New geometric presentations for $\mathrm{Aut}\, G_2(3)$ and $G_2(3)$

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1 Introduction

The purpose of this article is to provide new presentations for the groups $G_2(3)$ and $\mathrm{Aut}\, G_2(3)$. These presentations come from the amalgam of maximal parabolic subgroups arising in the action of $\mathrm{Aut}\, G_2(3)$ on a certain geometry.

The members of this amalgam are the well-known subgroups of $\hat{G} = \mathrm{Aut}\, G_2(3)$ (cf. [ATL]): $\hat{L} = 2^3 \cdot L_3(2) : 2$, $\hat{N} = 2^{1+4}.(S_3 \times S_3)$, and $M = G_2(2)$. Notice that M is fully contained in $G = O^2(\hat{G}) \cong G_2(3)$, while $\hat{L}$ and $\hat{N}$ are not. This explains our hat notation. According to this notation we set $L = \hat{L} \cap G \cong 2^3 \cdot L_3(2)$ and $N = \hat{N} \cap G \cong 2^{1+4}.(3 \times 3).2$.

We choose the subgroups $\hat{L}$ and M so that $D = \hat{L} \cap M$ is a maximal parabolic subgroup in M. Then D has a unique normal subgroup 2^2 (contained in $O_2(L) \cong 2^3$). Let z be an involution from that normal subgroup. We choose $\hat{N} = C_{\hat{G}}(z)$. This uniquely specifies the amalgam $\hat{\mathcal{A}} = \hat{L} \cup \hat{N} \cup M$. Let $e \in O_2(\hat{L}) \setminus O_2(L)$ and set $K = M^e$. Let $\mathcal{B} = L \cup N \cup M \cup K$. Clearly, $\hat{G} = \langle \hat{\mathcal{A}} \rangle$ and $G = \langle \mathcal{B} \rangle$.

Theorem 1. $\hat{G} = \mathrm{Aut}\, G_2(3)$ *is the universal completion of the amalgam* $\hat{\mathcal{A}}$.

As a corollary of this theorem we get our second main result.

Theorem 2. $G = G_2(3)$ *is the universal completion of the amalgam* $\mathcal{B}$.

As we have already mentioned, the amalgam $\hat{\mathcal{A}}$ is the amalgam of maximal parabolics with respect to the action of $\hat{G}$ on a certain geometry $\hat{\Gamma}$. In this sense, Theorem 1 is equivalent, via Tits' Lemma [T] (also cf. [IS], Theorem 1.4.5), to the simple connectedness of the geometry $\hat{\Gamma}$.

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Our interest in presentations for the groups $\mathrm{Aut}\, G_2(3)$ and $G_2(3)$ comes from the need to find, in a computer-free way, a presentation for a larger group, the sporadic Thompson group Th. In a forthcoming paper we will establish that Th acts on a similar simply connected geometry Λ. In fact, $\mathrm{Aut}\, G_2(3)$ arises in Th in the normalizer of a suitable subgroup X of order three. Furthermore, our geometry $\hat{\Gamma}$ is just the fixed subgeometry for the action of X in Λ.

The structure of the paper is as follows. In Sections 2 and 3 we specify the amalgam $\hat{\mathcal{A}}$ abstractly in terms of certain conditions (G1)–(G4), and we prove the uniqueness of such an amalgam. In Section 4 we switch from an arbitrary completion $\hat{G}$ of $\hat{\mathcal{A}}$ to its universal completion and define the geometry $\hat{\Gamma}$ as the coset geometry. Finally, in Section 5 we determine the exact number of points in $\hat{\Gamma}$, which gives us that the order of the universal completion $\hat{G}$ coincides with the order of $\mathrm{Aut}\, G_2(3)$. In view of the uniqueness of $\hat{\mathcal{A}}$, the group $\mathrm{Aut}\, G_2(3)$ is a factor group of $\hat{G}$, and hence Theorem 1 follows.

Our notation for groups follows that of [ATL]. The definitions, terminology and basic facts concerning geometries can be found, for example, in [IS]. Our result in Theorem 2 can be approached differently in terms of intransitive geometries (see [GVM]). We thank R. Gramlich for pointing this out.

2 Set-up and basic properties

We let $\hat{G}$ be a group generated by its subgroups $\hat{L}$ and M so that the following hold:

(G1) $\hat{L}$ has an index two subgroup L such that $L \cong 2^3.L_3(2)$; M has structure $U_3(3).2$.

(G2) $D = \hat{L} \cap M$ is contained in L, it is the full preimage in L of a maximal parabolic from $L/O_2(L)$.

We first list some basic properties that follow from (G1) and (G2). Let $E = O_2(L)$.

Lemma 2.1. *The following hold:*

(1) *The action of L on E is nontrivial and L is a nonsplit extension of E by* $L_3(2)$*; also,* $O_2(\hat{L}) \cong 2^4$.

(2) *D has a normal subgroup of order* 4 *in E and has a trivial center.*

(3) $M \cong G_2(2)$.

Proof: By (G2) we have that 2^6 divides $|D|$, which means that D contains a maximal parabolic subgroup (for characteristic two) from $O^2(M) \cong U_3(3) \cong G_2(2)'$. Hence, comparing with [ATL], we see that D cannot have 2^3 in the center. This means that the action of L on E is nontrivial, hence E is the natural module for $\bar{L} = L/E \cong L_3(2)$. Since the outer automorphism of $L_3(2)$ interchanges the natural and the dual natural modules, $O_2(\hat{L})$ has order 2^4, and it can only be elementary abelian.

Now looking at L, we see that D has two 2-dimensional chief factors. Comparing with the information on $U_3(3)$ from [ATL], we see that $D \cap O^2(M)$ must have structure $4^2 : S_3$, and thus it has a trivial center. Turning again to L, we see that D is the normalizer of a subgroup 2^2 from E. This yields (2). Moreover, since D has trivial center, so does M, and (3) follows, too.

Finally if L were the split extension of E, then D would have a normal elementary abelian subgroup of order 16. Since this is not the case, L does not split. This completes the proof of (1). ■

We remark that there is a unique nonsplit extension $2^3 \cdot L_3(2)$, and so part (1) of the above lemma specifies L up to isomorphism.

