Explicit formula for a fundamental class of functions

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Abstract

The purpose of this paper is to prove an analogue of A. Weil's explicit formula for a fundamental class of functions, i.e. the class of meromorphic functions that have an Euler sum representation and satisfy certain a functional equation. The advance of this explicit formula is that it enlarges the class of allowed test functions, from the class of functions with bounded Jordan variation to the class of functions of ϕ-bounded variation. A condition posed to the test function at zero is also reconsidered.

1 Introduction

It is well known that the Euler product formula $\sum \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$, where s is a complex number with $\mathrm{Re}\,(s) > 1$ and the product is taken over all primes served as a starting point for Riemann's investigation of a function widely known as the Riemann ζ function ever since. The function ζ satisfies the functional equation $\varsigma\,(s) = \eta\,(s)\,\varsigma\,(1-s)$,where $\eta\,(s) = \pi^{-s+\frac{1}{2}} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}$. The function $\eta\,(s)$ is a ratio of two functions that possess a representation in a form of canonical Weierstrass product.

A generalization of these properties led J. Jorgenson and S. Lang [7] to the concept of a fundamental class of functions - the class of meromorphic functions whose logarithmic derivative has an Euler sum representation and that satisfy a functional equation with fudge factors of a regularized product type.

Explicit formulas in classical number theory assert that the sum of a certain test function over the prime powers is equal to the sum of its transform usually taken

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over the zeros of the Riemann zeta function. A. Weil in [9] was the first to point out that these formulas can be stated more generally. He proved that the sum of a suitably smooth test function taken over the prime powers of the number field k is equal to the sum of its Mellin transform taken over the non-trivial zeros of the related Hecke $L-$ function, plus an analytic term at infinity. This term is called Weil's functional and may be viewed as a functional evaluated on a test function.

The purpose of this paper is to prove an analogue of the Weil explicit formula for a fundamental class of functions and a larger class of test functions. The advance in this explicit formula consists in weakening growth conditions posed on the test function F. Namely, J. Jorgenson and S. Lang in [7] required that F satisfy the following conditions:

JL1 $F \in BV(\mathbb{R}) \cap L^1(\mathbb{R})$

JL2 There exists $\varepsilon > 0$ such that $F(x) - F(0) = O(|x|^{\varepsilon}) \quad (x \to 0)$

JL3 $F(x)\, e^{\left(\frac{\sigma_0}{2}+a'\right)|x|} \in BV(\mathbb{R})$ for some $a' > 0$ and $\sigma_0 > 0$ to be defined in the Euler sum property.

In [2] we proved a general Parseval formula that provides an evaluation of the term at infinity under less restrictive conditions than JL1 and JL2.

In this paper we will derive an explicit formula under the conditions of [2] and will further weaken the condition JL3, replacing Jordan variation by $\phi-$variation.

2 Generalized variation, Stieltjes integral and Fourier transform

The universal class of this paper is the class W of regulated functions [4] i.e. functions possessing the one - sided limits at each point. For $f \in W$, we always suppose $2f(x) = f(x+0) + f(x-0)$. If I is an interval with endpoints a and b $(a < b)$, we write $f(I) = f(b) - f(a)$.

Let ϕ be a continuous function defined on $[0,\infty)$ and strictly increasing from 0 to ∞. A function f is said to be of $\phi-$bounded variation on I if

$$V_{\phi}(f, I) = \sup \sum_{n} \phi(|f(I_n)|) < \infty,$$

where the supremum is taken over all systems $\{I_n\}$ of nonoverlapping subintervals of I.

Example. *$\phi(u) = u$ gives us Jordan variation, and $\phi(u) = u^p$, $p > 1$, corresponds to Wiener variation. In the latter case $V_p(f)$ traditionally denotes the $p-$th root of $V_{\phi}(f)$ for $\phi(u) = u^p$, $p > 1$.*

Lemma 2.1. a) *Functions $F^{A,\alpha}(x) = \int_0^x e^{-\alpha u} \cos Au\,du$, and $G^{A,\alpha}(x) = \int_0^x e^{-\alpha u} \sin Au\,du$, $(A > 0, \alpha > 0)$ both belong to $V_p(\mathbb{R}^+)$, $p > 1$ and for their $p-$variation the following estimates hold*

$$V_p\left(F^{A,\alpha}\right) = O_{\alpha}\left(\left(\frac{1}{A}\right)^{1-\frac{1}{p}}\right), \qquad V_p\left(G^{A,\alpha}\right) = O_{\alpha}\left(\left(\frac{1}{A}\right)^{1-\frac{1}{p}}\right) \qquad (A \to \infty).$$

b) *Functions $F^{A,\alpha}$ and $G^{A,\alpha}$ $(A > 0, \alpha > 0)$ are bounded on $\mathbb{R}^+$.*

Proof. For simplicity, we will prove the statement for the function $G^{A,\alpha}$.

