Classical Subspaces of Symplectic Grassmannians

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1 Introduction and Basic Concepts

We assume the reader is familiar with the concepts of a *partial linear rank two incidence geometry* $\Gamma = (\mathcal{P}, \mathcal{L})$ (also called a point-line geometry) and the Lie incidence geometries. For the former we refer to articles in [B] and for the latter see the paper [Co].

The *collinearity graph* of Γ is the graph $(\mathcal{P}, \Delta)$ where Δ consists of all pairs of points belonging to a common line. For a point $x \in \mathcal{P}$ we will denote by $\Delta(x)$ the collection of all points collinear with x. For points $x, y \in \mathcal{P}$ and a positive integer t a *path of length* t from x to y is a sequence $x_0 = x, x_1, \ldots, x_t = y$ such that $\{x_i, x_{i+1}\} \in \Delta$ for each $i = 0, 1, \ldots, t-1$. The *distance* from x to y, denoted by $d(x, y)$ is defined to be the length of a shortest path from x to y if some path exists and otherwise is $+\infty$.

By a *subspace* of Γ we mean a subset S such that if $l \in \mathcal{L}$ and $l \cap S$ contains at least two points, then $l \subset S$. $(\mathcal{P}, \mathcal{L})$ is said to be *Gamma space* if, for every $x \in \mathcal{P}, \{x\} \cup \Delta(x)$ is a subspace. A subspace S is *singular* provided each pair of points in S is collinear, that is, S is a clique in the collinearity graph of Γ. For a Lie incidence geometry with respect to a "good node" every singular subspace, together with the lines it contains, is isomorphic to a projective space, see [Co]. Clearly the intersection of subspaces is a subspace and consequently it is natural to define the subspace generated by a subset X of $\mathcal{P}$, $\langle X \rangle_\Gamma$, to be the intersection of all subspaces of Γ which contain X. Note that if $(\mathcal{P}, \mathcal{L})$ is a Gamma space and X is a clique then $\langle X \rangle_\Gamma$ will be a singular subspace.

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1.1 The Grassmannian Geometries

Let $\mathbb{F}$ be a field. Let V be a vector space of dimension m over $\mathbb{F}$. For $1 \leq i \leq m-1$, let $L_i(V)$ be the collection of all $i-$dimensional subspaces of V. Now fix $j, 2 \leq j \leq m-2$ and set $\mathcal{P} = L_j(V)$.

For pairs (C, A) of incident subspaces of V with $dim(A) = a$, $dim(C) = c$ let $S(C, A)$ consist of all the $j-$subspaces B of V such that $A \subset B \subset C$.

Finally, let $\mathcal{L}$ consist of all the sets $S(C, A)$ where $dim\ A = j-1, dim\ C = j+1$. The rank two incidence geometry $(\mathcal{P}, \mathcal{L})$ is the incidence geometry of $j-$Grassmannians of V, denoted by $\mathcal{G}_j(V)$. We also use the notation $\mathcal{G}_{m,j}(\mathbb{F})$ for the isomorphism type of this geometry.

1.2 The Symplectic Grassmannians

Now let W be a space of dimension $2n$ over the field $\mathbb{F}$, f a non-degenerate alternating form on W so that (W, f) is a non-degenerate symplectic space. For $X \subset W$ let $X^{\perp} = \{w \in W : f(x, w) = 0, \forall x \in X\}$. Recall that a subspace U of W is *totally isotropic* if $U \subset U^{\perp}$.

For $1 \leq k \leq n$, let $\mathcal{I}_k$ consist of all totally isotropic $k-$dimensional subspaces of W. Fix k with $2 \leq k \leq n-1$ and set $P = \mathcal{I}_k$. For a pair of subspaces $C \subset D \subset C^{\perp}$ (so C is totally isotropic) where dim $C = c < k < d = D = d$ let $T(D, C)$ consist of all the $k-$dimensional totally isotropic subspaces U such that $C \subset U \subset D$. When $c = k-1, d = k+1$ we set $l(D, C) = T(D, C)$ and set $L = \{l(D, C) : C \subset D$, dim $C = k-1$, dim $D = k+1\}$. In this way we obtain another rank 2 incidence geometry $G = (P, L)$ which we refer to as the symplectic $k-$Grassmannians of W. We denote the isomorphism type of this geometry by $C_{n,k}(\mathbb{F})$. Note that two totally isotropic $k-$subspaces are on a line if they span a totally isotropic $k+1$ dimensional totally isotropic subspace. We remark that the automorphism group of the geometry (P, L) is isomorphic to $PSP_{2n}(\mathbb{F})$.