Let B be a Sylow 2-subgroup in D. Let L_1 be the full preimage in L of the second maximal parabolic from $\bar{L} = L/E \cong L_3(2)$ containing $\bar{B}$, and also let M_1 be the second maximal parabolic from $M \cong G_2(2)$ containing B. Let $\hat{L}_1$ be the normalizer of L_1 in $\hat{L}$. We have $[\hat{L}_1 : L_1] = 2$.

Now we can introduce a new condition on $\hat{G}$. Clearly, $\hat{L}_1 \cap M_1 = L_1 \cap M_1 = B$. Let $\hat{N} = \langle \hat{L}_1, M_1 \rangle$.

(G3) $\hat{L}_1$ and M_1 are permutable, that is, $\hat{N} = \hat{L}_1 M_1$.

Let $N = \langle L_1, M_1 \rangle$.

Lemma 2.2. *We have* $[\hat{N} : N] = 2$ *and* $N = L_1 M_1$*, so* L_1 *and* M_1 *are permutable. Furthermore,* $Q = O_2(N)$ *coincides with* $O_2(L_1)$ *and* $O_2(M_1)$*, and it is an extraspecial subgroup* 2_+^{1+4}*. Finally,* $\bar{N} = N/Q \cong 3^2 : 2$ *where the involution inverts all elements of order three.*

Proof: Both $O_2(L_1)$ and $O_2(M_1)$ have order 2^5 and the factor group is S_3 in both cases. Consider $\bar{N}$. By (G3) we know that this is a group of order $2^s 3^2$ which has no normal 2 subgroup. This forces that the Sylow 3-subgroup is normal. In particular, this means that all 2-dimensional chief factors of L_1 and M_1 are contained in Q. Looking at L_1 as a subgroup of L we see that $O_2(L_1)$ is contained in Q. This forces the equalities $O_2(L_1) = B \cap Q = O_2(M_1)$. In particular, $Q_0 = B \cap Q$ is normal in N. Since $O_2(N/Q_0)$ has size at most two, N/Q_0 has a normal Sylow 3-subgroup, which means that $Q_0 = Q$. It also means L_1 and M_1 are permutable and that $\bar{N}$ is as claimed in the lemma.

It remains to show that Q is extraspecial of plus type. First of all, the center $Z(Q)$ is of order two. Also, the element of order 3 from L_1 acts fixed-point-freely on $Q/Z(Q)$. This shows that $Q/Z(Q)$ is either elementary abelian or homocyclic 4^2. The latter structure is impossible because an element of order 3 from M_1 has only one 2-dimensional factor in $Q/Z(Q)$. Thus, $Q/Z(Q)$ is elementary abelian, so Q is extraspecial. Moreover, since Q contains $E \cong 2^3$, Q must be of plus type. ■

3 Uniqueness of the amalgam

In this section we prove that the amalgam $\mathcal{A} = L \cup M \cup N$ is unique up to isomorphism and also that $\hat{\mathcal{A}} = \hat{L} \cup M \cup \hat{N}$ is unique provided that the following extra condition holds.

(G4) $\hat{L}$ and $\hat{N}$ have no common direct factor of order two.

We remark that we do not impose condition (G4) until the end of this section. We also remark that $\hat{L}$ and $\hat{N}$ with a common direct factor are possible, but they lead to a not so interesting configuration, where that factor is in the center of $\hat{G}$ (which can in that case be infinite).

Lemma 3.1. *The outer automorphism group of D is of order two. Every automorphism of D is induced by an automorphism of L.*

Proof: Let $R = O_2(D)$ and $Z = Z(R)$. Then $Z \cong 2^2$ is contained in E. Notice that $D/R \cong S_3$ acts nontrivially on Z. According to [ATL], D has a normal subgroup $T \cong 4^2$, which clearly contains Z. We claim that Z, E and T are characteristic in D. Indeed, it is clear for Z. Considering D/Z, we see that E/Z and T/Z are the only normal subgroups of D/Z of orders 2 and 2^2, respectively. So E and T are also characteristic.

Let $A = \operatorname{Aut} D$. Since the center of D is trivial, we can identify D with $\operatorname{Inn} D$. Since Z is characteristic, the images of A and D in $\operatorname{Aut} E$ coincide with the same maximal parabolic. This shows that $A = DC_A(E)$. Furthermore, since $E \leq C_A(E)$ and E is transitive on the Sylow 3-subgroups of D, we get that $A = D(C_A(E) \cap N_A(S))$, where S is an arbitrary Sylow 3-subgroup of D. Let $F = C_A(E) \cap N_A(S)$. Clearly, an element centralizing E cannot invert S, so $F = C_A(ES)$.

We claim that $|F| \leq 4$. Observe first that no nontrivial element of A centralizes RS. Indeed, RS is characteristic in D and $C_D(RS) = 1$. Hence, all elements of D induce different automorphisms of RS. Thus, $C_A(RS) = 1$. This means, since $RS = TES$, that F acts faithfully on T. Let $T = \langle t_1 \rangle \times \langle t_2 \rangle$. Since F centralizes $Z = \Omega_1(T)$, we have that t_1 and t_1^f differ by an involution (from Z) for each $f \in F$. Furthermore, the action of f on t_1 fully identifies the action of f on the entire T, since f commutes with S, and the latter acts on T fixed-point-freely. Consequently, $|F| \leq 4$.

Clearly, $F \cap D = F \cap E$ has order two. Since $A = DF$, this implies that $|\operatorname{Out} D| = [A : D] \leq 2$. Thus, to complete the proof of the lemma it suffices to find an outer automorphism of D in $\operatorname{Aut} L$.

