Inclusions of irreducible spherical buildings of equal rank ≥ 3

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Abstract

We classify all inclusions of irreducible spherical buildings of equal rank $\geq$ 3, with the only restriction that the characteristic of the defining field is different from 2 if there are residues which are Moufang quadrangles.

1 Introduction

Spherical buildings are certain incidence geometric objects which are a powerful tool to visualize isotropic absolutely simple algebraic groups (over arbitrary fields). There is a vast literature on these objects, and there are a number of good introductory texts available on the subject [1, 4, 7].

All irreducible spherical buildings of rank at least 3 have been classified by J. Tits in his famous '74 Lecture Notes [5]. In fact, all irreducible spherical buildings of rank at least 3 satisfy the so-called Moufang condition, and so do all of their residues. In particular, all rank 2 residues of any irreducible spherical building of rank at least 3 are Moufang polygons (a generalized polygon is the same as a rank 2 spherical building). Only rather recently, the Moufang polygons have been classified [6], and a new description of all irreducible spherical buildings of rank at least 3, in terms of their rank 2 residues, has been obtained (see Theorem 3.5 below). Depending on the problem, this "local" approach may certainly have many advantages over the classical "global" approach. The study of inclusions of spherical buildings is a good example where the local methods turn out to be very effective, in particular since the classification of inclusions of Moufang polygons has already been achieved in many cases [2].

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Our goal is to classify all inclusions of irreducible spherical buildings of equal rank greater than or equal to 3. We would like to clarify at this point that by an "inclusion", we mean a type-preserving inclusion of incidence geometries. Therefore, two buildings of equal rank, one of which is a subbuilding of the other, also have equal Tits diagram (i.e., they have isomorphic Weyl groups).

It is already clear from [2] that the study of inclusions of Moufang quadrangles defined over a field of characteristic 2 is essentially more difficult, and therefore, we restrict to characteristic $\neq 2$ in the case that there is a rank 2 residue which is a Moufang quadrangle — but this is the only restriction we make.

We will start by looking at the rank 2 case, that is, the case of the Moufang polygons. We will recall the facts from [2] which we will need in the treatment of the higher rank case, and we will show one additional result about a certain dual inclusion which occurs in the rank ≥ 3 case. We can then use the results for rank 2 to do a case by case study of the inclusions of spherical buildings of higher rank.

2 The rank 2 case

We start by recalling the description of the Moufang generalized polygons in terms of their root groups, as in [6].

Definition 2.1. *A* generalized n-gon *is a connected bipartite graph with diameter n and girth $2n$. A* generalized polygon *is a generalized n-gon for some finite $n \geq 2$. A generalized polygon Γ is called* thick *if $|\Gamma_x| \geq 3$ for all vertices x of Γ. A circuit of Γ of length $2n$ is called an* apartment *of Γ. A path of length n in Γ is called a* root *or a* half-apartment *of Γ.*

Definition 2.2. *If $\alpha = (v_0, \dots, v_n)$ is a root of a generalized n-gon Γ, then the group of all automorphisms of Γ which fix all the vertices of $\Gamma_{v_1} \cup \cdots \cup \Gamma_{v_{n-1}}$ is called a* root group *of Γ (corresponding to the root α) and is denoted by U_α. If U_α acts regularly on the set of apartments through α, then α is called a* Moufang root. *If all roots of Γ are Moufang roots, then Γ is called a* Moufang n-gon.

We assume that Γ is a thick Moufang n-gon for some $n \geq 3$, and we will fix an (arbitrary) apartment Σ which we label by the integers modulo $2n$ such that $i+1 \in \Gamma_i$ and $i+2 \neq i$ for all i. We define $U_i := U_{(i,i+1,\dots,i+n)}$ for all i, and we set $U_{[i,j]} = \langle U_i, U_{i+1}, \dots, U_j \rangle$ for all $i \leq j < i+n$ and $U_{[i,i-1]} = 1$ for all i.

Definition 2.3. *Let $\hat{U}_{[1,n]}$ be a group generated by non-trivial subgroups $\hat{U}_1, \dots, \hat{U}_n$ for some $n \geq 3$. The $(n+1)$-tuple $(\hat{U}_{[1,n]}, \hat{U}_1, \dots, \hat{U}_n)$ is called a* root group sequence *if there exists a Moufang n-gon Γ and a labeled apartment $\Sigma = (0, \dots, 2n-1)$ in Γ such that there exists an isomorphism from $\hat{U}_{[1,n]}$ to $U_{[1,n]}$ mapping $\hat{U}_i$ to U_i for all $i \in \{1, \dots, n\}$. We will denote the root group sequence $(U_{[1,n]}, U_1, \dots, U_n)$ itself by $\Theta(\Gamma, \Sigma)$. The number n will be called the* length *of the root group sequence.*

Definition 2.4. *If $\Theta = (U_{[1,n]}, U_1, \dots, U_n)$ is a root group sequence, then the sequence $(U_{[1,n]}, U_n, \dots, U_1)$ is also a root group sequence. It is called the* opposite of *Θ and is denoted by Θ^{op}.*

Definition 2.5. *Consider two root group sequences* $\Theta = (U_{[1,n]}, U_1, \ldots, U_n)$ *and* $\Theta' = (U'_{[1,n]}, U'_1, \ldots, U'_n)$. *An* isomorphism *from* Θ *to* Θ' *is an isomorphism from* $U_{[1,n]}$ *to* $U'_{[1,n]}$ *mapping* U_i *to* U'_i *for all* $i \in \{1, \ldots, n\}$. *We say that* Θ *and* Θ' *are isomorphic, and we write* $\Theta \cong \Theta'$, *if there exists an isomorphism from* Θ *to* Θ'. *An* anti-isomorphism *from* Θ *to* Θ' *is an isomorphism from* Θ *to* Θ'^{op}.

Theorem 2.6. *Let* Θ *be a root group sequence of length* n. *Then there is, up to isomorphism, a unique Moufang* n*-gon* Γ *such that* $\Theta \cong \Theta(\Gamma, \Sigma)$ *for some labeled apartment* Σ *of* Γ. *We denote this Moufang* n*-gon by* $\Gamma(\Theta)$.

Proof. See [6, (7.6) and (7.7)]. ∎

Definition 2.7. *Let* $\Theta = (U_{[1,n]}, U_1, \ldots, U_n)$ *be a root group sequence. For each* $i \in \{1, \ldots, n\}$, *let* U'_i *be a non-trivial subgroup of* U_i, *and let* $U'_{[1,n]}$ *denote the subgroup of* $U_{[1,n]}$ *generated by* $U'_1, \ldots, U'_n$. *If the* n*-tuple* $\Theta' = (U'_{[1,n]}, U'_1, \ldots, U'_n)$ *is again a root group sequence, then* Θ' *will be called a* subsequence *of* Θ.

The study of subpolygons of Moufang polygons is equivalent to the study of subsequences of root group sequences.

Theorem 2.8. (i) *Let* $\tilde{\Gamma}$ *be a Moufang* n*-gon and let* Γ *be a sub-*n*-gon of* $\tilde{\Gamma}$*; then* Γ *is also a Moufang* n*-gon. Let* Σ *be an arbitrary labeled apartment of* Γ, *then* $\Theta := \Theta(\Gamma, \Sigma)$ *is a subsequence of* $\tilde{\Theta} := \Theta(\tilde{\Gamma}, \Sigma)$.

(ii) *Let* $\tilde{\Theta}$ *be a root group sequence and let* Θ *be a subsequence of* $\tilde{\Theta}$. *Then* $\Gamma(\Theta)$ *is isomorphic to a subpolygon of* $\Gamma(\tilde{\Theta})$.

Proof. See [2, Theorem 2.9]. ∎

We now gather the results on subpolygons of Moufang polygons which we will need for the higher rank buildings. We refer to [2] for a description of the operators Θ_T and Θ_Q and to [3] for more details on quadrangular systems.

Theorem 2.9. *Let* Γ *and* $\tilde{\Gamma}$ *be two Moufang triangles. Then* Γ *is isomorphic to a subtriangle of* $\tilde{\Gamma}$ *if and only if there exists an alternative division ring* $\tilde{A}$ *and a subring* A *of* $\tilde{A}$ *such that* $\Gamma \cong \Gamma(\Theta_T(A))$ *and* $\tilde{\Gamma} \cong \Gamma(\Theta_T(\tilde{A}))$.