1.3 Grassmannian Subspaces of Symplectic Grassmannians

When $E \subset F \subset W$, $dim\ E = e, dim\ F = f$ satisfy $e < k-1, f > k+1$ with E, F totally isotropic, the collection $T(E, F)$ is a subspace of (P, L) and is isomorphic to an ordinary Grassmannian geometry $\mathcal{G}_{f-e,k-e}(\mathbb{F})$. Such a subspace is called “parabolic” since the stablizer in $Aut(P, L)$ is a parabolic subgroup of $Aut(P, L)$. It is natural to ask: Is every subspace of $C_{n,k}(\mathbb{F})$ which is isomorphic to some $\mathcal{G}_{m,j}(\mathbb{F})$ parabolic?

Actually, this is not quite the case as the following example demonstrates:

Assume $char(\mathbb{F}) = 2$ and $n - k \geq 2$. Let U be a totally isotropic subspace of dimension $k-1$. Then $T(U^{\perp}, U)$ is a subspace of G and is isomorphic to a symplectic polar space of rank $n-k+1 \geq 3, C_{n-k+1,1}(\mathbb{F})$. Since the characteristic is two this is isomorphic to the orthogonal polar space, $B_{n-k+1,1}(\mathbb{F})$ (the space of singular points and totally singular lines in a non-singular orthogonal space on a vector space of dimension $2(n-k+1)+1$. In turn this contains subspaces which are isomorphic to the hyperbolic orthogonal space on a vector space of dimension six, $D_{3,1}(\mathbb{F})$. However, this is isomorphic to $\mathcal{G}_{4,2}(\mathbb{F})$ via the Klein correspondence. As we shall show in our main theorem, apart from the parabolic subspaces, these are the only other examples of Grassmannians subspaces of a symplectic Grassmannian:

Main Theorem: *Let S be a subspace of $C_{n,k}(\mathbb{F}), S \cong \mathcal{G}_{m,j}(\mathbb{F})$.*

Then either S is parabolic or else $char(\mathbb{F}) = 2, (m,j) = (4,2)$ and S is a subspace of $T(U^{\perp}, U)$ for some totally isotropic subspace $U, dim\ U = k-1$. Moreover, if Y is the subspace spanned by all the elements of S then $dim\ Y/U = 6$.

Before proceeding to the proofs we introduce some notation:

Notation: Since we will generate all kinds of subspaces, of W the symplectic space, of the geometry (P, L), etc. we need to distinguish between these. When X is some collection of subspaces or vectors from W we will denote the subspace of W spanned by X by $\langle X\rangle_{\mathbb{F}}$. When X is a subset of P we will denote the subspace (P, L) generated by X by $\langle X\rangle_G$. And, when X is a subset of $(\mathcal{P}, \mathcal{L})$ we will denote the subspace of this geometry generated by X by $\langle X\rangle_\Gamma$.

For a point $p \in P$ we will denote by $\Delta(x)$ the collection of all points of P which are collinear with x in (P, L) (including p). For a point $p \in \mathcal{P}$ we will use $\gamma(p)$ to indicate the points of $\mathcal{P}$ where are collinear with p.

2 Properties of Grassmannians

In this short section we recall some properties of a Grassmannian incidence geometry $\mathcal{G}_j(V) \cong \mathcal{G}_{m,j}(\mathbb{F})$. We omit the proofs because they are either well known or entirely straightforward to prove.

Lemma 2.1. *i) There are two classes of maximal singular subspaces of $(\mathcal{P}, \mathcal{L})$ with representatives $S(V, D)$ where $dim\ D = j-1$ and $S(E, 0)$ where $dim\ E = j+1$. $S(V, D) \cong \mathbb{PG}_{m-j}(\mathbb{F})$ and $S(E, 0) \cong \mathbb{PG}_j(\mathbb{F})$. Those of the first class will be referred to as type one and the second class as type two.*

ii) If M_1 and M_2 are maximal singular subspaces and $M_1 \cap M_2$ is a line then M_1 and M_2 are in different classes. If $M_1 \cap M_2$ is a point then they are in the same class.