Since $\left(G^{A,\alpha}\right)'(x) = 0$ for $x = \frac{k\pi}{A}$, $k \in \mathbb{Z}_{\geq 0}$ we have that

$$\begin{aligned} V_p^p\left(G^{A,\alpha}, \mathbb{R}^+\right) &= \sum_{k=0}^{\infty} \left| \int_{\frac{k\pi}{A}}^{\frac{(k+1)\pi}{A}} e^{-\alpha u} \sin Au du \right|^p \leq \sum_{k=0}^{\infty} \frac{1}{\alpha^p} e^{-\alpha p \frac{(k+1)\pi}{A}} \left(1 - e^{-\frac{\alpha\pi}{A}}\right)^p \\ &= \frac{1}{\alpha^p} e^{-\frac{\alpha\pi}{A} p} \frac{\left(1 - e^{-\frac{\alpha\pi}{A}}\right)^p}{\left(1 - e^{-\frac{\alpha\pi}{A} p}\right)} \end{aligned}$$

Now, we get that $V_p^p\left(G^{A,\alpha}, \mathbb{R}^+\right) \leq \frac{1}{\alpha^p}\left(1 - e^{-\frac{\alpha\pi}{A}}\right)^{p-1}$. The observation that $\left(1 - e^{-\frac{\alpha\pi}{A}}\right) = O_\alpha\left(\frac{1}{A}\right)$ $(A \to \infty)$ completes the proof.

b) Obvious.

A function f is said to be of harmonic bounded variation on I $(f \in HBV(I))$ if

$$\sum \frac{|f(I_n)|}{n} < \infty$$

for every choice of nonoverlapping subintervals $I_n \subset I$. The supremum of these sums is called the harmonic variation of f on I and denoted by $V_H(f, I)$. $HBV(I)$ is a linear space and furthermore, the product of two functions from $HBV(I)$ lies again in $HBV(I)$.

The following remark [1] will be useful in the proof of the explicit formula.

Remark. ϕ is usually taken to be a continuous, strictly increasing convex function on $[0, \infty)$ satisfying two asymptotic conditions

(0_1) $\lim\limits_{x \to 0^+} \frac{\phi(x)}{x} = 0$ and

(∞_1) $\lim\limits_{x \to \infty} \frac{\phi(x)}{x} = \infty$.

Under these conditions the inclusion $\phi BV(I) \subset HBV(I)$ holds if $\sum \frac{1}{n}\phi^{-1}\left(\frac{1}{n}\right) < \infty$. The additional assumption

(Δ_2) There exist positive constants x_0 and d $(d \geq 2)$ such that for $0 \leq x \leq x_0$ we have $\phi(2x) \leq d\phi(x)$

turns ϕBV into a linear space.

The next straightforward lemma is a direct consequence of the definition of $\phi-$variation.

Lemma 2.2. If $f \in \phi BV(\mathbb{R}) \cap L^1(\mathbb{R})$ then $f(x) \to 0$ $(|x| \to \infty)$.

In the case of functions of $\phi-$bounded variation L.C.Young proved the following theorem on existence and approximation of the Stieltjes integral (in generalized Moore-Pollard sense, defined in [10]).

Theorem 2.A. [10] *Let f and g be respectively of bounded $\phi-$ and $\psi-$ variation on $[a, b]$ and let ϕ and ψ satisfy the condition $\sum \phi^{-1}\left(\frac{1}{n}\right)\psi^{-1}\left(\frac{1}{n}\right) < \infty$. Then the Stieltjes integral of f and g exists and for $a \leq \xi \leq b$ we have*

$$\int_a^b [f(x) - f(\xi)]\, dg(x) \leq c \sum \phi^{-1}\left(\frac{A}{n}\right)\psi^{-1}\left(\frac{B}{n}\right),$$

where $A = V_\phi(f,[a,b])$ *and* $B = V_\psi(g,[a,b])$.