It is well-known that $\operatorname{Out} L$ is of order two. Namely, $\operatorname{Aut} L$ is an extension of an indecomposable module 2^4 by $L_3(2)$. Clearly, D has index two in $O_2(\operatorname{Aut} L)D$ (we identify L with $\operatorname{Inn} L$). So either D has an outer automorphism in $\operatorname{Aut} L$, or D centralizes a subgroup of order two from $O_2(\operatorname{Aut} L)$. However, the latter possibility cannot hold because the module on 2^4 is indecomposable and D induces on it a full Sylow 2-subgroup of $L_3(2)$. ■

Corollary 3.2. *The amalgam $L \cup M$ is unique up to isomorphism.*

Proof: Follows from Lemma 3.1 and Goldschmidt's Lemma (see (2.7) in [G], or Proposition 8.3.2 in [IS]). ■

Notice that the subgroups L_1 and M_1 are uniquely determined within our unique amalgam $L \cup M$, once B is chosen. So the uniqueness of $\mathcal{A}$ follows from our next lemma.

Lemma 3.3. *The free product with intersection $L_1 *_B M_1$ has a unique factor group such that* (1) *L_1 and M_1 map isomorphically into this factor group, and* (2) *the factor group is the product of the images of L_1 and M_1.*

Proof: Let $F = L_1 *_B M_1$. Suppose F has two such factor groups and let U and V denote the corresponding kernels. Recall that $Q = O_2(L_1) = O_2(M_1)$ is extraspecial. We identify L_1 and M_1 with the corresponding subgroups in F. Under this identification, Q is normal in F and it trivially intersects both U and V. Hence $U, V \leq C_F(Q)$. Since F/U is the product of the images of L_1 and M_1 and since L_1 and M_1 act differently on $Q/Z(Q)$, we have that the centralizer in F/U of the image of Q is $Z(Q)$, which is of order two. Hence $[V : U \cap V] = 2$.

Let $\bar{F} = F/Q(U \cap V)$. This group has structure $2.3^2.2$ and it is generated by $\bar{L}_1$ and $\bar{M}_1$. The latter two groups are both isomorphic to S_3 and they share $\bar{B} \cong 2$. This is a contradiction, since $\bar{F}$ clearly has a normal Sylow 3-subgroup. ■

Corollary 3.4. *The amalgam* $\mathcal{A}$ *is unique up to isomorphism.* ■

We now turn to the amalgam $\hat{\mathcal{A}}$. If G is any group generated by a copy of our unique amalgam $\mathcal{A}$ then consider, as $\hat{G}$, the direct product of G with a group of order two. Define $\hat{L}$ and $\hat{N}$ to be the extensions of L and N by the direct factor 2. It is clear that the resulting amalgam $\hat{\mathcal{A}}$ satisfies (G1)-(G3), but not (G4).

So from this point on we assume that (G4) holds.

Lemma 3.5. *The amalgam* $\hat{\mathcal{A}}$ *is unique up to isomorphism.*

Proof: Suppose a is an automorphism of N centralizing L_1. Let $x \in N \setminus L_1$ be an element of order three. Clearly, $N = \langle L_1, x\rangle$. Notice that x and $x' = x^a$ act the same way on Q (since $Q \leq L_1$). We have already seen that $C_N(Q) = Z(Q)$, so x and x' differ by an element from $Z(Q)$, which means that $x = x'$. This shows that a must in fact be trivial.

Next we identify the structure of $\hat{L}$. If $C_{\hat{L}}(L) \neq 1$ then $\hat{L} = L \times Z$, where Z has order two. Clearly, $Z \leq \hat{L}_1$, so $Z \leq \hat{N}$ and $\hat{N} = NZ$. Furthermore, the involution from Z acts trivially on L_1 and by the above it acts trivially on N. Therefore, Z is a common direct factor in both $\hat{L}$ and $\hat{N}$, contradicting (G4). Thus, $C_{\hat{L}}(L) = 1$, which means that $\hat{L} \cong \mathrm{Aut}\, L$. In particular, the amalgam $\hat{L} \cup M \cup N$ is unique up to isomorphism.

Pick now any element $x \in \hat{L}_1 \setminus L_1$. Since $\hat{N} = N\langle x\rangle$, it remains to see that the action of x uniquely extends from L_1 to N. If there were two such actions on N then they would differ by an automorphism of N that is trivial on L_1. However, as we proved above, such an automorphism is trivial on the entire N, and so the action of x on N is unique. ■

We also record two useful facts. First, we showed in this proof that $\hat{L} \cong \mathrm{Aut}\, L$. Secondly, the element x in the last paragraph of the proof can be chosen to normalize both L_1 and D (and hence also B). If this x normalizes M_1 then it normalizes $M = \langle D, M_1\rangle$. However, $M \cong G_2(2)$ has no outer automorphisms. This yields that x acts on M as some element $y \in D \cap M_1 = B$. However, this means that $xy^{-1} \in \hat{L} \setminus L$ acts trivially on B, which is not possible since $\hat{L} \cong \mathrm{Aut}\, L$. The contradiction shows that x does not normalize M_1. This forces that $\hat{N}/Q \cong S_3 \times S_3$ and $Q = O_2(\hat{N})$.

Notice that almost all proofs in this section are purely amalgamic. The only exception is in the preceding paragraph, where the fact that $\hat{\mathcal{A}}$ is embedded in a group was used to conclude that x acts on M. (Luckily, our unique amalgam $\hat{\mathcal{A}}$ can be found in $\mathrm{Aut}\, G_2(3)$.)