Proof. See [2, Theorem 3.3]. ∎

Definition 2.10. *Let* Γ *be a Moufang quadrangle, and let* $\Sigma = (0, \ldots, 7)$ *be a labeled apartment of* Γ. *Let* $U_0, \ldots, U_7$ *be the root groups associated to* Σ. *Then we write* $V_i := [U_{i-1}, U_{i+1}] \leq U_i$ *for all* i, *and we let* $Y_i := C_{U_i}(U_{i-2}) \leq U_i$ *for each* i. *By [6, (21.20.i)], we have* $Y_i = C_{U_i}(U_{i+2})$ *as well.*

Theorem 2.11. *By relabeling the vertices of* Σ *by the transformation* $i \mapsto 5 - i$ *if necessary, we can assume that*

(i) $Y_i \neq 1$, $[U_i, U_i] \leq V_i \leq Y_i \leq Z(U_i)$ *for all odd* i *;*

(ii) U_i *is abelian for all even* i *.*

Proof. See [6, (21.28)]. ∎

Theorem 2.12. *Let Γ and $\tilde{\Gamma}$ be two Moufang quadrangles.*

(i) *Suppose that Γ is a subquadrangle of $\tilde{\Gamma}$, let Σ be an apartment of Γ, and assume that Σ is labeled in such a way that the statements of Theorem 2.11 hold for the root groups $\tilde{U}_i$ of $\tilde{\Gamma}$.*

(a) *If the statements of Theorem 2.11 hold for the root groups U_i of Γ, then there exist quadrangular systems $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ and $\tilde{\Omega} = (\tilde{V}, \tilde{W}, \tau_{\tilde{V}}, \tau_{\tilde{W}}, \tilde{\epsilon}, \tilde{\delta})$ for which $\Gamma \cong \Gamma(\Theta_Q(\Omega))$ and $\tilde{\Gamma} \cong \Gamma(\Theta_Q(\tilde{\Omega}))$, such that there exist group monomorphisms ϕ from V into $\tilde{V}$ and ψ from W into $\tilde{W}$ satisfying the conditions*

$$\begin{aligned} \tau_{\tilde{V}}(\phi(v), \psi(w)) &= \tau_{\tilde{V}}(\phi(\tau_V(v, w)), \psi(\delta))\,, \\ \tau_{\tilde{W}}(\psi(w), \phi(v)) &= \tau_{\tilde{W}}(\psi(\tau_W(w, v)), \phi(\epsilon))\,, \end{aligned} \tag{1}$$

for all $v \in V$ and all $w \in W$; we then say that Γ is semi-algebraically included in $\tilde{\Gamma}$.

(a') *If the morphisms ϕ and ψ in* (a) *can be chosen in such a way that $\phi(\epsilon) = \tilde{\epsilon}$ and $\psi(\delta) = \tilde{\delta}$, then we say that Γ is algebraically included in $\tilde{\Gamma}$. This is always the case if none of the root groups of Γ and Γ' is a 2-torsion group.*

(b) *If the statements of Theorem 2.11 hold for the root groups U_{5-i} of Γ, then there exist quadrangular systems $\Omega = (V, W, \tau_V, \tau_W, \epsilon, \delta)$ and $\tilde{\Omega} = (\tilde{V}, \tilde{W}, \tau_{\tilde{V}}, \tau_{\tilde{W}}, \tilde{\epsilon}, \tilde{\delta})$ for which $\Gamma \cong \Gamma(\Theta_Q(\Omega))$ and $\tilde{\Gamma} \cong \Gamma(\Theta_Q(\tilde{\Omega}))$, such that there exist group monomorphisms ϕ from V into $\tilde{W}$ and ψ from W into $\tilde{V}$ satisfying the conditions*

$$\begin{aligned} \tau_{\tilde{V}}(\psi(w), \phi(v)) &= \tau_{\tilde{V}}(\psi(\tau_W(w, v)), \phi(\epsilon))\,, \\ \tau_{\tilde{W}}(\phi(v), \psi(w)) &= \tau_{\tilde{W}}(\phi(\tau_V(v, w)), \psi(\delta))\,, \end{aligned} \tag{2}$$

for all $v \in V$ and all $w \in W$; we then say that Γ is dually included in $\tilde{\Gamma}$.

(ii) *If Γ is (semi-)algebraically or dually included in $\tilde{\Gamma}$, then Γ is isomorphic to a subquadrangle of $\tilde{\Gamma}$.*

Proof. See [2, Theorem 6.6 and Lemma 6.16(i)]. ∎

In the light of (a') above, we will assume from now on, until the end of this section, that none of the root groups is 2-torsion. Under this assumption, all algebraic inclusions of Moufang quadrangles have been classified in [2]. We recall the results which we will need, and we refer once again to [2] for more details about the notations and definitions.

Theorem 2.13. *Let $\Omega \cong \Omega_I(K, \sigma)$ and $\tilde{\Omega} \cong \Omega_I(\tilde{K}, \tilde{\sigma})$ for some involutory sets (K, σ) and $(\tilde{K}, \tilde{\sigma})$.*

(i) *Suppose that (K, σ) is proper. Then Ω is isomorphic to a subsystem of $\tilde{\Omega}$ if and only if there exists a field monomorphism α from K into $\tilde{K}$ such that $\alpha \circ \sigma = \tilde{\sigma} \circ \alpha$.*

(ii) *Suppose that* (K, σ) *is not proper. Then* Ω *is isomorphic to a subsystem of* $\tilde{\Omega}$ *if and only if there exists a field monomorphism or a field anti-monomorphism* α *from* K *into* $\tilde{K}$ *such that* $\alpha \circ \sigma = \tilde{\sigma} \circ \alpha$.

Theorem 2.14. *Let* $\Omega \cong \Omega_Q(K, V_0, q, e)$ *and* $\tilde{\Omega} \cong \Omega_Q(\tilde{K}, \tilde{V}_0, \tilde{q}, \tilde{e})$ *for some anisotropic quadratic spaces* (K, V_0, q) *and* $(\tilde{K}, \tilde{V}_0, \tilde{q})$ *with base points* e *and* $\tilde{e}$, *respectively. Then* Ω *is isomorphic to a subsystem of* $\tilde{\Omega}$ *if and only if there exists a vector space monomorphism* (β, α) *from* (K, V_0) *into* $(\tilde{K}, \tilde{V}_0)$ *such that* $\alpha(e) = \tilde{e}$ *and* $\tilde{q}(\alpha(v)) = \beta(q(v))$ *for all* $v \in V_0$.

Theorem 2.15. *Let* $\Omega \cong \Omega_I(K, \sigma)$ *and* $\tilde{\Omega} \cong \Omega_P(\tilde{K}, \tilde{\sigma}, \tilde{V}_0, \tilde{\pi})$ *for some involutory set* (K, σ) *and some anisotropic pseudo-quadratic space* $(\tilde{K}, \tilde{\sigma}, \tilde{V}_0, \tilde{\pi})$. *Then* Ω *is isomorphic to a subsystem of* $\tilde{\Omega}$ *if and only if* Ω *is already isomorphic to a subsystem of* $\Omega_I(\tilde{K}, \tilde{\sigma})$.

Theorem 2.16. *Let* $\Omega \cong \Omega_P(K, \sigma, V_0, \pi)$ *and* $\tilde{\Omega} \cong \Omega_P(\tilde{K}, \tilde{\sigma}, \tilde{V}_0, \tilde{\pi})$ *for some proper anisotropic pseudo-quadratic spaces* (K, σ, V_0, π) *and* $(\tilde{K}, \tilde{\sigma}, \tilde{V}_0, \tilde{\pi})$. *Denote the skew-hermitian forms corresponding to* Ω *and* $\tilde{\Omega}$ *by* h *and* $\tilde{h}$, *respectively. Then* Ω *is isomorphic to a subsystem of* $\tilde{\Omega}$ *if and only if there exists a vector space monomorphism* (β, α) *from* (K, V_0) *into* $(\tilde{K}, \tilde{V}_0)$ *such that* $\beta \circ \sigma = \tilde{\sigma} \circ \beta$ *and* $\beta(h(a, b)) = \tilde{h}(\alpha(a), \alpha(b))$ *for all* $a, b \in V_0$.

Remark 2.17. *It follows from the defining commutator relations of the Moufang quadrangles that*

(i) *a Moufang quadrangle of quadratic form type can never be dually included in a Moufang quadrangle of involutory type;*

(ii) *a Moufang quadrangle of involutory type can never be dually included in a Moufang quadrangle of quadratic form type;*

(iii) *a Moufang quadrangle of proper pseudo-quadratic form type can never be included (neither semi-algebraically nor dually) in a Moufang quadrangle of involutory type or in a Moufang quadrangle of quadratic form type.*

There is one case of a dual inclusion which we have to examine in detail. We will need the following definition.

