Minimal covering of all chords of a conic in $PG(2, q)$, q even

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Abstract

In this paper we determine the minimal blocking sets of chords of an irreducible conic $\mathcal{C}$ in the desarguesian projective plane $PG(2, q)$, q even. Similar results on blocking sets of external lines, as well as of nonsecant lines, are given in [1], [3], and [2].

1 Introduction

In this paper a purely combinatorial question concerning a conic $\mathcal{C}$ in $PG(2, q)$ with q even, is investigated, namely the classification of all point sets of minimum size in $PG(2, q)$ that meet every chord of $\mathcal{C}$. It is easy to see that such a point set $\mathcal{B}$ (also called a *minimal blocking set of chords* of $\mathcal{C}$) has size q. Now, we describe a procedure for the construction of minimal blocking sets of chords. Assume that the conic $\mathcal{C}$ has (affine) equation $Y = X^2$, that is, $\mathcal{C}$ is a parabola in the affine plane $AG(2, q)$. For every $a \in GF(q)$,

$$\varphi_a : (X, Y) \longrightarrow (X + a, Y + a^2)$$

is a translation of the affine plane $AG(2, q)$. The center of φ_a, viewed as an elation in the projective closure $PG(2, q)$ of $AG(2, q)$, is the infinite point $B_a = (1, a, 0)$. The translation group of $\mathcal{C}$ is $T = \{\varphi_a \mid a \in GF(q)\}$ and it is isomorphic to the additive group $(GF(q), +)$ of $GF(q)$. Take a subgroup $G = \{\varphi_a \mid a \in H\}$ of T where H is a subgroup in $(GF(q), +)$, and define Γ to be the set of all centers of all non-trivial translations in G. If $P = (u, u^2)$ is an affine point in $\mathcal{C}$, the orbit of P under G is

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$\Delta_u = \{(a+u,(a+u)^2) \mid a \in H\}$. Then, $\mathcal{B}(G,u) = (\mathcal{C} \setminus \Delta_u) \cup \Gamma$ is a blocking set of chords of $\mathcal{C}$.

The following theorem is the main result in the present paper.

Theorem 1.1. *Let $\mathcal{C}$ be an irreducible conic of $PG(2,q)$, with $q = 2^h$, and let $\mathcal{L}$ be the set of all chords of $\mathcal{C}$. Any point set $\mathcal{B}$ of $PG(2,q)$ meeting every line of $\mathcal{L}$ has size at least q. If equality holds then $\mathcal{B} = \mathcal{B}(G,u)$ for some $\mathcal{B}(G,u)$ arising from an additive subgroup H of $GF(q)$ as in the above construction.*

Two blocking sets of chords are linearly equivalent if there is a linear collineation preserving $\mathcal{C}$ which sends one to the other. Since T acts transitively on the affine points of $\mathcal{C}$, while fixing the infinite line pointwise, there is a translation in T which sends $\mathcal{B}(G,u)$ to $\mathcal{B}(G,u')$ for any two $u, u' \in GF(q)$. So, $\mathcal{B}(G,u)$ and $\mathcal{B}(G,u')$ are equivalent. Furthermore, for two additive subgroups H and H' of $GF(q)$, the corresponding blocking sets $\mathcal{B}(G,u)$ and $\mathcal{B}(G',u)$ are equivalent if and only if there is an affinity preserving $\mathcal{C}$ which sends Δ to Δ'. This occurs when G and G' have not only the same order, but they are also conjugate subgroups in the affine group $AGL(1,q)$ of the parabola $\mathcal{C}$.

2 Proof of Theorem 1.1

We keep the notations defined in the introduction. Furthermore, if A and B are two distinct points, AB stands for the line through them.

We begin by noting that $\mathcal{B}$ can coincide with $\mathcal{C}$, and if this occurs then $\mathcal{B}$ has size $q+1$. We assume that $\mathcal{C}$ has a point not lying on $\mathcal{B}$. If $A \in \mathcal{C} \setminus \mathcal{B}$ then every chord of $\mathcal{C}$ through A meets $\mathcal{B}$ in some point. Since distinct chords through A meet $\mathcal{B}$ in distinct points, $|\mathcal{B}| \geq q$ follows.

From now on, we assume that the size of $\mathcal{B}$ attains the lower bound. Since each point outside $\mathcal{C}$ and different from its nucleus lies on exactly $\frac{1}{2}q$ chords of $\mathcal{C}$, any point set of size q which is disjoint from $\mathcal{C}$ meets at most $\frac{1}{2}q^2$ chords. Hence $\mathcal{B}$ contains some point from $\mathcal{C}$.