Using Young's results we proved the formula of partial integration for functions of $\phi-$bounded variation.

Theorem 2.B. [2] *Let* f *and* g *be respectively of bounded* $\phi-$ *and* $\psi-$ *variation on* $[a,b]$ *and* $\sum \phi^{-1}\left(\frac{1}{n}\right)\psi^{-1}\left(\frac{1}{n}\right) < \infty$. *If for every* $\delta > 0$ *there exists a division of* $[a,b]$ *into subintervals on each of which at least one of the functions* f *and* g *has its oscillation less than* δ, *then*

$$\int_a^b f dg = f(b)g(b) - f(a)g(a) - \int_a^b g df.$$

In particular, the formula of partial integration is valid if one of the functions f *and* g *is continuous.*

Now, let f be an integrable function over $\mathbb{R}$ and $\widehat{f}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-itx}dx$ be its Fourier transform. For such an f and $A > 0$, we define

$$f_A(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\frac{\sin A(x-y)}{x-y}dy = \frac{1}{\sqrt{2\pi}}\int_{-A}^{A} \widehat{f}(t)e^{itx}dx.$$

For integrable functions of harmonic bounded variation we have proved the following Fourier inversion theorem:

Theorem 2.C. [2] *If* $f \in HBV(\mathbb{R}) \cap L^1(\mathbb{R})$, *then* f_A *is bounded independently of* A, $f_A(x) \to f(x)$ $(A \to \infty)$ *everywhere and convergence is uniform on compact sets of points of continuity of* f.

3 Regularized products and series. Fundamental class of functions

In this section we will recall a necessary background material from the theory of regularized products and series, introduced by J. Jorgenson and S. Lang in [6] and the theory of functions that are of regularized product and series type, developed in [7].

J. Jorgenson and S. Lang consider two sequences $L = \{\lambda_k\}$ and $A = \{a_k\}$ of complex numbers ($\lambda_0 = 0$ and $\lambda_k \neq 0$ for $k \geq 1$) and refer to the case $a_k \in \mathbb{Z}$ as a spectral case.

To the sequences L and A they formally associate a Dirichlet series and a theta-series or a theta function $\theta(t) = a_0 + \sum_k a_k e^{-\lambda_k t}$.

These series are usually subject to the certain convergence conditions, such as DIR1-DIR3 and AS1-AS3, given in [6, pp. 9-29]. We will recall only the condition AS2 posed on the theta function.

AS2. There exist a sequence $\{p\} = \{p_j\}$ of complex numbers with $\mathrm{Re}(p_0) \leq \mathrm{Re}(p_1) \leq ...$ increasing to infinity and a sequence of polynomials B_p such that for every complex q we have

$$\theta(t) - P_q(t) = O\left(t^{\mathrm{Re}(q)}\left|\log t\right|^{m(q)}\right) \qquad (t \to 0),$$

where $P_q(t) = \sum_{\mathrm{Re}(p)<\mathrm{Re}(q)} t^p B_p(\log t)$.

To the pair (L, A) in a spectral case, they associate, under conditions AS1-AS3, a regularized product, and in a general case, a regularized harmonic series. Many properties of regularized products and series, similar to those of zeta and gamma functions are proved in [6, pp. 30-52]. One of them is a generalization of Gauss formula [6, Th. 4.1., p. 49] we will use in the evaluation of the Weil's functional.

The asymptotic condition AS2 is closely related to the definition of the reduced order (M, m) of a regularized product, since M is an integer such that $-1 \leq M + \mathrm{Re}(p_0) < 0$, and m depends upon the degree of B_p. For a precise definition see [6].

Furthermore, J. Jorgenson and S. Lang generalized the concept of regularized series and products introducing functions of regularized series and product type [7, pp. 36-37]. The function is of a regularized product (RP) type if it is a product of a rational function, a function $e^{P(z)}$ (P is a polynomial) and finitely many regularized products. Similarly, they defined functions of regularized harmonic series (RHS) type. The logarithmic derivative of a RP type function is a RHS type function.