Definition 2.18. *Let* K *be a commutative field, and let* L *be a (skew) field containing* K *as a subfield, with an involution* σ *(which might or might not act trivially on* K*). Assume that there exists a non-zero element* $b \in L$ *(which might or might not belong to* K*) with* $b + b^\sigma = 0$, *such that* $bs = s^\sigma b$ *for all* $s \in K$ *(or, equivalently, such that* $\mathrm{Tr}(bK) = 0$*). Then we call* (L, σ, b) *a* skew envelope *of* K.

Let (K, V_0, q) be an anisotropic quadratic space with $\mathrm{char}(K) \neq 2$, and let (L, σ, b) be a skew envelope of K. Let f be the bilinear form from $V_0 \times V_0$ to K corresponding to q, and denote its corresponding symmetric matrix by M; then $f(u, v) = uMv^T$ for all $u, v \in V_0$. (The matrix M is allowed to be infinite dimensional.) Let $X := V_0 \otimes L$ be the right vector space over L obtained by extending the scalars of V_0 to L, and let h be the map from $X \times X$ to L defined by $h(u, v) = u^\sigma bMv^T$ for all $u, v \in X$. It is clear that $h(us, vt) = s^\sigma h(u, v)t$ for all $u, v \in X$ and all $s, t \in L$.

Since M has all its entries in K, it follows from the fact that (L, σ, b) is a skew envelope of K that $(bM)^\sigma = M^\sigma b^\sigma = -M^\sigma b = -bM = (-bM)^T$. Also observe that $(AB)^{\sigma T} = B^{\sigma T} A^{\sigma T}$ for any two matrices A, B with entries in the skew field L. Hence

$$h(u,v)^\sigma = h(u,v)^{\sigma T} = (u^\sigma \cdot bM \cdot v^T)^{\sigma T} = v^\sigma(-bM)u^T = -h(v,u)$$

for all $u, v \in X$, and therefore h is a skew-hermitian form on X. It follows that the map $\pi : X \to L$ defined by $\pi(u) := h(u,u)/2$ for all $u \in X$ is a pseudo-quadratic form on X (which might or might not be anisotropic); see [6, (11.28)].

Definition 2.19. *Let (K, V_0, q) be an anisotropic quadratic space with* $\mathrm{char}(K) \neq 2$*, and let (L, σ, b) be a skew envelope of K. Then the pseudo-quadratic space (L, σ, X, π) as constructed above will be called an* enveloping pseudo-quadratic space *for the quadratic space (K, V_0, q), with* skew factor $b \in L$.

Theorem 2.20. *Let $\Omega \cong \Omega_Q(K, V_0, q, e)$ and $\tilde\Omega \cong \Omega_P(\tilde K, \tilde\sigma, \tilde V_0, \tilde\pi)$ for some anisotropic quadratic space (K, V_0, q) with base point e and some proper anisotropic pseudo-quadratic space $\tilde\Xi = (\tilde K, \tilde\sigma, \tilde V_0, \tilde\pi)$. Then $\Gamma(\Omega)$ is dually included in $\Gamma(\tilde\Omega)$ if and only if there exists a subspace $\tilde V_1$ of $\tilde V_0$ such that $\tilde\Xi_1 := (\tilde K, \tilde\sigma, \tilde V_1, \tilde\pi_{|\tilde V_1})$ is isomorphic to an enveloping pseudo-quadratic space for the quadratic space (K, V_0, q).*

Proof. Let $\tilde F := \mathrm{Fix}_{\tilde K}(\tilde\sigma)$. Denote the bilinear form corresponding to Ω by f. By the definition of the operators Ω_Q and Ω_P, we have that $V = [V_0]$, $W = [K]$, $\tilde V = [\tilde K]$ and $\tilde W = [\tilde T]$, where $(\tilde T, \boxplus)$ is the group with underlying set $\{(a,t) \in \tilde V_0 \times \tilde K \mid \tilde\pi(a) - t \in \tilde F\}$, and with group multiplication $(a,t) \boxplus (b,s) = (a+b, t+s+\tilde h(b,a))$ for all $(a,t), (b,s) \in \tilde T$.

First assume that there exists a subspace $\tilde V_1$ of $\tilde V_0$ such that the pseudo-quadratic space $\tilde\Xi_1 := (\tilde K, \tilde\sigma, \tilde V_1, \tilde\pi_{|\tilde V_1})$ is isomorphic to an enveloping pseudo-quadratic space for the quadratic space (K, V_0, q). We have to show that $\Gamma(\Omega)$ is dually included in $\Gamma(\Omega_P(\tilde\Xi_1))$; we can assume without loss of generality that $\tilde\Xi_1 = \tilde\Xi$, and that $\tilde\Xi_1$ is itself an enveloping pseudo-quadratic space for (K, V_0, q). Let $b \in \tilde K$ be its skew factor. We define a map ϕ from $V = [V_0]$ to $\tilde W = [\tilde T]$ and a map ψ from $W = [K]$ to $\tilde V = [\tilde K]$ by setting $\phi[v] := [v, bq(v)]$ and $\psi[t] := [t]$ for all $v \in V_0$ and all $t \in K$. Note that $v^{\tilde\sigma} b = bv$ for all $v \in V_0$, and hence $\pi(v) = v^{\tilde\sigma} bMv^T/2 = bf(v,v)/2 = bq(v)$ for all $v \in V_0$, and therefore $\phi[v] \in [\tilde T]$ as required. Since $h(u,v) = bf(u,v)$ for all $u, v \in V_0$, the maps ϕ and ψ are group morphisms; they are obviously injective. It finally follows immediately from the definitions of the τ-maps corresponding to the operators Ω_Q and Ω_P that the conditions (2) are satisfied, and hence $\Gamma(\Omega)$ is dually included in $\Gamma(\Omega_P(\tilde\Xi_1))$.

Now assume that $\Gamma(\Omega)$ is dually included in $\Gamma(\Omega_P(\tilde\Xi))$, with corresponding group monomorphisms ϕ and ψ as in (2). We define a map α from V_0 to $\tilde V_0$, a map β from V_0 to $\tilde K$ and a map γ from K to $\tilde K$ by setting $\phi[v] = [\alpha(v), \beta(v)]$ for all $v \in V_0$ and $\psi[t] = [\gamma(t)]$ for all $t \in K$. Note that α cannot be identically zero, since that would imply that $\Gamma(\Omega)$ is already dually included in $\Gamma(\Omega_I(\tilde K, \tilde\sigma))$, contradicting Remark 2.17(ii). Also observe that α and γ are additive maps, and that

$$\beta(u+v) = \beta(u) + \beta(v) + h(\alpha(u), \alpha(v)) \tag{3}$$

for all $u, v \in V_0$. Using the explicit descriptions of the τ-maps, the conditions (2) can be rewritten as

$$\beta(v)\gamma(t) = \beta(\epsilon)\gamma(tq(v)), \quad (4)$$
$$\alpha(v)\gamma(t) = \alpha(vt)\gamma(1), \quad (5)$$
$$\gamma(t)^{\tilde{\sigma}}\beta(v)\gamma(t) = \gamma(1)^{\tilde{\sigma}}\beta(vt)\gamma(1), \quad (6)$$

for all $v \in V_0$ and all $t \in K$. Let χ be the map from K to $\tilde{K}$ given by $\chi(t) := \gamma(t)\gamma(1)^{-1}$; then $\chi(1) = 1$, and by (5), $\alpha(vt) = \alpha(v)\chi(t)$ for all $v \in V_0$ and all $t \in K$. It follows that $\alpha(v)\chi(st) = \alpha(v)\chi(s)\chi(t)$ for all $v \in V_0$ and all $t \in K$, and since α is not identically zero, this implies that χ is multiplicative and therefore a field monomorphism. Let $L := \chi(K)$; then L is a commutative subfield of $\tilde{K}$. Moreover, it follows from (4) that $\beta(v) = \beta(\epsilon)\chi(q(v))$ for all $v \in V_0$; let $b := \beta(\epsilon) \in \tilde{K}$, then this can be written as

$$\beta(v) = b\chi(q(v)) \quad (7)$$

for all $v \in V_0$. In particular, we have that $\beta(vt) = \beta(v)\chi(t)^2$ for all $v \in V_0$ and all $t \in K$. It then follows from (6) that $\chi(t)^{\tilde{\sigma}}\beta(v)\chi(t) = \beta(v)\chi(t)^2$, and hence $s^{\tilde{\sigma}}b = bs$ for all $s \in L$.