Therefore, $\mathcal{B}$ splits into two non-empty subsets, namely $\Gamma = \mathcal{B} \setminus \mathcal{C}$ and $\Sigma = \mathcal{B} \cap \mathcal{C}$. Set $\Delta = \mathcal{C} \setminus \Sigma$. Note that every chord of Δ meets Γ, and that $|\Gamma| = |\Delta| - 1$. Since $\mathcal{B}$ has size q, a counting argument shows that

($*$) *every chord of Δ meets $\mathcal{B}$ in exactly one point.*

Now, fix one point A of Δ and consider the $|\Delta|-1$ secants through A. They all contain one point of Γ. Since $|\Gamma| = |\Delta| - 1$, and since all secants to $\mathcal{C}$ through A contain precisely one point of $\mathcal{B}$, it follows that a point of Γ only lies on secants to Δ. This implies that $|\Delta|$ is even.

It is easily seen that if $|\Delta| < 4$, then either

(i) $\mathcal{B}$ consists of all points of $\mathcal{C}$ minus one;

(ii) $\mathcal{B}$ arises from $\mathcal{C}$ and a chord r of $\mathcal{C}$ by replacing their two common points by a point of r outside $\mathcal{C}$.

From now on we assume $|\Delta| \geq 4$.

Lemma 2.1. *Let A_1, A_2, A_3 be distinct points of Δ. Then the points $B_1 = \Gamma \cap A_2A_3$, $B_2 = \Gamma \cap A_1A_3$ and $B_3 = \Gamma \cap A_1A_2$ are collinear.*

Proof. This proof follows the idea introduced by B. Segre in [5]. We choose a homogeneous coordinate system in such a way that $A_1 = (1,0,0)$, $A_2 = (0,1,0)$ and $A_3 = (0,0,1)$. Then

$$B_1 = (0, b_1, 1),\ \ B_2 = (1, 0, b_2),\ \ B_3 = (b_3, 1, 0),$$

for some $b_1, b_2, b_3 \in GF(q)$ with $b_1b_2b_3 \neq 0$.

Let $P = (p_0, p_1, p_2)$ be a point of $\mathcal{B}$ other than B_1, B_2, B_3. Then lines through P and the A_i's are respectively

$$X_2 = \alpha_P X_3, \qquad X_3 = \beta_P X_1 \qquad X_1 = \gamma_P X_2,$$

where $\alpha_P = p_1/p_2$, $\beta_P = p_2/p_0$, $\gamma_P = p_0/p_1$ and so, by Ceva's Theorem,

$$\alpha_P \beta_P \gamma_P = 1. \tag{1}$$

Let $\tau_1 : X_2 = t_1X_3$, $\tau_2 : X_3 = t_2X_1$ and $\tau_3 : X_1 = t_3X_2$ be the tangents to $\mathcal{C}$ at the points A_i. For P in $\mathcal{B}$ other than B_1, B_2, B_3, the coefficients α_P of the $q-3$ lines through A_1 and P assume exactly once all non-zero values in $GF(q)$ other than t_1 and b_1. The product of all the non-zero elements of $GF(q)$ is -1, so

$$t_1 b_1 \prod_P \alpha_P = -1. \tag{2}$$

Similarly, for the $q-3$ lines through A_2 and A_3 other than the sides of the triangle of reference and the tangents t_2 and t_3, $t_2b_2 \prod_P \beta_P = -1$ and $t_3b_3 \prod_P \gamma_P = -1$. Hence,

$$t_1t_2t_3b_1b_2b_2 \prod_P \alpha_P\beta_P\gamma_P = (-1)^3 = -1.$$

Since, $\alpha_P\beta_P\gamma_P = 1$ for each P by (1), then

$$t_1b_1t_2b_2t_3b_3 = 1.$$

Furthermore, as q is even, the tangent lines τ_1, τ_2, τ_3 are concurrent at the nucleus of $\mathcal{C}$. Thus $t_1t_2t_3 = 1$. Therefore, $b_1b_2b_3 = -1 = 1$ and this implies that the points B_1, B_2, B_3 are collinear. ∎

Now, we assume that Γ contains two points B_1, B_2 not lying on the same tangent to $\mathcal{C}$. Since $\mathcal{B}$ is a blocking set with respect to the chords of $\mathcal{C}$, some chord ℓ of Δ passes through B_1. Let A_2, A_3 denote the common points of ℓ and Δ. The line B_2A_3 necessarily meets Δ in a further point A_1. Let B_3 be the common point of Γ and the line A_1A_2. By Lemma 2.1, B_1, B_2, B_3 are collinear points.

