define a (possibly directed) graph on the affine points of $PG(1, q^2)$, i.e., the elements of $K = \mathbb{F}_{q^2}$. Let $(x, y) \in K^2$ be an arc if $y - x$ is a non-zero square in $F = \mathbb{F}_q$. Denoting this graph by γ, we set $V = \gamma^G$ and $W' = V(G, V)$.

7.3 Incomplete projective model

Let $F = \mathbb{F}_q$, $K = \mathbb{F}_{q^2}$, and denote by V the set of points of $PG(3, q)$. We can express such points in the form $\langle (a, b) \rangle$ where $a, b \in K$, a, b not both zero.

Let $C = F^* \leq K^*$. Let β be a primitive element of F, and for $i = 1, \ldots, q + 1$, let $C_i = \beta^i C$ be the cosets of C in K^*. Set $C_0 = \{0\}$.

Define the function $\phi : V \times V \to \{0, 1, \ldots, q + 1\}$ by setting

$$\phi(\langle (a, b) \rangle, \langle (c, d) \rangle) = i \iff \begin{vmatrix} a & b \\ c & d \end{vmatrix} \in C_i$$

This yields a color graph $\Gamma = (V, \phi)$ on $q^3 + q^2 + q + 1$ vertices of rank $q + 2$.

7.4 Main results

Below we state all results from which one may conclude the existence of an infinite series of Siamese objects. We mention that proofs of these results depend on an amalgamation of ideas, facts and techniques from algebraic combinatorics, finite geometry and group theory (especially, a familiarity with the classical groups). In particular, we refer the reader to [2], [4], [5], [8], [14], [18], [22].

Proposition 13. (a) *The association scheme W and the complete projective model W' are isomorphic.*

(b) *The incomplete projective model Γ is a merging of classes of the complete projective model.* ■

Proposition 14. Γ *is a Siamese color graph which admits W as a Siamese association scheme.* ■

Corollary 15. *Consider the inclusion $PSL(2, q^2) \leq P\Gamma L(4, q)$, implied by the inclusion $PSp(4, q) \leq P\Gamma L(4, q)$. Then the orbits of $PSL(2, q^2)$ on the set of lines of $PG(3, q)$ (inherited from the action of $P\Gamma L(4, q)$ on this set) give a Siamese partition of $PG(3, q)$.* ■

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