Lemma 2.2. *Let M be a maximal singular subspace of $\mathcal{P}$ of type one. Then $\langle M\rangle_{\mathbb{F}} = V$.*

Now let U be a hyperplane of V and X a one space of V, X not contained in U. Set $\mathcal{P}(U) = \{x \in \mathcal{P} : x \subset U\}$ and $\mathcal{P}_X = \{x \in \mathcal{P} : X \subset x\}$.

Lemma 2.3. *i) $\mathcal{P}(U)$ is a subspace of $\mathcal{P}$ and $\mathcal{P}(U) \cong \mathcal{G}_{m-1,j}(\mathbb{F})$.*

ii) $\mathcal{P}_X$ is a subspace of $\mathcal{P}$ and $\mathcal{P}_X \cong \mathcal{G}_{m-1,j-1}(\mathbb{F})$.

iii) If $x \in \mathcal{P}(U)$ then $\gamma(x) \cap \mathcal{P}_X$ is a maximal singular subspace of $\mathcal{P}_X$ isomorphic to $\mathbb{PG}_{m-j-1}(\mathbb{F})$. Furthermore, $\langle x, \gamma(x) \cap \mathcal{P}_X \rangle_{\mathcal{G}_{m,j}}$ is a maximal singular subspace of $\mathcal{P}$.

iv) If $y \in \mathcal{P}_X$ then $\gamma(y) \cap \mathcal{P}(U)$ is a maximal singular subspace of $\mathcal{P}(U)$ isomorphic to $\mathbb{PG}_{j-1}(\mathbb{F})$. Futhermore, $\langle y, \gamma(y) \cap \mathcal{P}(U) \rangle_{\mathcal{G}_{m,j}}$ is a maximal singular subspace of $\mathcal{P}$.

v) If $x_1, x_2 \in \mathcal{P}(U)$ are collinear then $\gamma(x_1) \cap \gamma(x_2) \cap \mathcal{P}_X$ is a point. Similarly, if $y_1, y_2 \in \mathcal{P}_X$ are collinear then $\gamma(y_1) \cap \gamma(y_2) \cap \mathcal{P}(U)$ is a point.

Lemma 2.4. *The diameter of the collinearity graph of $\mathcal{G}_j(V)$ is $min\{j, m-j\}$. For $x, y \in \mathcal{P}, d(x, y) = dim\ (x/x \cap y) = dim\ (y/x \cap y)$.*

3 Properties of Symplectic Grassmannians

In this short section we review some properties of symplectic Grassmannians. As with the case of ordinary Grassmannians we omit the proofs because these are either well known or easy to prove.

Lemma 3.1. *i) The symplectic Grassmannian space $(P, L) \cong C_{n,k}(\mathbb{F})$ has two classes of maximal singular subspaces with representatives $T(B, 0)$ where B is a totally isotropic subspace of $W, dim\ B = k+1$, and $T(C, A)$ where A and C are incident totally isotropic subspaces of W, where $dim\ A = k-1, dim\ C = n$. In the former case $T(B, 0) \cong \mathbb{PG}_k(\mathbb{F})$ and in the latter $T(C, A) \cong \mathbb{PG}_{n-k}(\mathbb{F})$. We refer to the first as type one maximal singular subspaces and the latter as type two.*

ii) If M_1 and M_2 are maximal singular subspaces of different types then either $M_1 \cap M_2$ is empty or a line.

iii) If M_1 and M_2 are type one maximal singular subspaces then $M_1 \cap M_2$ is either empty or a point.

Definition

A symp of (P, L) is a maximal geodesically closed subspace which is isomorphic to a polar space.