By (3), we have that $\beta(\epsilon+\epsilon) = \beta(\epsilon)+\beta(\epsilon)+h(\alpha(\epsilon), \alpha(\epsilon))$, and hence $h(\alpha(\epsilon), \alpha(\epsilon)) = 4\beta(\epsilon) - 2\beta(\epsilon) = 2\beta(\epsilon)$. On the other hand, the fact that $(\alpha(\epsilon), \beta(\epsilon)) \in \tilde{T}$ implies that $h(\alpha(\epsilon), \alpha(\epsilon)) = \beta(\epsilon) - \beta(\epsilon)^{\sigma}$; see [6, (11.19)]. It follows that $\beta(\epsilon) = -\beta(\epsilon)^{\tilde{\sigma}}$, that is, $b + b^{\tilde{\sigma}} = 0$. We conclude that $(\tilde{K}, \tilde{\sigma}, b)$ is a skew envelope of L.

Let $\tilde{W} := \alpha(V_0) \subseteq \tilde{V}_0$. Then $\tilde{W}$ is closed under scalar multiplication by elements of L, hence $\tilde{W}$ is a vector space over L. Let $\bar{q} := \chi \circ q \circ \alpha^{-1}$; then $\bar{q}$ is a quadratic form from $\tilde{W}$ to L with corresponding bilinear form $\bar{f} := \chi \circ f \circ (\alpha^{-1}, \alpha^{-1})$. It then follows from (3) and (7) that

$$h(w, z) = b\bar{f}(w, z) \quad (8)$$

for all $w, z \in \tilde{W}$. Let $\tilde{V}_1 := \tilde{W} \otimes \tilde{K}$ be the right vector space over $\tilde{K}$ obtained by extending the scalars of $\tilde{W}$ to $\tilde{K}$; then $\tilde{V}_1$ is a $\tilde{K}$-subspace of $\tilde{V}_0$. The restriction of h to $\tilde{V}_1 \times \tilde{V}_1$ is completely determined by (8) and the fact that h is a skew-hermitian form, and it follows from this equation that $(\tilde{K}, \tilde{\sigma}, \tilde{V}_1, \pi_{|\tilde{V}_1})$ is an enveloping pseudo-quadratic space for the quadratic space $(L, \tilde{W}, \bar{q})$. Since the quadratic spaces (K, V_0, q) and $(L, \tilde{W}, \bar{q})$ are isomorphic, the result follows. ■

3 Passing on to higher rank

Now that we have assembled all the required results in the rank 2 case, we are ready to pass on to higher rank. So let Γ be a spherical building of rank $n \geq 3$; then Γ is a Moufang building, and each of its rank 2 residues is either a generalized digon or a Moufang polygon.

Let Σ be an arbitrary fixed apartment of Γ, and let c be an arbitrary chamber of Σ. Let $I := \{1, \ldots, n\}$, and let W be the Weyl group of Γ, with corresponding generating set $S = \{s_i \mid i \in I\}$ and corresponding Coxeter matrix $[m_{ij}]_{i,j \in I}$, so that

$$W = \langle s_1, \ldots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \rangle$$

where $m_{ii} = 1$ for all $i \in I$ and $2 \leq m_{ij} < \infty$ for all $i, j \in I$ with $i \neq j$.

For each $i \in I$, there is a unique root α_i of Σ such that $c \in \alpha_i$ and such that the unique chamber of Σ which is i-adjacent to c is not contained in α_i. For all $i, j \in I$ with $i \neq j$, we let Δ_{ij} be the $\{i, j\}$-residue of Γ containing c; then Δ_{ij} is a generalized m_{ij}-gon, which is a Moufang polygon if $m_{ij} \geq 3$.

Lemma 3.1. *Let $i, j \in I$ with $i \neq j$, let $\Delta := \Delta_{ij}$, and let $m := m_{ij}$. Then the intersection $\Sigma \cap \Delta$ is an apartment of the generalized m-gon Δ. There are exactly m roots $\omega_1 = \alpha_i, \omega_2, \ldots, \omega_m = \alpha_j$ of Σ containing c but not $\Sigma \cap \Delta$, and if $m \geq 3$, they can be ordered so that $(\omega_1 \cap \Delta, \omega_2 \cap \Delta, \ldots, \omega_n \cap \Delta)$ is the root sequence of Δ from $\alpha_i \cap \Delta$ to $\alpha_j \cap \Delta$. Let $U_+ := U_{\omega_1} U_{\omega_2} \ldots U_{\omega_m}$; then U_+ is a subgroup of* $\mathrm{Aut}(\Gamma)$ *acting faithfully on Δ; in particular, we can identify the root groups U_{ω_k} with their image in* $\mathrm{Aut}(\Delta)$. *Then $(U_+, U_{\omega_1}, \ldots, U_{\omega_m})$ restricted to Δ is a root group sequence for the Moufang m-gon Δ.*

Proof. See [6, (40.9)]. ∎

If we want to emphasize the fact that these root groups are related to Δ_{ij}, then we will write $U_+^{(ij)}$ for U_+, and $U_k^{(ij)}$ for U_{ω_k}. With this notation, we have that $U_1^{(ij)} = U_{\alpha_i}$ and $U_{m_{ij}}^{(ij)} = U_{\alpha_j}$ for all $i, j \in J$ with $i \neq j$.

The following two definitions are based on [6, (40.14) and (40.15)].

Definition 3.2. *Let Π be the Coxeter graph of Γ; then I is the vertex set of Π. Let E denote the set of directed edges of Π, that is, the set of ordered pairs (i, j) with $i, j \in I$, $i \neq j$, such that $m_{ij} \geq 3$. Let π be the labeling of Π which assigns to each vertex $i \in I$ the group U_{α_i}, and to each edge (i, j) of E the root group sequence $(U_+^{(ij)}, U_1^{(ij)}, \ldots, U_{m_{ij}}^{(ij)})$ of the Moufang m_{ij}-gon Δ_{ij}. In particular, $\pi(j, i) = \pi(i, j)^{\mathrm{op}}$. Then π is called a* root group labeling of Π, *and (Π, π) is called the* root group system of Γ based at (Σ, c).

By the remark following [6, (40.15)], the root group system of Γ is, up to isomorphism, independent of the choice of Σ and c (where isomorphism of root group systems is defined in the natural way). Moreover, the building Γ is completely determined, up to isomorphism, by one of its root group systems; see [6, (40.17)].

Definition 3.3. *Let Π be a Coxeter graph, and let π and $\tilde{\pi}$ be two root group labelings of Π. Then we call (Π, π) a* subsystem *of $(\Pi, \tilde{\pi})$ if $\pi(i)$ is a subgroup of $\tilde{\pi}(i)$ for each $i \in I$, and if the root group sequence $\pi(i, j)$ is a subsequence of $\tilde{\pi}(i, j)$ as defined in Definition 2.7, for all $i, j \in I$, $i \neq j$.*

Let Γ and $\tilde{\Gamma}$ be two spherical buildings of rank $n \geq 3$ with the same Weyl group (and hence with the same Coxeter diagram). Suppose that Γ is a subbuilding of $\tilde{\Gamma}$, and let Σ be an apartment of Γ and c be a chamber of Σ. Then it is clear that the root group system of Γ based at (Σ, c) is a subsystem of the root group system of $\tilde{\Gamma}$ based at (Σ, c). On the other hand, if Γ has a root group system which is isomorphic to a subsystem of a root group system of $\tilde{\Gamma}$, then it follows from [6, (40.20)] that Γ is isomorphic to a subbuilding of $\tilde{\Gamma}$. Therefore, the study of the inclusions of buildings with the same Coxeter diagram is equivalent to the study of subsystems of root group systems.

We now recall the classification of all spherical buildings of rank at least 3, in terms of their root group systems, as in [6]. We first give the definition of quadratic and honorary involutory sets, which we will need in the description of the buildings of type C_n and F_4.