4 The geometries Γ and $\hat{\Gamma}$

The first order of business in this section is to choose a more suitable group $\hat{G}$. Let $\hat{U}$ be the universal completion of the amalgam $\hat{\mathcal{A}}$. Since $\hat{G}$ is generated by $\hat{\mathcal{A}}$, we have that $\hat{G}$ is isomorphic to a factor group of $\hat{U}$. In particular, $\hat{\mathcal{A}}$ embeds into $\hat{U}$ isomorphically. Furthermore, if we prove Theorem 1 for the group $\hat{U}$ in place of $\hat{G}$ then, clearly, $\hat{U}$ has no proper nontrivial factor groups and so $\hat{G} = \hat{U}$, yielding the claim for $\hat{G}$. Thus, without loss of generality, we can assume from now on that *$\hat{G} = \hat{U}$ is the universal completion of $\hat{\mathcal{A}}$.*

Fix an arbitrary involution e in $\hat{L}_1 \setminus L_1$ that normalizes D. Such an involution can be chosen, for example, in $\hat{E} = O_2(\hat{L}) \cong 2^4$ (this is how it was chosen in the introduction). Set $K = M^e$ and $K_1 = M_1^e$. Notice that e normalizes L and N. We remarked at the end of the preceding section that $M_1 \neq M_1^e = K_1$ and so $M \neq M^e = K$. Clearly, e interchanges M and K. Thus, it induces an automorphism of the amalgam $\mathcal{B} = L \cup N \cup M \cup K$. Let $G = \langle \mathcal{B} \rangle$. Notice that since $K = M^e = \langle D^e, M_1^e \rangle = \langle D, K_1 \rangle$ and since $K_1 \leq N$, we have that $K \leq \langle \mathcal{A} \rangle$ and so $G = \langle \mathcal{A} \rangle$.

Lemma 4.1. *We have $[\hat{G} : G] = 2$; namely, $\hat{G} = G\langle e \rangle$. Furthermore, G is the universal completion of $\mathcal{B}$.*

Proof: Clearly, e normalizes G and $\langle G, e \rangle$ contains the entire $\hat{\mathcal{A}}$, which means that $\hat{G} = G\langle e \rangle$. Thus, $[\hat{G} : G] \leq 2$. To show that the index is exactly two, consider the universal completion U of $\mathcal{B}$. Since e induces an automorphism of $\mathcal{B}$, it also induces an automorphism on U. Set $\hat{U}$ to be the semidirect product of U and $\langle e \rangle$, defined with respect to this automorphism. Clearly, U contains a copy of $\mathcal{B}$ left invariant by e. Extending the images of L and N by e we also find a copy of $\hat{\mathcal{A}}$ that generates $\hat{U}$. Thus, $\hat{U}$ is a homomorphic image of $\hat{G}$, since $\hat{G}$ is the universal completion of $\hat{\mathcal{A}}$. It is clear, that this homomorphism isomorphically maps G onto U. Thus, $\hat{G} \neq G$. Notice that this also establishes that G is the universal completion of $\mathcal{B}$. ∎

We are ready to define the geometry Γ. We define it in the group-theoretic manner. The elements of Γ are the right cosets in G of the subgroups L, N, M, and K. Thus, Γ contains elements of four types. We call the cosets of L *points* and the cosets of N *lines*. For reasons that will become apparent later the cosets of M and K are called *M- and K-hexagons*, respectively. Two cosets are *incident* elements of Γ if they have a nonempty intersection.

Clearly, G acts on Γ by right multiplication. Also, e acts on Γ by conjugation. It is easy to see that these actions agree, so that the entire group $\hat{G}$ acts on Γ. The elements from $\hat{G} \setminus G$ interchange the two types of hexagons in Γ, so with respect to the action of $\hat{G}$ the rank four geometry Γ should be viewed as a rank three geometry $\hat{\Gamma}$, where M- and K-hexagons are united into one type—*hexagons*. Notice that Γ and $\hat{\Gamma}$ have the same set of elements. However, in order to satisfy the axioms of diagram geometry, we need to modify the incidence relation. Namely, we postulate that *two hexagons are never incident in* $\hat{\Gamma}$. The incidence between points and lines, and between hexagons and other elements is the same as in Γ.

We now proceed to establish the basic properties of Γ and $\hat{\Gamma}$.

Lemma 4.2. $\hat{\Gamma}$ *is a residually connected geometry, on which* $\hat{G}$ *acts flag-transitively.*

Proof: Suppose F is a maximal flag in $\hat{\Gamma}$. Clearly, F consists of a point, a line, and a hexagon. Acting, if necessary, by e, we can assure that the hexagon in F is an M-hexagon. Thus, $F = \{Lg_1, Ng_2, Mg_3\}$. Without loss of generality, $g_2 = 1$. Then $Ng_2 = N$; furthermore, $N \cap Lg_1$ and $N \cap Mg_3$ are cosets in N of L_1 and M_1, respectively. Since $N = L_1M_1$, the triple intersection $Lg_1 \cap N \cap Mg_3$ is nonempty, hence it contains an element g. Acting on F by g^{-1} we obtain the standard flag $\{L, N, M\}$. This proves flag-transitivity.

Observe that the stabilizers in $\hat{G}$ of the cosets L, N, and M are $\hat{L}$, $\hat{N}$, and M, respectively. So $\hat{\mathcal{A}}$ is simply the amalgam of maximal parabolics in $\hat{G}$. Connectedness of $\hat{\Gamma}$ now follows from the fact that $\hat{G}$ is generated by $\hat{\mathcal{A}}$. Similarly, the full residual connectedness follows from the fact that each of the three maximal parabolics is generated by its intersections with the other two maximal parabolics. ∎

Since $\hat{N} = \hat{L}_1M_1$, the geometry $\hat{\Gamma}$ has a string diagram. Furthermore, since $D = \hat{L} \cap M$ and $M_1 = \hat{N} \cap M$ are the two maximal parabolic subgroups in $M \cong G_2(2)$, containing the Borel subgroup B, we see that the residue of the coset M (and hence also of every coset Mg or Kg) is a natural generalized hexagon geometry of $G_2(2)$. This explains our name for these elements.

Finally, we record some numeric data. Every point is incident to exactly $[\hat{L} : \hat{L}_1] = 7$ lines and $[\hat{L} : D] = 14$ hexagons. It is easy to see that the 14 hexagons are evenly split between M- and K-hexagons—seven of each. Every line is incident to $[\hat{N} : \hat{L}_1] = 3$ points and $[\hat{N} : M_1] = 6$ hexagons (again three of each kind). Every hexagon is incident to $[M : D] = 63$ points and $[M : M_1] = 63$ lines.