Lemma 3.2. *There are two classes of symps in (P, L). One class has representative $T(E, D)$ where $D \subset E$ are totally isotropic subspaces, $dim\ D = k-2, dim\ E = k+2$. In this case $T(E, D) \cong D_{3,1}(\mathbb{F})$ the polar space of a non-degenerate six dimensional orthogonal space with maximal Witt index. The second class has representative $T(C^\perp, C)$ where C is a totally isotropic subspace, $dim\ C = k-1$. In this case $T(C^\perp, C)$ is isomorphic to the polar space of a non-degenerate symplectic space of dimension $2(n-k+1)$. We refer to the former as a type one symp and the latter as a type two symp.*

Lemma 3.3. *There are two classes of points at distance two in $G = (P, L)$. For one pair (x, y) as subspaces of $W, dim\ (x \cap y) = k-2$ and $x \perp y$. The unique symp on $\{x, y\}$ is $T(x+y, x \cap y)$. For a representative (x, y) of the second type, $dim\ (x \cap y) = k-1$ and $(x+y)/(x \cap y)$ is a non-degenerate two space. The unique symp on such a pair is $T((x \cap y)^\perp, x \cap y)$.*

4 Proof of the Main Theorem

In this section we prove our main theorem. Our proof is by induction on $N = n + k + m + min\{j, m - j\}$.

Lemma 4.1. *If $S \cong \mathcal{G}_{4,2}(\mathbb{F})$ then the main theorem holds.*

Proof: Assume $S \cong \mathcal{G}_{4,2}(\mathbb{F})$. Since S is a polar space it is contained in some symp $\mathcal{S}$ of (P, L). By Lemma (3.2) there are two possibilities for $\mathcal{S}$: either there are totally isotropic subspaces $D \subset E, dim\ D = k - 2, dim\ E = k + 2$ with $\mathcal{S} = T(E, D)$ or there is a totally isotropic subspace $C, dim\ C = k - 1$ such that $\mathcal{S} = T(C^{\perp}, C)$. In the former case, since $S \cong T(E, D)$ we get equality and the main theorem holds. In the latter case, let $U = \langle S \rangle_{\mathbb{F}}$ a vector subspace of $C^{\perp}$. The map taking $x \in S$ to x/C is an embedding of the polar space S into $\mathbb{PG}(U/C)$. Since S is strongly hyperbolic (see [CS]) it follows that $dim\ U/C = 6$. Because we have an embedding from the orthogonal polar space S into the symplectic polar space $T(C^{\perp}, C)$ it must also be the case that $char(\mathbb{F}) = 2$ which is one of the conclusions of the theorem. ∎

Lemma 4.2. *Assume that $min\{j, m - j\} = 2$. Then the theorem holds.*

Proof: Let S' be a subspace of $S, S' \cong \mathcal{G}_{5,2}(\mathbb{F})$ and let $\mathcal{D}$ be a symp of S'. Since $\mathcal{D}$ is a polar space it is contained in a symp of G. Suppose $\mathcal{D}$ is contained in a type two symp $T(C^{\perp}, C)$, C a totally isotropic subspace of $W, dim\ C = k - 1$. Now for every point $x \in S' \setminus \mathcal{D}, \Delta(x) \cap \mathcal{D}$ is a maximal singular subspace of $\mathcal{D}$ (and isomorphic to $\mathbb{PG}_2(\mathbb{F})$). The subspace $\Delta(x) \cap \mathcal{D} \subset T(C^{\perp}, C)$ is a projective plane. Let M be a maximal singular subspace of $T(C^{\perp}, C)$ containing $\Delta(x) \cap \mathcal{D}$. Then it follows from Lemma (3.1) that M is a type two maximal singular subspace of G. Now by Lemma (3.1) (ii), $x \in M \subset T(C^{\perp}, C)$. Since the point $x \in S' \setminus \mathcal{D}$ was arbitrary, it follows $S' \subset T(C^{\perp}, C)$. However, since S' is not a polar space we have a contradiction.