We say that the triangle $A_1A_2A_3$ is associated with the pair $\{B_1, B_2\}$. Furthermore, we note that for every chord ℓ of Δ through B_1 there are two distinct triangles associated with $\{B_1, B_2\}$ and sharing ℓ, according as the point A_1 arises from the line B_2A_3 or from B_2A_2.

We first prove that different triangles associated with $\{B_1, B_2\}$ define different points in Γ on the line B_1B_2 .

Let $A_1^*A_2^*A_3^*$ be a triangle associated with $\{B_1, B_2\}$ and different from $A_1A_2A_3$ such that $B_3^* = A_1^*A_2^* \cap \Gamma = A_1A_2 \cap \Gamma = B_3$. There are three possibilities.

case (1): *The triangles $A_1A_2A_3$ and $A_1^*A_2^*A_3^*$ have no common vertex.*

By Desargues' Theorem the lines $A_1A_1^*$, $A_2A_2^*$, $A_3A_3^*$ are concurrent at a point V which is neither on $\mathcal{C}$ nor the nucleus of $\mathcal{C}$.

Let h denote the involutory perspectivity with center V which preserves $\mathcal{C}$. Since h sends A_i to A_i^* for $i = 1, 2, 3$, it turns out that h fixes B_1, B_2 and B_3. As q is even, h is an elation and its axis is a tangent line t to $\mathcal{C}$. Since the fixed points of h lie on t, it turns out that both B_1 and B_2 lie on t, a contradiction.

case (2): *The triangles $A_1A_2A_3$ and $A_1^*A_2^*A_3^*$ share one side.*

We suppose that $A_2A_3 = A_2^*A_3^*$. Therefore $B_2 \in A_1A_3 \cap A_1^*A_2$ and $B_3 \in A_1A_2 \cap A_1^*A_3$. As q is even, the diagonal points of the complete quadrilateral $A_1A_2A_3A_1^*$ are collinear, see Thm. 2.24 in [4] pag. 43, and hence, we get $B_1 \in A_2A_3 \cap A_1A_1^*$. Let h be the involutory perspectivity preserving $\mathcal{C}$ with center at one of the diagonal points, say B_1. As before h is an elation whose axis is the tangent t to $\mathcal{C}$ through B_1. Since the points B_2 and B_3 are also fixed by h, it follows that B_1 and B_2 must lie on t, a contradiction.

case (3): *The triangles $A_1A_2A_3$ and $A_1^*A_2^*A_3^*$ have a common vertex .*

We may assume $A_2 = A_2^*$. Then, also $A_3 = A_3^*$ and we are in the case (2). Therefore distinct triangles associated with the same pair $\{B_1, B_2\}$ define distinct points in $B_1B_2 \cap \Gamma$ other than B_1 and B_2. Since through B_1 there are $|\Delta|/2$ chords of Δ, we get at least $|\Delta| + 2$ distinct points in Γ, a contradiction. Hence, the points in Γ are collinear and lie on the tangent to $\mathcal{C}$ through B_1.

We now show that the points in Γ are the centers of the non-trivial elations of a group of elations fixing $\mathcal{C}$ and fixing Δ. We choose a homogeneous coordinate system in such a way that $\mathcal{C}$ has equation $X_2X_3 = X_1^2$ and that the line B_1B_2 is the infinite line ℓ_∞: $X_3 = 0$.

Let $B_a = (1, a, 0)$ be a point in ℓ_∞ with $a \in GF(q)$. We denote by φ_a the elation with center at B_a preserving $\mathcal{C}$ which maps the point $(t, t^2, 1) \in \mathcal{C}$ to the point $(t + a, t^2 + a^2, 1) \in \mathcal{C}$. Let $G = \{\varphi_a | \Delta^{\varphi_a} = \Delta\}$.

Clearly, a non-trivial elation φ_a is in G if and only if $B_a = (1, a, 0)$ is in Γ. We show that for each $B_a, B_b \in \Gamma$, the elation $\varphi_a \circ \varphi_b$ has center in Γ. In fact $\varphi_a \circ \varphi_b$ maps the point $(t, t^2, 1) \in \Delta$ to the point $(t + a + b, t^2 + a^2 + b^2, 1) \in \Delta$, whence $\varphi_a \circ \varphi_b = \varphi_{a+b}$.

Therefore, the points in Γ are the centers of the non-trivial elations of a group of elations fixing $\mathcal{C}$ and fixing Δ, and G is a group of order $|\Gamma| + 1$, isomorphic to a subgroup of the additive group $(\mathrm{GF}(q), +)$. More precisely, G is an elementary abelian group of order 2^s, for some $s \leq h$.

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