5 The collinearity graph Δ

Let Δ be the collinearity graph of Γ, which is clearly the same as the collinearity graph of $\hat{\Gamma}$. We prove Theorem 1 as follows: In view of the uniqueness of the amalgam $\hat{\mathcal{A}}$, it coincides with the amalgam found in $\mathrm{Aut}\, G_2(3)$. This means that $\mathrm{Aut}\, G_2(3)$ is a factor group of the universal completion $\hat{G}$. In particular, the number of points in Γ, that is, $|\Delta|$, is at least $|\mathrm{Aut}\, G_2(3)|/|\hat{L}| = 3159$ (cf. [ATL]). If we show that $|\Delta| \leq 3159$ then $|\Delta| = 3159$ and $|\hat{G}| = |\mathrm{Aut}\, G_2(3)|$. Thus, we need to bound the size of Δ.

We first establish a few facts about Δ and about the action of the point stabilizer on it. It is convenient to identify every line and every hexagon with a subgraph of Δ on the points incident to that line or hexagon. Thus, every line becomes a 3-clique in Δ and every hexagon becomes a subgraph on 63 points in Δ, isomorphic to the collinearity graph of the $G_2(2)$ generalized hexagon.

Let point p be chosen as the coset L. Then, clearly, $G_p = L$ and $\hat{G}_p = \hat{L}$. Also, it is clear that $\hat{L}$ induces the group $L_3(2)$ on the seven lines through p.

Lemma 5.1. Γ *is a partial linear space, that is, two points from* Γ *lie on at most one common line.*

Proof: Suppose two lines share a point. Without loss of generality, that point is p. Since L induces the 2-transitive group $L_3(2)$ on the seven lines through p, any two of these lines lie in a common hexagon. The latter is a partial linear space, so the two lines share only one point. ■

In particular, this lemma shows that different lines produce different 3-cliques. Also, the following holds.

Corollary 5.2. *Every point is collinear with exactly* 14 *other points.* ■

The point stabilizer in $G_2(2)$, acting on the 63 points of the generalized hexagon, has orbits of lengths 1, 6, 24, and 32 (the orbits consist of the points at distance 0, 1, 2, and 3 from the original point, respectively). Since the valency of Δ is only 14, it shows that hexagons are induced subgraphs of Δ. This, in turn, implies that the neighborhood of p, $\Delta(p)$, has no further edges, other than the edges in the seven lines through p. It follows that the lines are maximal cliques in Δ and that every 3-clique in Δ is a line.

Since $\hat{L}$ induces $L_3(2)$ on the set of seven lines through p, this set carries the structure π_p of a Fano plane (projective plane of order two). Similarly for any point x we have a Fano plane π_x whose points are the seven lines through x. To distinguish the points and lines of π_x from those of Γ, we will call the former *Fano points* and *Fano lines*. Thus Fano points are lines on a given vertex x and Fano lines are some triples of lines on x. Clearly, π_x is invariant under the action of $\hat{G}_x$.

Lemma 5.3. *If* H *is a hexagon containing a point* x*, then the three lines in* H *on* x *are the Fano points of a Fano line in* π_x*. Conversely, for any Fano line there is exactly one M-hexagon and exactly one K-hexagon containing those three lines.*

Proof: We can assume x to be p and H to be the hexagon corresponding to the coset M. Notice that $D = \hat{L} \cap M$ stabilizes a Fano line in π_p. Furthermore, D has orbits of lengths 3 and 4 on the Fano points. Since H is invariant under D and it only contains three lines on p, we have the first claim of the lemma. Since L is transitive on the seven Fano lines and it does not interchange M- and K-hexagons, the second claim also follows. ■

Suppose ℓ_1, ℓ_2, and ℓ_3 are lines on x, that are the Fano points of a Fano line from π_x. Then we will call $\ell_1 \cup \ell_2 \cup \ell_3$ a *claw based at* x.

Lemma 5.4. *If* H_1 *and* H_2 *are hexagons then every connected component of* $H_1 \cap H_2$ *is either a line or a claw.*

Proof: Suppose x is a common point of H_1 and H_2. Then H_1 and H_2 correspond to either the same Fano line in π_x (in which case they are hexagons of different type) or they correspond to two different Fano lines. In the first case, H_1 and H_2 share a claw, in the second case they share just a line in the neighborhood of x. Suppose the connected component of $H_1 \cap H_2$ is not a line and not a claw. Then it contains two claws based at neighbors y and z. Without loss of generality, $y = p$, the line ℓ through y and z corresponds to the coset N, and H_1 and H_2 correspond to the cosets M and K, respectively. Let g be an element of M (which is the stabilizer of H_1) that interchanges y and z. Then g also interchanges the claw based at y and the claw based at z. Since H_2 is the only K-hexagon that contains either of those claws, we obtain that g leaves H_2 invariant, that is, $g \in K$. However, since D is a maximal subgroup in both M and K and since $g \notin D$, this means that $M = \langle D, g\rangle = K$, a contradiction. ∎

In a hexagon the distance between a claw (neighborhood of a point) and a line never exceeds one. This means that if two hexagons share a claw then they have no further intersection.

Lemma 5.5. *If H and H' are hexagons of different type then either they are disjoint or $H \cap H'$ is a claw.*

Proof: Suppose H and H' are hexagons of different type and suppose they share a point x. Without loss of generality, H is an M-hexagon. Let $x_0 = x$ and let x_1, x_2, …, x_6 be the six points collinear with x in H. Let H_i be the K-hexagon that shares with H the claw based at x_i. By the preceding lemma, the hexagons H_i are pairwise disjoint; furthermore, they all contain x. Since x is contained in exactly seven K-hexagons, we conclude that $H' = H_i$ for some i. Thus, H and H' share a claw. As we discussed before this lemma, the claw is the entire intersection of H and H'. ∎

If H and H' are of the same type then it is still possible that $H \cap H'$ is disconnected. However, each of the connected components is just a line.

We now consider the action of the point stabilizer on the neighborhood of the point. Namely, we study the actions of L and $\hat{L}$ on $\Delta(p)$. Recall that $E = O_2(L) \cong 2^3$ and $\hat{E} = O_2(\hat{L}) \cong 2^4$.