As a consequence of this argument, all the symps of S are type one symps of G. From this it follows that if $x, y \in S, d(x, y) = 2$ then as subspaces of W we have $x \perp y$ by Lemma (3.3). Since the diameter of S is two by Lemma (2.4) it then follows that $Y = \langle S \rangle_{\mathbb{F}}$ is a totally isotropic subspace of W. Consequently, $S \subset T(Y, 0)$. By Theorem (2.15) of [CKS] it follows that S is parabolic and the theorem holds. ∎

The completion of the proof

It now follows that we may assume that $m \geq 6$ and $min\{j, m-j\} \geq 3$. We continue with the notation of the introduction where V was introduced as an $m-$ dimensional vector space and $(\mathcal{P}, \mathcal{L})$ is the Grassmannian geometry of $j-$dimensional subspaces of V. Let $\tau : \mathcal{P} \to S$ be an isomorphism of geometries. As in section two let U be a hyperplane of V and X a one-dimensional subspace of V such that X is not contained in U and set $\mathcal{P}(U) = \{x \in \mathcal{P} : x \subset U\}$ and $\mathcal{P}_X = \{x \in \mathcal{P} : X \subset x\}$. Also, set $S_1 = \tau(\mathcal{P}(U))$ and $S_2 = \tau(\mathcal{P}_X)$.

Since $S_1 \cong \mathcal{G}_{m-1,j}(\mathbb{F})$ and $(m-1)+min\{j, m-1-j\} < m+min\{j, m-j\}$ it follows by our induction hypothesis that $S_1 = T(B_1, A_1)$ where $A_1 \subset B_1$ are totally isotropic subspaces with $dim\ A_1 = a_1, dim\ B_1 = b_1$ and $m-1 = b_1 - a_1, j = k - a_1$.

Similarly, since $S_2 \cong \mathcal{G}_{m-1,j-1}(\mathbb{F})$ and $(m-1)+min\{j-1, (m-1)-(j-1)\} < m+min\{j, m-j\}$ it follows that $S_2 = T(B_2, A_2)$ where $A_2 \subset B_2$ are totally isotropic subspaces with $dim\ A_2 = a_2, dim\ B_2 = b_2$ and $m-1 = b_2 - a_2, j-1 = k - a_2$.

Let $x \in S_1, y \in S_2, x, y$ collinear. Then by Lemma (2.3), $U_1 = \langle x, S_2 \cap \Delta(x)\rangle_G$ and $U_2 = \langle y, S_1 \cap \Delta(x)\rangle_G$ are maximal singular subspaces of S which meet in a line.

Let M_i be a maximal singular subspace of P containing $U_i, i = 1, 2$. Then M_1 and M_2 come from different classes of G by Lemma (3.1). Consequently, at least one of M_1, M_2 is of type 2. For the sake of argument, assume M_1 is of type 2. Then there is a maximal totally isotropic subspace B and a $(k-1)-$dimensional subspace $A \subset B$ such that $M_1 = T(B, A)$.

$M_1 \cap S_2 = T(B, A) \cap T(B_2, A_2) = S_2 \cap \Delta(x)$ is a maximal singular subspace of S_2. It follows that $B_2 \subset B$ and that $M_1 \cap S_2 = T(B_2, A)$. Then $B_2 = \langle M_1 \cap S_2\rangle_{\mathbb{F}} = \langle S_2 \cap \Delta(x)\rangle_{\mathbb{F}}$ by Lemma (2.2) which implies that $B_2 \subset x^{\perp}$ since $y' \in \Delta(x)$ implies $x \perp y'$.

Now assume that $x' \in S_1$ such that x', x are collinear. Then by Lemma (2.3) it follows that $S_2 \cap \Delta(x)$ and $S_2 \cap \Delta(x')$ are in the same class of maximal singular subspaces of S_2. Therefore it also follows that $B_2 \subset (x')^{\perp}$.

Since the collinearity graph of S_1 is connected, it follows that for all $z \in S_1, B_2 \subset z^{\perp}$. Since $\langle S_1\rangle_{\mathbb{F}} = B_1$ we have $B_1 \perp B_2$.

Set $D = B_1 + B_2$, a totally isotropic subspace. Now $S_1, S_2 \subset T(D, 0)$. Since $\langle S_1, S_2\rangle_G = S$ (as follows from [BB], [CoSh], [RS]), it follows that $S \subset T(D, 0)$, for, if x, y are collinear points of P and $x, y \subset D$, then for every $z \in T(x+y, x \cap y)$ also $z \subset D$. Now we are done by Theorem (2.15) of [CKS].

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