Definition 3.4. *[6, (38.11)]. An involutory set (K, K_0, σ) is called* quadratic *if K_0 is a subfield of the center Z of K and K is quadratic over K_0, i.e., there exist functions T and N from K to K_0 such that $a^2 - T(a)a + N(a) = 0$ for all $a \in K_0$. By [6, (20.3)], (K, K_0, σ) is quadratic if and only if (exactly) one of the following holds:*

(i) $K = Z$, $\mathrm{char}(K) = 2$, $K^2 \subseteq K_0 \neq K$ *and* $\sigma = 1$;

(ii) $K = Z = K_0$ *and* $\sigma = 1$;

(iii) $K = Z$, K/K_0 *is a separable quadratic extension and σ is the non-trivial element of* $\mathrm{Gal}(K/K_0)$;

(iv) *K is a quaternion division algebra over K_0, $K_0 = Z$, and σ is the standard involution of K/K_0 .*

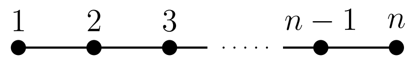
Moreover, a triple (K, K_0, σ) is called an honorary involutory set *if*

(v) *K is a Cayley-Dickson division algebra (a.k.a. an octonion division algebra) with center K_0, and σ is the standard involution of K/K_0.*

(Note that K is only an alternative division ring in this case, and not a skew field, hence the name "honorary" involutory set.) In all five cases, we set $N_{K/K_0}(a) := a^\sigma a$ and $T_{K/K_0}(a) := a^\sigma + a$. We will say that a quadratic or honorary involutory set is of type (n) *according as to which of these five cases it belongs.*

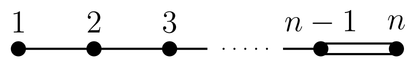
Theorem 3.5. *Let Γ be a spherical building of rank $n \geq 3$, let Σ be an apartment of Γ, and let c be a chamber of Σ. Let (Π, π) be the root group system of Γ based at (Σ, c). Then up to isomorphism, one of the following holds.*

(i) $\Pi = \mathsf{A}_n$ $(n \geq 3)$:



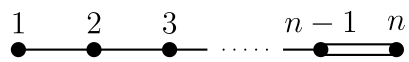
There is a (skew) field K such that $\pi(i) = (K, +)$ for all $i \in \{1, \ldots, n\}$, and $\pi(i, i+1) = \mathcal{T}(K)$ for all $i \in \{1, \ldots, n-1\}$. We will denote this building by $\mathsf{A}_n(K)$.

(ii) $\Pi = \mathsf{B}_n$ $(n \geq 3)$:



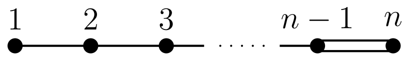
There is an anisotropic quadratic space (K, V_0, q) such that $\pi(i) = (K, +)$ for all $i \in \{1, \ldots, n-1\}$, $\pi(n) = (V_0, +)$, $\pi(i, i+1) = \mathcal{T}(K)$ for all $i \in \{1, \ldots, n-2\}$, and $\pi(n-1, n) = \mathcal{Q}(\Omega_Q(K, V_0, q))$. We will denote this building by $\mathsf{B}_n(K, V_0, q)$.

(iii) $\Pi = \mathsf{C}_n$ $(n \geq 3)$:



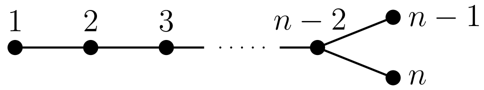
There is an involutory set (K, K_0, σ) which is either proper or quadratic or, but only if $n = 3$, honorary, such that $\pi(i) = (K, +)$ for all $i \in \{1, \ldots, n-1\}$, $\pi(n) = (K_0, +)$, $\pi(i, i+1) = \mathcal{T}(K)$ for all $i \in \{1, \ldots, n-2\}$, and $\pi(n-1, n) = \mathcal{Q}(\Omega_I(K, K_0, \sigma)^{\mathrm{op}})$. We will denote this building by $\mathsf{C}_n(K, K_0, \sigma)$.

(iv) $\Pi = \mathsf{BC}_n$ $(n \geq 3)$:



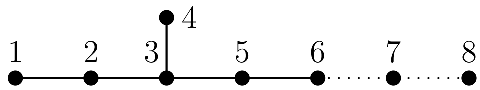
There is an anisotropic pseudo-quadratic space (K, K_0, σ, X_0, p) *with corresponding group* T *(as defined in the proof of Theorem 2.20) such that* $\pi(i) = (K, +)$ *for all* $i \in \{1, \dots, n-1\}$, $\pi(n) = T$, $\pi(i, i+1) = \mathcal{T}(K)$ *for all* $i \in \{1, \dots, n-2\}$, *and* $\pi(n-1, n) = \mathcal{Q}(\Omega_P(K, K_0, \sigma, X_0, p)^{\mathrm{op}})$. *We will denote this building by* $\mathsf{BC}_n(K, K_0, \sigma, X_0, p)$.

(v) $\Pi = \mathsf{D}_n$ $(n \geq 4)$:



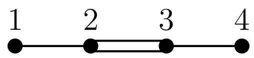
There is a commutative field K *such that* $\pi(i) = (K, +)$ *for all* $i \in \{1, \dots, n\}$, *and* $\pi(i, j) = \mathcal{T}(K)$ *for all* $(i, j) \in E$. *We will denote this building by* $\mathsf{D}_n(K)$.

(vi) $\Pi = \mathsf{E}_n$ $(n \in \{6, 7, 8\})$:



There is a commutative field K *such that* $\pi(i) = (K, +)$ *for all* $i \in \{1, \dots, n\}$, *and* $\pi(i, j) = \mathcal{T}(K)$ *for all* $(i, j) \in E$. *We will denote this building by* $\mathsf{E}_n(K)$.

(vii) $\Pi = \mathsf{F}_4$:



There is a quadratic or honorary involutory set (K, K_0, σ) *such that* $\pi(1) = \pi(2) = (K, +)$, $\pi(3) = \pi(4) = (K_0, +)$, $\pi(1, 2) = \mathcal{T}(K)$, $\pi(3, 4) = \mathcal{T}(K_0)$, *and* $\pi(2, 3) = \mathcal{Q}(\Omega_I(K, K_0, \sigma)^{\mathrm{op}}) = \mathcal{Q}(\Omega_Q(K_0, K, N_{K/K_0})^{\mathrm{op}})$. *We will denote this building by* $\mathsf{F}_4(K, K_0, \sigma)$.

Proof. See [6, Chapter 40]. ∎

Remark 3.6. *If* (K, K_0, σ) *is an involutory set with* $\mathrm{char}(K) \neq 2$, *then* K_0 *is completely determined by* K *and* σ; *in fact,* $K_0 = \{a + a^\sigma \mid a \in K\} = \mathrm{Fix}_K(\sigma)$ *in this case. We will therefore often omit* K_0 *in our notation and simply write* (K, σ) *instead.*

Remark 3.7. *Let* (K, V_0, q) *be an anisotropic quadratic space. Then strictly speaking,* $\Omega_Q(K, V_0, q)$ *is not defined, since it also depends on the choice of a base point* $e \in V_0^*$. *However, for any* $v \in V_0^*$, *the root group sequence* $\mathcal{Q}(\Omega_Q(K, V_0, q(v)^{-1}q, v))$ *is, up to isomorphism, independent of the choice of* v, *which justifies the use of the notation* $\mathcal{Q}(\Omega_Q(K, V_0, q))$ *in Theorem 3.5; see [2, Remark 6.14].*

We are now ready to start the study of the inclusions of spherical buildings. The case where the diagram has only single edges is straightforward.

Theorem 3.8. (i) *Let* Γ *and* $\tilde{\Gamma}$ *be two spherical buildings of type* A_n, $n \geq 3$. *Then* Γ *is isomorphic to a subbuilding of* $\tilde{\Gamma}$ *if and only if there exists a (skew) field* $\tilde{K}$ *and a subfield* K *of* $\tilde{K}$ *such that* $\Gamma \cong \mathsf{A}_n(K)$ *and* $\tilde{\Gamma} \cong \mathsf{A}_n(\tilde{K})$.

(ii) *Let* Γ *and* $\tilde{\Gamma}$ *be two spherical buildings of type* D_n, $n \geq 4$. *Then* Γ *is isomorphic to a subbuilding of* $\tilde{\Gamma}$ *if and only if there exists a commutative field* $\tilde{K}$ *and a subfield* K *of* $\tilde{K}$ *such that* $\Gamma \cong \mathsf{D}_n(K)$ *and* $\tilde{\Gamma} \cong \mathsf{D}_n(\tilde{K})$.

(iii) *Let* Γ *and* $\tilde{\Gamma}$ *be two spherical buildings of type* E_n, $n \in \{6,7,8\}$. *Then* Γ *is isomorphic to a subbuilding of* $\tilde{\Gamma}$ *if and only if there exists a commutative field* $\tilde{K}$ *and a subfield* K *of* $\tilde{K}$ *such that* $\Gamma \cong \mathsf{E}_n(K)$ *and* $\tilde{\Gamma} \cong \mathsf{E}_n(\tilde{K})$.

Proof. This follows immediately from Definition 3.3 and Theorem 2.9. ∎

In the other cases, there is (exactly) one residue which is a Moufang quadrangle; we will therefore assume from now on that the characteristic of the defining field K is different from 2. We start with a very specific case which we will need on two different occasions.