Lemma 5.6. *The group E acts trivially on $\Delta(p)$, while L acts on it transitively. Moreover, $\hat{E}$ acts trivially on π_p but nontrivially on each line through p.*

Proof: Clearly, $\hat{E}$ acts trivially on π_p. Hence it stabilizes every line through p. Let ℓ be the line corresponding to the coset N. Then the stabilizer of ℓ in $\hat{G}$ is $\hat{N}$. Recall that $Q = O_2(N)$ coincides with $O_2(\hat{N})$ and that $\hat{N}/Q \cong S_3 \times S_3$. (See the discussion after Lemma 3.5.) Let $e \in \hat{E} \setminus E$ and consider the subgroup L_1. If t is an element of order three from L_1 then t acts trivially on the line ℓ. Notice that e and t commute modulo Q. Since $e \notin Q$, we obtain that e does not centralize any other 3-element from $\hat{N}/Q$. This implies that e acts nontrivially on ℓ. Since $\hat{E}$ is normal in $\hat{L}$, the same is also true for every line through p.

It remains to see that L is transitive on $\Delta(p)$. Notice first that L induces $L_3(2)$ on the seven lines through p, hence it acts transitively on them. Moreover, L_1, the joint stabilizer of p and ℓ, cannot act trivially on the other two points of ℓ because in that case N would act trivially on ℓ. ∎

This lemma uniquely determines the action of L and $\hat{L}$ on $\Delta(p)$. In L, the stabilizer of a point $q \in \ell$ (where ℓ is again the line stabilized by $\hat{N}$) is the unique index two subgroup in L_1.

In the remainder of this section we prove the following proposition.

Proposition 5.7. $|\Delta| = 3159$.

We prove it in a sequence of lemmas. Let H be the M-hexagon corresponding to the coset M. Our approach is to decompose Δ with respect to H. Let Δ_i be the set of vertices at distance i from H. Then, clearly, $\Delta_0 = H$ and $|\Delta_0| = 63$. Also, notice that the stabilizer of Δ_0 in $\hat{G}$ is M.

By contradiction, we assume until the end of the proof of Proposition 5.7 *that* $|\Delta| > 3159$.

Lemma 5.8. *If $x \in \Delta_0$ then x has exactly six neighbors in Δ_0, while the remaining eight are in Δ_1. The group M_x acts transitively on those eight points. In particular, M is transitive on Δ_1.*

Proof: Without loss of generality we can assume that $x = p$ and so $M_x = D$. Since H is an induced subgraph, p has exactly six neighbors in Δ_0 and so the remaining eight must be in Δ_1. Moreover, D acts transitively on the 4 lines not lying in H. Let ℓ one of those four lines. Since E acts trivially on $\Delta(p)$, consider the action of $\bar{L} = L/E$. The stabilizer in $\bar{L}$ of a point $q \in \ell$ is a subgroup A_4 which intersects $\bar{D}$ in just a group of order three. This shows that $\bar{D}_x$ has index eight in $\bar{D}$. Hence $\bar{D}$ is transitive on the eight points. ■

It follows from this lemma that $|\Delta_1| \leq 63 \cdot 8 = 504$. Let x be a point in Δ_1 adjacent to p. As above, the joint stabilizer X in M of p and x is the extension of E by a group of order three.

Lemma 5.9. *The group X has orbits of lengths* 1, 1, 12 *on $\Delta(x)$. In particular, x has one neighbor in Δ_0, one in Δ_1 and twelve in Δ_2. Moreover, M is transitive on Δ_2.*

Proof: Let $L' = G_x$ and let $\bar{L}' = L'/E'$ where $E' = O_2(L')$. Since E is not normal in N, E cannot be equal to E'. Hence $\bar{X} \cong A_4$. This subgroup can be identified as the index two subgroup in the stabilizer in $\bar{L}'$ of the line through x and p. It coincides with the full stabilizer in $\bar{L}'$ of p.

Let q be a point in $\Delta(x)$ that does not lie on the line through p and x. We claim that the joint stabilizer U' in L' of p and q is E'. It suffices to show that U' fixes all Fano points in π_x. For a point $y \in \Delta(x)$ let $\tilde{y}$ denote the Fano point containing y. Suppose that U' moves a Fano point $\tilde{r}$ for some $r \in \Delta(x)$. Then U' moves the Fano line through $\tilde{p}$ and $\tilde{r}$, or it moves the Fano line through $\tilde{q}$ and $\tilde{r}$. Suppose the former holds. Since U' fixes the Fano line through $\tilde{p}$ and $\tilde{q}$, it must induce a transposition on the three Fano lines through $\tilde{p}$. However, this contradicts to the fact that U' is contained in the stabilizer of p and hence it can only induce a group of even permutations on the three Fano lines through $\tilde{p}$. This is a contradiction. Similarly, U' cannot move a Fano line through $\tilde{q}$. Thus, we have shown that $U' = E'$. This implies that $X_q \leq E'$, hence $[X : X_q] = 12$. This proves the claim about the orbits. Clearly, the orbit with twelve points cannot be contained in $\Delta_0 \cup \Delta_1$, since in that case $\Delta_2 = \emptyset$ and Δ is too small. Now all the claims follow. ■

Notice that since p is the only point in Δ_0 adjacent to x, we have that $X = M_x$. Furthermore, since every point in Δ_0 is adjacent to eight points in Δ_1 and every point in Δ_1 is adjacent to just one point in Δ_0, we compute that $|\Delta_1| = 63 \cdot 8 = 504$.

Next we look at the hexagons that intersect Δ_0.

Lemma 5.10. *Suppose $H' \neq H$ is a hexagon intersecting $H = \Delta_0$ nontrivially.*

(1) *If H' is a K-hexagon that $H \cap H'$ is a claw based at some point $h \in H$. Furthermore, the points in H' that are at distance $i \geq 1$ from h are contained in Δ_{i-1}.*

(2) *If H' is an M-hexagon then $H \cap H'$ is a line ℓ. Furthermore, the points in H' that are at distance $i \geq 0$ from ℓ are contained in Δ_i.*

Proof: If H' is a K-hexagon then $H \cap H'$ is a claw by Lemma 5.5. Clearly, the points in H' that are at distance two from h, the base of the claw, are in Δ_1. Since, for every point in Δ_1, six lines on it go to Δ_2, we have that the points of H', that are at distance three from h, are contained in Δ_2. It remains to notice that three is the diameter of H'.