Lemma 3.9. *Let* $\Gamma \cong \mathsf{C}_3(K,\sigma)$ *and* $\tilde{\Gamma} \cong \mathsf{C}_3(\tilde{K},\tilde{\sigma})$ *for some quadratic or honorary involutory sets* (K,σ) *with norm* N *and* $(\tilde{K},\tilde{\sigma})$ *with norm* $\tilde{N}$. *Then* Γ *is isomorphic to a subbuilding of* $\tilde{\Gamma}$ *if and only if there exists a ring monomorphism* β *from* K *into* $\tilde{K}$ *such that an element* $a \in K$ *lies in* $K_0 = \mathrm{Fix}_K(\sigma)$ *if and only if* $\beta(a)$ *lies in* $\tilde{K}_0 = \mathrm{Fix}_{\tilde{K}}(\tilde{\sigma})$. *Moreover, if* (K,σ) *is of type* (iv) *or* (v), *then every ring monomorphism* β *from* K *into* $\tilde{K}$ *satisfies this condition; in fact,* $\beta \circ \sigma = \tilde{\sigma} \circ \beta$ *in this case.*

Proof. Let $\Gamma \cong \mathsf{C}_3(K,\sigma)$ and $\tilde{\Gamma} \cong \mathsf{C}_3(\tilde{K},\tilde{\sigma})$ for some quadratic or honorary involutory sets (K,K_0,σ) and $(\tilde{K},\tilde{K}_0,\tilde{\sigma})$ with norms $N = N_{K/K_0}$ and $\tilde{N} = N_{\tilde{K}/\tilde{K}_0}$, respectively. Let $T = T_{K/K_0}$ and $\tilde{T} = T_{\tilde{K}/\tilde{K}_0}$; then T and $\tilde{T}$ are not identically zero, since we assumed the characteristic of K and $\tilde{K}$ to be different from 2.

Assume first that there exists a ring monomorphism $\beta : K \hookrightarrow \tilde{K}$ such that $\beta(K_0) \subseteq \tilde{K}_0$ and $\beta^{-1}(\tilde{K}_0) \subseteq K_0$. Since (K,K_0,σ) is quadratic or honorary, we have $a^2 - T(a)a + N(a) = 0$ for all $a \in K$. Hence

$$\beta(a)^2 - \beta(T(a))\beta(a) + \beta(N(a)) = 0\,. \tag{9}$$

On the other hand, a similar argument for $(\tilde{K},\tilde{K}_0,\tilde{\sigma})$ shows that

$$\beta(a)^2 - \tilde{T}(\beta(a))\beta(a) + \tilde{N}(\beta(a)) \tag{10}$$

for all $a \in K$. If $a \notin K_0$ (and hence also $\beta(a) \notin \tilde{K}_0$), then it follows from these two equations that $\beta(N(a)) = \tilde{N}(\beta(a))$; if $a \in K_0$ (and hence also $\beta(a) \in \tilde{K}_0$), then $\beta(N(a)) = \beta(a^2) = \beta(a)^2 = \tilde{N}(\beta(a))$. Hence $\beta \circ N = \tilde{N} \circ \beta$. Let $\alpha := \beta_{|K_0}$, then (α,β) is a vector space monomorphism from (K_0,K) into $(\tilde{K}_0,\tilde{K})$ such that $\alpha \circ N = \tilde{N} \circ \beta$, and hence, by Theorem 2.14, $\pi(2,3) = \mathcal{Q}(\Omega_Q(K_0,K,N,1)^{\mathrm{op}})$ is isomorphic to a subsequence of $\tilde{\pi}(2,3) = \mathcal{Q}(\Omega_Q(\tilde{K}_0,\tilde{K},\tilde{N},\tilde{1})^{\mathrm{op}})$. Moreover, by Theorem 2.9, $\pi(1,2) = \mathcal{T}(K)$ is isomorphic to a subsequence of $\tilde{\pi}(1,2) = \mathcal{T}(\tilde{K})$. We conclude that Γ is isomorphic to a subbuilding of $\tilde{\Gamma}$.

Assume now that Γ is isomorphic to a subbuilding of $\tilde{\Gamma}$; then $\Omega_I(K,\sigma)$ is isomorphic to a subsystem of $\Omega_I(\tilde{K},\tilde{\sigma})$. We will distinguish between two cases, depending on the type of (K,σ) as in Definition 3.4. If (K,σ) is of type (ii) or (iii), then K is abelian, and hence it follows from Theorem 2.13 that there exists a field monomorphism α from K into $\tilde{K}$ such that $\alpha \circ \sigma = \tilde{\sigma} \circ \alpha$. Since α is injective, this implies that $a = a^\sigma$ if and only if $\alpha(a) = \alpha(a)^{\tilde{\sigma}}$, for all $a \in K$, as required. So we may assume that (K,σ) is of type (iv) or (v). Then we use the fact that $\pi(1,2) \cong \mathcal{T}(K)$

is isomorphic to a subsequence of $\tilde{\pi}(1,2) \cong \mathcal{T}(\tilde{K})$ to deduce that there exists a ring monomorphism $\beta : K \hookrightarrow \tilde{K}$. Observe that $K_0 = Z(K)$ and $\tilde{K}_0 = Z(\tilde{K})$ in this case. If $a \in K \setminus K_0$, then it follows from this observation and from the injectivity of β that $\beta(a) \in \tilde{K} \setminus \tilde{K}_0$, and hence $\beta^{-1}(\tilde{K}_0) \subseteq K_0$. Now let $t \in K_0^*$ be arbitrary. Since $\mathrm{char}(K) \neq 2$, there exists an element $a \in K \setminus K_0$ such that $T(a) = t$. Again, we use the quadratic equations (9) and (10) for a to deduce that $\beta(T(a)) = \tilde{T}(\beta(a))$, and hence $\beta(t) \in \tilde{K}_0$; so we have shown that $\beta(K_0) \subseteq \tilde{K}_0$ as well. Also $\beta(N(a)) = \tilde{N}(\beta(a))$, that is, $\beta(aa^\sigma) = \beta(a)\beta(a)^{\tilde{\sigma}}$ for all $a \in K$, and since β is a morphism, it follows that $\beta(a^\sigma) = \beta(a)^{\tilde{\sigma}}$ for all $a \in K$, and we are done. ∎

The case of buildings of type F_4 is now an immediate corollary of the previous lemma.

Theorem 3.10. *Let $\Gamma \cong \mathsf{F}_4(K,\sigma)$ and $\tilde{\Gamma} \cong \mathsf{F}_4(\tilde{K},\tilde{\sigma})$ for some quadratic or honorary involutory sets (K,σ) with norm N and $(\tilde{K},\tilde{\sigma})$ with norm $\tilde{N}$. Then Γ is isomorphic to a subbuilding of $\tilde{\Gamma}$ if and only if there exists a ring monomorphism β from K into $\tilde{K}$ such that an element $a \in K$ lies in $K_0 = \mathrm{Fix}_K(\sigma)$ if and only if $\beta(a)$ lies in $\tilde{K}_0 = \mathrm{Fix}_{\tilde{K}}(\tilde{\sigma})$. Moreover, if (K,σ) is of type* (iv) *or* (v)*, then every ring monomorphism β from K into $\tilde{K}$ satisfies this condition.*

Proof. Let $\Gamma \cong \mathsf{F}_4(K,\sigma)$ and $\tilde{\Gamma} \cong \mathsf{F}_4(\tilde{K},\tilde{\sigma})$ for some quadratic or honorary involutory sets (K,σ) and $(\tilde{K},\tilde{\sigma})$.

Assume first that there exists a ring monomorphism $\beta : K \hookrightarrow \tilde{K}$ such that $\beta(K_0) \subseteq \tilde{K}_0$ and $\beta^{-1}(\tilde{K}_0) \subseteq K_0$. Then it follows from Lemma 3.9 that $\mathsf{C}_3(K,\sigma)$ is isomorphic to a subbuilding of $\mathsf{C}_3(\tilde{K},\tilde{\sigma})$, so in particular, $\pi(1,2)$ is isomorphic to a subsequence of $\tilde{\pi}(1,2)$ and $\pi(2,3)$ is isomorphic to a subsequence of $\tilde{\pi}(2,3)$. Since $\beta(K_0) \subseteq \tilde{K}_0$, we have that K_0 is isomorphic to a subfield of $\tilde{K}_0$, and hence $\pi(3,4) \cong \mathcal{T}(K_0)$ is isomorphic to a subsequence of $\tilde{\pi}(3,4) \cong \mathcal{T}(\tilde{K}_0)$ as well. We conclude that Γ is isomorphic to a subbuilding of $\tilde{\Gamma}$.

Assume now that Γ is isomorphic to a subbuilding of $\tilde{\Gamma}$. Then in particular, $\mathsf{C}_3(K,\sigma)$ is isomorphic to a subbuilding of $\mathsf{C}_3(\tilde{K},\tilde{\sigma})$, and hence the result follows from Lemma 3.9. ∎

We are left with the case that the Coxeter diagram of the building is of type $\mathsf{B}_n = \mathsf{C}_n = \mathsf{BC}_n$.

Lemma 3.11. *A building of type B_n cannot be isomorphic to a subbuilding of a building of type C_n; a building of type C_n cannot be isomorphic to a subbuilding of a building of type B_n; a building of type BC_n can neither be isomorphic to a subbuilding of a building of type B_n nor to a subbuilding of a building of type C_n.*

Proof. This follows from Remark 2.17 by looking at the residues $\pi(n-1,n)$ and $\tilde{\pi}(n-1,n)$. ∎

Theorem 3.12. *Let $\Gamma \cong \mathsf{B}_n(K,V_0,q)$ and $\tilde{\Gamma} \cong \mathsf{B}_n(\tilde{K},\tilde{V}_0,\tilde{q})$ for some anisotropic quadratic spaces (K,V_0,q) and $(\tilde{K},\tilde{V}_0,\tilde{q})$. Then Γ is isomorphic to a subbuilding of $\tilde{\Gamma}$ if and only if there exists a vector space monomorphism (β,α) from (K,V_0) into $(\tilde{K},\tilde{V}_0)$ and a fixed element $\lambda \in \tilde{K}^*$ such that $\tilde{q}(\alpha(v)) = \lambda\beta(q(v))$ for all $v \in V_0$.*

Proof. Let $\Gamma \cong \mathsf{B}_n(K, V_0, q)$ and $\tilde{\Gamma} \cong \mathsf{B}_n(\tilde{K}, \tilde{V}_0, \tilde{q})$ for some anisotropic quadratic spaces (K, V_0, q) and $(\tilde{K}, \tilde{V}_0, \tilde{q})$. Then $\pi(n-1,n) \cong \mathcal{Q}(\Omega_Q(K, V_0, q(e)^{-1}q, e))$ for all $e \in V_0^*$ and $\tilde{\pi}(n-1,n) \cong \mathcal{Q}(\Omega_Q(\tilde{K}, \tilde{V}_0, \tilde{q}(\tilde{e})^{-1}\tilde{q}, \tilde{e}))$ for all $\tilde{e} \in \tilde{V}_0^*$; see Remark 3.7.

Assume first that there exists a vector space monomorphism (β, α) from (K, V_0) into $(\tilde{K}, \tilde{V}_0)$ and a fixed element $\lambda \in \tilde{K}^*$ such that $\tilde{q}(\alpha(v)) = \lambda\beta(q(v))$ for all $v \in V_0$. Then β is a field monomorphism from K into $\tilde{K}$, and it already follows that $\pi(i, i+1)$ is isomorphic to a subsequence of $\tilde{\pi}(i, i+1)$ for all $i \in \{1, \dots, n-2\}$. Now let $e \in V_0^*$ be arbitrary, and let $\tilde{e} := \alpha(e) \in \tilde{V}_0^*$. Let $q' := q(e)^{-1}q$ and let $\tilde{q}' := \tilde{q}(\tilde{e})^{-1}\tilde{q}$. Then $\pi(n-1,n) \cong \mathcal{Q}(\Omega_Q(K, V_0, q', e))$ and $\tilde{\pi}(n-1,n) \cong \mathcal{Q}(\Omega_Q(\tilde{K}, \tilde{V}_0, \tilde{q}', \tilde{e}))$; moreover,

$$\tilde{q}'(\alpha(v)) = \tilde{q}(\tilde{e})^{-1}\tilde{q}(\alpha(v)) = (\lambda\beta(q(e)))^{-1} \cdot \lambda\beta(q(v)) = \beta(q'(v))$$

for all $v \in V_0$. It follows from Theorem 2.14 that $\pi(n-1,n)$ is isomorphic to a subsequence of $\tilde{\pi}(n-1,n)$. Hence Γ is isomorphic to a subbuilding of $\tilde{\Gamma}$.

Assume now that Γ is isomorphic to a subbuilding of $\tilde{\Gamma}$, and let $e \in V_0^*$ and $\tilde{e} \in \tilde{V}_0^*$ be arbitrary. Then it follows from Theorem 2.14 applied on $\pi(n-1,n) \cong \mathcal{Q}(\Omega_Q(K, V_0, q(e)^{-1}q, e))$ and $\tilde{\pi}(n-1,n) \cong \mathcal{Q}(\Omega_Q(\tilde{K}, \tilde{V}_0, \tilde{q}(\tilde{e})^{-1}\tilde{q}, \tilde{e}))$ that there exists a vector space monomorphism (β, α) from (K, V_0) into $(\tilde{K}, \tilde{V}_0)$ such that $\alpha(e) = \tilde{e}$ and $\beta(q(e)^{-1}q(v)) = \tilde{q}(\tilde{e})^{-1}\tilde{q}(\alpha(v))$ for all $v \in V_0$. If we set $\lambda := \beta(q(e))^{-1}\tilde{q}(\tilde{e})$, then we can rewrite this as $\tilde{q}(\alpha(v)) = \lambda\beta(q(v))$ for all $v \in V_0$, which is what we had to show. ∎

Theorem 3.13. *Let $\Gamma \cong \mathsf{C}_n(K, \sigma)$ and $\tilde{\Gamma} \cong \mathsf{C}_n(\tilde{K}, \tilde{\sigma})$ for some involutory sets (K, σ) and $(\tilde{K}, \tilde{\sigma})$ which are proper, quadratic or honorary. Then Γ is isomorphic to a subbuilding of $\tilde{\Gamma}$ if and only if there exists a ring monomorphism α from K into $\tilde{K}$ such that $\alpha \circ \sigma = \tilde{\sigma} \circ \alpha$.*

Proof. Let $\Gamma \cong \mathsf{C}_n(K, \sigma)$ and $\tilde{\Gamma} \cong \mathsf{C}_n(\tilde{K}, \tilde{\sigma})$ for some involutory sets (K, σ) and $(\tilde{K}, \tilde{\sigma})$; let $K_0 := \mathrm{Fix}_K(\sigma)$ and $\tilde{K}_0 = \mathrm{Fix}_{\tilde{K}}(\tilde{\sigma})$.

Assume first that there exists a ring monomorphism α from K into $\tilde{K}$ such that $\alpha \circ \sigma = \tilde{\sigma} \circ \alpha$. In particular, $\pi(i, i+1) \cong \mathcal{T}(K)$ is isomorphic to a subsequence of $\tilde{\pi}(i, i+1) \cong \mathcal{T}(\tilde{K})$ for all $i \in \{1, \dots, n-2\}$. If (K, σ) is proper or quadratic, then it follows from 2.13 that $\pi(n-1,n) \cong \mathcal{Q}(\Omega_I(K, \sigma)^{\mathrm{op}})$ is isomorphic to a subsequence of $\tilde{\pi}(n-1,n) \cong \mathcal{Q}(\Omega_I(\tilde{K}, \tilde{\sigma})^{\mathrm{op}})$. If (K, σ) is honorary, then K is not associative, so neither is $\tilde{K}$; hence $(\tilde{K}, \tilde{\sigma})$ is honorary as well. In this case, it follows from Lemma 3.9 that $\pi(n-1,n) = \mathcal{Q}(\Omega_I(K, K_0, \sigma)^{\mathrm{op}})$ is isomorphic to a subsequence of $\tilde{\pi}(n-1,n) = \mathcal{Q}(\Omega_I(\tilde{K}, \tilde{K}_0, \tilde{\sigma})^{\mathrm{op}})$. In all cases, we can conclude that Γ is isomorphic to a subbuilding of $\tilde{\Gamma}$.