Similarly, suppose H' is an M-hexagon. By Lemma 5.4, every connected component of $H \cap H'$ is a line. Let ℓ be one of them. Then the points in H' that are at distance one from ℓ are in Δ_1 and the points that are at distance two are in Δ_2. It remains to notice that every point from H' is at distance at most two from ℓ. ■

Corollary 5.11. *If H_1 and H_2 are of the same type then either they are disjoint or $H_1 \cap H_2$ is a line.* ■

Let $y \in \Delta_2$. Without loss of generality we can assume that y is adjacent to x. Let $Y = M_y$.

Lemma 5.12. *The following hold.*

(1) *Three lines on y have a point in Δ_1 and two other points in Δ_2; these lines are the Fano points of a Fano line from π_y.*

(2) *Three further lines on y are fully contained in Δ_2.*

(3) *The seventh line on y has two points in Δ_3.*

(4) *$Y \cong S_3$ has orbits 3, 3, 6, and 2 on $\Delta(y)$; in particular, M is transitive on Δ_3.*

Proof: First of all, y lies in a K-hexagon H' containing x and p. So (1) follows from Lemma 5.10 (1). Let $x = x_1$, x_2, and x_3 be the three neighbors of y in $\Delta_1 \cap H'$. Furthermore, let $p = p_1$, p_2, and p_3 be the unique neighbors of x_1, x_2, and x_3 in Δ_0. Assuming that y is adjacent to a fourth point $z \in \Delta_1$, we obtain that y is contained in a second K-hexagon H'' meeting Δ_0. However, H' and H'' must share a line on y, that is, H' and H'' share y, some x_i, and hence also p_i. This means that H' and H'' share a claw, which is a contradiction, since H' and H'' are of the same type. Thus, y has exactly three neighbors in Δ_1. It follows that $|\Delta_2| = \frac{504 \cdot 12}{3} = 2016$. Since

$63 + 504 + 2016 < 3159$, we conclude that Δ_3 is nonempty and hence y is adjacent to some points in Δ_3.

Let H_i be the M-hexagon containing y, x_i, and p_i. These three hexagons are pairwise distinct. In view of Lemma 5.10 (2) each of H_i contains two lines on y that are fully in Δ_2. Since there should still be at least one line reaching into Δ_3, we obtain that there are exactly three lines ℓ_1, ℓ_2, and ℓ_3 on y, that are fully contained in Δ_2. Notice that every ℓ_j is contained in two hexagons H_i.

It remains to study M_y and its action on $\Delta(y)$. Let H' be as above and let the claw $H \cap H'$ be based at a point h. Let K' be the stabilizer of H'. Since $K' \cong G_2(2)$ and since h and y are at distance three in H', we have that $K'_{hy} \cong S_3$. We claim that $M_y = K'_{hy}$. Clearly, M_y stabilizes H', and hence it also stabilizes $H \cap H'$ and h. Thus, $M_y \leq K'_{hy}$. On the other hand, K'_{hy} stabilizes the claw $H \cap H'$ and hence it also stabilizes the only M-hexagon containing this claw, M. Thus, $K'_{hy} \leq M_y$. We have established that $M_y \cong S_3$.

All subgroups S_3 in $L_3(2)$ are conjugate. Each of them stabilizes a unique point and a unique line in the corresponding Fano plane. For M_y acting on π_y, those are the Fano point, that is the line on y reaching into Δ_3, and the Fano line corresponding to H'. It is easy to see that M_y has two orbits of size three on the six neighbors of y in H', and that the remaining orbits have lengths 6 and 2. The latter orbit consists of the two points on the line reaching into Δ_3. Now all claims of the lemma follow.■

We record that, as we have shown, $|\Delta_2| = 2016$.

Before we study Δ_3 we need to get information about the M-hexagons containing points from Δ_2. Let H' be such a hexagon and let y' be a point in $H' \cap \Delta_2$. Recall that the stabilizer $M_{y'}$ is isomorphic to S_3. This stabilizer has three orbits on the Fano lines in $\pi_{y'}$, hence on the M-hexagons containing y'. The first orbit consists of one Fano line, which has, as its three Fano points, the three lines reaching into Δ_1. The second orbit consists of three Fano lines, each of which has one Fano point, that is a line going into Δ_1, and two other Fano points, that are fully in Δ_2. The third orbit consists of three Fano lines, each of which has as one Fano point a line going to Δ_1, as second Fano point a line contained in Δ_2, and as the last Fano point the only line going into Δ_3. Notice that if H' corresponds to a Fano line in the second orbit then H' meets Δ_0 in a line (see Lemma 5.10 (2)).

Suppose H' is not of that kind. Then $H' \cap \Delta_2 = A_1 \cup A_3$, where A_i consists of the points, for which H' corresponds to a Fano line in orbit i. Potentially, either of A_i can be empty. However, we can choose H' so that $A_1 \neq \emptyset$. Let $B = H' \cap \Delta_1$ and $C = H' \cap (\cup_{i \geq 3} \Delta_i)$. Clearly, $B \neq \emptyset$.

Lemma 5.13. *The following hold.*

(1) *$M_{H'}$ acts transitively on A_1, and if $a \in A_1$ then $M_{a,H'} = M_a \cong S_3$.*

(2) *$M_{H'}$ acts transitively on B, and if $b \in B$ then $M_{b,H'} \cong \mathbb{Z}_6$.*

Proof: Take $a_1, a_2 \in A_1$ and let $g \in M$ be such that $a_1^g = a_2$. Clearly g will map the Fano line in π_{a_1} corresponding to H' to the similar Fano line in π_{a_2}. Thus it will stabilize H'. Moreover, if we repeat this argument for $a_1 = a_2 = a$ we get that $M_{a,H'} = M_a \cong S_3$, proving (1).