Assume now that Γ is isomorphic to a subbuilding of $\tilde{\Gamma}$. Then in particular, we have that $\pi(n-1,n) \cong \mathcal{Q}(\Omega_I(K, \sigma)^{\mathrm{op}})$ is isomorphic to a subsequence of $\tilde{\pi}(n-1,n) \cong \mathcal{Q}(\Omega_I(\tilde{K}, \tilde{\sigma})^{\mathrm{op}})$. We will distinguish between three different cases, depending on the type of the involutory set (K, σ). Suppose first that (K, σ) is proper; then it follows from Theorem 2.13(i) that there exists a ring monomorphism α from K into $\tilde{K}$ such that $\alpha \circ \sigma = \tilde{\sigma} \circ \alpha$. Suppose next that (K, σ) is quadratic; in particular, (K, σ) is not proper, and moreover, (K, K_0, σ) is one of the types (ii), (iii) or (iv) in Definition 3.4. Then it follows from Theorem 2.13(ii) that there exists a ring monomorphism or a ring anti-monomorphism α from K into $\tilde{K}$ such that $\alpha \circ \sigma = \tilde{\sigma} \circ \alpha$. If (K, σ) is of type (ii) or (iii), then K is abelian, and therefore every ring anti-monomorphism is also a ring monomorphism. Hence we may assume that (K,σ) is of type (iv) and that α is an anti-monomorphism. Let E be a separable quadratic extension field of K_0 which is contained in K, and let e be an element of $(E^\perp)^*$, i.e. an element of K^* such that $T(e,E)=0$. Then every element $x\in K$ can be written in a unique way as $x=a+eb$ for some $a,b\in E$, and $x^\sigma=a^\sigma-eb$. Now let

$$\gamma : K \to K : a+eb \mapsto a+eb^\sigma$$

for all $a,b\in E$. Then it is easily checked that γ is an anti-isomorphism of K which commutes with σ. But then $\alpha\circ\gamma$ is a field monomorphism from K into $\tilde{K}$ such that $(\alpha\circ\gamma)\circ\sigma=\tilde\sigma\circ(\alpha\circ\gamma)$, so $\alpha\circ\gamma$ satisfies the requirements. Suppose finally that (K,σ) is honorary. Then it follows from the fact that $\pi(1,2)\cong\mathcal{T}(K)$ is isomorphic to a subsequence of $\tilde\pi(1,2)\cong\mathcal{T}(\tilde K)$ that there exists a ring monomorphism from K into $\tilde K$, and again we can conclude that $(\tilde K,\tilde\sigma)$ is also an honorary involutory set. It follows from Lemma 3.9 that $\beta\circ\sigma=\tilde\sigma\circ\beta$, so we are done. ∎

Theorem 3.14. *Let* $\Gamma\cong\mathsf{BC}_n(K,\sigma,X_0,p)$ *and* $\tilde\Gamma\cong\mathsf{BC}_n(\tilde K,\tilde\sigma,\tilde X_0,\tilde p)$ *for some proper anisotropic pseudo-quadratic spaces* (K,σ,X_0,p) *and* $(\tilde K,\tilde\sigma,\tilde X_0,\tilde p)$. *Then* Γ *is isomorphic to a subbuilding of* $\tilde\Gamma$ *if and only if there exists a vector space monomorphism* (β,α) *from* (K,V_0) *into* $(\tilde K,\tilde V_0)$ *such that* $\beta\circ\sigma=\tilde\sigma\circ\beta$ *and* $\beta(h(a,b))=\tilde h(\alpha(a),\alpha(b))$ *for all* $a,b\in V_0$.

Proof. Let $\Gamma\cong\mathsf{BC}_n(K,\sigma,X_0,p)$ and $\tilde\Gamma\cong\mathsf{BC}_n(\tilde K,\tilde\sigma,\tilde X_0,\tilde p)$ for some proper anisotropic pseudo-quadratic spaces (K,σ,X_0,p) and $(\tilde K,\tilde\sigma,\tilde X_0,\tilde p)$.

Assume first that there exists a vector space monomorphism (β,α) from (K,V_0) into $(\tilde K,\tilde V_0)$ such that $\beta\circ\sigma=\tilde\sigma\circ\beta$ and $\beta(h(a,b))=\tilde h(\alpha(a),\alpha(b))$ for all $a,b\in V_0$. Then in particular, β is a field monomorphism from K into $\tilde K$, and hence $\pi(i,i+1)\cong\mathcal{T}(K)$ is isomorphic to a subsequence of $\tilde\pi(i,i+1)\cong\mathcal{T}(\tilde K)$ for all $i\in\{1,\dots,n-2\}$. By Theorem 2.16, our assumptions also imply that $\pi(n-1,n)\cong\mathcal{Q}(\Omega_P(K,\sigma,X_0,p)^{\mathrm{op}})$ is isomorphic to a subsequence of the root group sequence $\tilde\pi(n-1,n)\cong\mathcal{Q}(\Omega_P(\tilde K,\tilde\sigma,\tilde X_0,\tilde p)^{\mathrm{op}})$; hence Γ is isomorphic to a subbuilding of $\tilde\Gamma$.

Assume now that Γ is isomorphic to a subbuilding of $\tilde\Gamma$. Then in particular, $\pi(n-1,n)\cong\mathcal{Q}(\Omega_P(K,\sigma,X_0,p)^{\mathrm{op}})$ is isomorphic to a subsequence of $\tilde\pi(n-1,n)\cong\mathcal{Q}(\Omega_P(\tilde K,\tilde\sigma,\tilde X_0,\tilde p)^{\mathrm{op}})$, and hence the result follows again from Theorem 2.16. ∎

Theorem 3.15. *Let* $\Gamma\cong\mathsf{B}_n(K,V_0,q)$ *and* $\tilde\Gamma\cong\mathsf{BC}_n(\tilde K,\tilde\sigma,\tilde X_0,\tilde p)$ *for some anisotropic quadratic space* (K,V_0,q) *and for some proper anisotropic pseudo-quadratic space* $(\tilde K,\tilde\sigma,\tilde X_0,\tilde p)$. *Then* Γ *is isomorphic to a subbuilding of* $\tilde\Gamma$ *if and only if there exists a subspace* $\tilde V_1$ *of* $\tilde V_0$ *such that* $\tilde\Xi_1:=(\tilde K,\tilde\sigma,\tilde V_1,\tilde\pi_{|\tilde V_1})$ *is isomorphic to an enveloping pseudo-quadratic space for the quadratic space* (K,V_0,q).

Proof. Let $\Gamma\cong\mathsf{B}_n(K,V_0,q)$ and $\tilde\Gamma\cong\mathsf{BC}_n(\tilde K,\tilde\sigma,\tilde X_0,\tilde p)$ for some anisotropic quadratic space (K,V_0,q) and some proper anisotropic pseudo-quadratic space $(\tilde K,\tilde\sigma,\tilde X_0,\tilde p)$.

Assume first that there exists a subspace $\tilde V_1$ of $\tilde V_0$ such that the pseudo-quadratic space $\tilde\Xi_1:=(\tilde K,\tilde\sigma,\tilde V_1,\tilde\pi_{|\tilde V_1})$ is isomorphic to an enveloping pseudo-quadratic space for the quadratic space (K,V_0,q). In particular, K is isomorphic to a subfield of $\tilde K$, and hence $\pi(i,i+1)\cong\mathcal{T}(K)$ is isomorphic to a subsequence of $\tilde\pi(i,i+1)\cong\mathcal{T}(\tilde K)$ for all $i\in\{1,\dots,n-2\}$. By Theorem 2.20, our assumptions also imply that

$\pi(n-1,n) \cong \mathcal{Q}(\Omega_Q(K,V_0,q))$ is isomorphic to a subsequence of $\tilde{\pi}(n-1,n) \cong \mathcal{Q}(\Omega_P(\tilde{K},\tilde{\sigma},\tilde{X}_0,\tilde{p})^{\mathrm{op}})$; hence Γ is isomorphic to a subbuilding of $\tilde{\Gamma}$.

Assume now that Γ is isomorphic to a subbuilding of $\tilde{\Gamma}$. Then in particular, $\pi(n-1,n) \cong \mathcal{Q}(\Omega_Q(K,V_0,q))$ is isomorphic to a subsequence of $\tilde{\pi}(n-1,n) \cong \mathcal{Q}(\Omega_P(\tilde{K},\tilde{\sigma},\tilde{X}_0,\tilde{p})^{\mathrm{op}})$, and again, the result follows from Theorem 2.20. ∎

Theorem 3.16. *Let* $\Gamma \cong \mathsf{C}_n(K,\sigma)$ *and* $\tilde{\Gamma} \cong \mathsf{BC}_n(\tilde{K},\tilde{\sigma},\tilde{X}_0,\tilde{p})$ *for some involutory set* (K,σ) *and some proper anisotropic pseudo-quadratic space* (K,σ,X_0,p) *and* $(\tilde{K},\tilde{\sigma},\tilde{X}_0,\tilde{p})$*. Then* Γ *is isomorphic to a subbuilding of* $\tilde{\Gamma}$ *if and only if* Γ *is already isomorphic to a subbuilding of* $\mathsf{C}_n(\tilde{K},\tilde{\sigma})$*.*

Proof. This follows immediately from Theorem 2.15, since the root group systems of the buildings $\mathsf{C}_n(\tilde{K},\tilde{\sigma})$ and $\mathsf{BC}_n(\tilde{K},\tilde{\sigma},\tilde{X}_0,\tilde{p})$ only differ in their $\pi(n-1,n)$-part. ∎

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