To see (2), we notice that if $b \in B$ then the Fano line in π_b corresponding to H', consists of three Fano points that are lines going to Δ_2. We have seen that M_x induces on $\Delta(x)$ the full stabilizer of p (isomorphic to A_4). This stabilizer acts transitively on the four Fano lines in π_x of the above type. Thus, if $g \in M$ and it maps b_1 to b_2, where $b_1, b_2 \in B$, then g can be corrected by an element from M_{b_2}, so that the resulting new element normalizes H', still taking b_1 to b_2. This shows that $M_{H'}$ is transitive on B. Clearly, $M_{b,H'}$ induces just $\mathbb{Z}_3$ on $\Delta(b)$, so (2) follows. ∎

Lemma 5.14. *We have that* $A_3 \neq \emptyset$. *Furthermore,* $M_{H'}$ *acts transitively on* A_3, *and if* $a \in A_3$ *then* $M_{a,H'} \cong \mathbb{Z}_2$.

Proof: Let $b \in B$ and let ℓ be a line on b that is contained in H'. Then $\ell \cap \Delta_2 = \{a_1, a_2\}$. We claim that these two points belong to different subsets A_i. Suppose not. Then by Lemma 5.13, there is an element $g \in M_{H'}$ that maps a_1 to a_2. This g can be chosen so that it fixes b. Indeed, if $i = 3$ then this is automatic, since b is the only neighbor of each of a_1 and a_2 in B. If $i = 1$ then $M_{a_2,H'}$ acts transitively on the three neighbors of a_2 in B, so g can be adjusted to fix b. However, now we have a contradiction. Since g fixes b, it preserves ℓ and hence it switches a_1 and a_2. This means that g induces on $\Delta(b)$ an element of even order. This contradicts to the fact that $M_{b,H'}$ induces on $\Delta(b)$ a group of order three.

Thus, a_1 and a_2 belong to different A_i. Say, $a_1 \in A_3$, making the latter nonempty. Since $M_{a_1} \cong S_3$ and since H' lies in the orbit of three M-hexagons, we obtain that $M_{a,H'} \cong \mathbb{Z}_2$. ∎

Lemma 5.15. *Let* $C = C_1 \cup C_2$ *where the points in* C_1 *have neighbors in* A_3 *and the points of* C_2 *have only neighbors in* C *(within* H'*). Then* $M_{H'}$ *acts transitively on* C_1. *In particular, every point from* C_1 *has the same number* k *of neighbors in* A_3.

Proof: We know that $M_{H'}$ acts transitively on A_3 and so it acts transitively on the lines of H' that go to Δ_3. Also, if $a \in A_3$, the stabilizer $M_{a,H'}$ acts nontrivially on the line on a that goes to Δ_3. This completes the proof. ∎

We are now ready to carry out some computations about the sizes of A_i, B, and C. Let $|A_1| = n$. Since every point from A_1 has three neighbors in B and every point from B has three neighbors in A_1, we have that $|B| = n$, too. Since every point from A_3 has one neighbor in B and since every point from B has three neighbors in A_3, we get that $|A_3| = 3n$. Similarly, $|C_1| = \frac{6n}{k}$ where k is as in Lemma 5.15. Thus, $|C_2| = 63 - 5n - \frac{6n}{k} \geq 0$. This immediately implies, since $k \leq 3$, that $3 \geq \frac{6n}{63-5n}$, and so $n \leq 9$. Observe now that $A_1 \cup B$ induces a graph of valency three with girth at least six. This implies that $n = |A_1| \geq 7$.

Therefore $7 \leq n \leq 9$. If $n = 8$ then $63 - 40 - \frac{48}{k} \geq 0$, which means that $k \geq \frac{48}{23} > 2$. It follows that $k = 3$. However, this means that every neighbor in H' of a point from C_1 is either in A_3 or in C_1, and so $C_2 = \emptyset$, which is impossible, since there are too few points in $A_1 \cup A_3 \cup B \cup C_1$.

If $n = 7$ then $|M_{H'}| = 7 \cdot 6$ and so $M_{H'}$ is solvable. By Hall Theorem, there is only one conjugacy class of subgroups of order six, contradicting Lemma 5.13. It now follows that $n = 9$, $k = 3$ and $C = C_1$.

We summarize this as follows.

Lemma 5.16. *Suppose H' is an M-hexagon that contains a point in Δ_2, but does not intersect Δ_0. Then H' contains exactly 9 points from Δ_1, $9+27$ points from Δ_2, and 14 points from Δ_3. Furthermore, if $z \in H' \cap \Delta_3$ then each of the three lines through z in H' contains one point from Δ_2 and two points from Δ_3 (z is one of the two).* ∎

Finally, we can bound the size of the graph Δ.

Lemma 5.17. *If $u \in \Delta_3$ then each of the seven lines through u in Δ has one point in Δ_2 and the two other points (including u itself) in Δ_3. In particular, $\Delta_4 = \emptyset$.*

Proof: Clearly, there is at least one line on u that contains a point from Δ_2. Consider an M-hexagon H' that contains that line. Clearly, H' contains a point from Δ_2 and it does not intersect Δ_0, since it also contains a point from Δ_3 (namely, u). By Lemma 5.16, each of the three lines through u in H' contains a point from Δ_2. However, every M-hexagon on u contains at least one of those three lines. Repeating the above argument, we see that each line on u contains a point from Δ_2. ∎

According to this lemma, $|\Delta_3| = \frac{2|\Delta_2|}{7} = 576$ and $|\Delta| = |\Delta_0|+|\Delta_1|+|\Delta_2|+|\Delta_3| = 63+504+2016+576 = 3159$. This concludes the proof of Proposition 5.7.

It remains to notice that $|G| = |M| \cdot 3159 = |G_2(3)|$ and so $|\hat{G}| = |\mathrm{Aut}\, G_2(3)|$. Since the unique amalgam $\hat{\mathcal{A}}$ occurs in $\mathrm{Aut}\, G_2(3)$ and generates it, we have that $\mathrm{Aut}\, G_2(3)$ is a factor group of $\hat{G}$. The equality of the orders now establishes Theorem 1.

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