The definition of the reduced order of both types of functions is given, as well as a very important theorem on asymptotic behaviour (in vertical strips) of functions of RHS type of a given order [7, Th. 6.2., p. 38].

Finally, in [7] they introduce a fundamental class of functions as a family of triples $\left(Z, \widetilde{Z}, \Phi\right)$ of functions that are meromorphic, such that $\log Z$ and $\log \widetilde{Z}$ have a Dirichlet series representation and which satisfy a functional equation $Z(s)\Phi(s) = \widetilde{Z}(\sigma_0 - s)$, with the factor Φ of a regularized product type. For a precise definition see [7, pp. 45-46].

4 Mellin transform estimates

Let f be a measurable function on $\mathbb{R}^+$. The Mellin transform of the function f is formally (without any convergence assumptions) defined as

$$Mf(s) = \int_0^\infty f(u)\, u^s \frac{du}{u}.$$

For $\sigma_0 \in \mathbb{R}$, let $M_{\frac{\sigma_0}{2}} f$ denote the translate of Mf by $\frac{\sigma_0}{2}$, i.e. $M_{\frac{\sigma_0}{2}} f(s) = Mf\left(s - \frac{\sigma_0}{2}\right)$.

If one put $F(x) = f(e^{-x})$, or $f(u) = F(-\log u)$, $u > 0$, letting $s = \sigma + it$ one formally gets that

$$M_{\frac{\sigma_0}{2}} f(s) = \int_{-\infty}^{\infty} F(x)\, e^{-\left(\sigma - \frac{\sigma_0}{2}\right)x} e^{-itx} dx.$$

With this notation we have the following lemma:

Lemma 4.1. *Let* $F(x)\, e^{\left(\frac{\sigma_0}{2} + a'\right)|x|} \in \phi BV(\mathbb{R}) \cap L^1(\mathbb{R})$, *for some* $a' > 0$. *Let* ϕ *be a continuous function on* $(0, \infty)$ *strictly increasing from zero to infinity, such that* $\sum \phi^{-1}(\frac{1}{n})(\frac{1}{n})^{\frac{1}{p}} < \infty$, for some $p > 1$. *For* $s = \sigma + it$ *and* $a' > a > 0$ *we have* *the asymptotic relation*

$$M_{\frac{\sigma_0}{2}} f(s) = O\left(\left(\frac{1}{|t|}\right)^{1-\frac{1}{p}}\right) \textit{ uniformly in } -a \le \sigma \le \sigma_0 + a.$$

Proof. Let us put $g(x) = F(x) e^{\left(\frac{\sigma_0}{2}+a'\right)|x|}$. Then

$$M_{\frac{\sigma_0}{2}} f(s) = \int_0^\infty g(-x) e^{-(\sigma_0+a'-\sigma)x} e^{itx} dx + \int_0^\infty g(x) e^{-(\sigma+a')x} e^{-itx} dx.$$

Since $-a' < -a \le \sigma \le \sigma_0 + a < \sigma_0 + a'$ we have that $\alpha = \sigma_0 + a' - \sigma > 0$ and $\beta = \sigma + a' > 0$. Now, we have:

$$\begin{aligned} M_{\frac{\sigma_0}{2}} f(s) &= \int_0^\infty g(-x) e^{-\alpha x} \cos(tx)\, dx + i \int_0^\infty g(-x) e^{-\alpha x} \sin(tx)\, dx + \\ &+ \int_0^\infty g(x) e^{-\beta x} \cos(tx)\, dx - i \int_0^\infty g(x) e^{-\beta x} \sin(tx)\, dx. \end{aligned}$$

Without loss of generality, we may assume $t > 0$. If $t < 0$ we will write $\cos(tx) = \cos(-tx)$ and $\sin(tx) = -\sin(-tx)$.

With the notation of Lemma 2.1. we have:

$$\begin{aligned} M_{\frac{\sigma_0}{2}} f(s) &= \int_0^\infty g(-x)\, dF^{t,\alpha}(x) + i \int_0^\infty g(-x)\, dG^{t,\alpha}(x) \\ &+ \int_0^\infty g(x)\, dF^{t,\beta}(x) - i \int_0^\infty g(x)\, dG^{t,\beta}(x) \\ &= I_1 + iI_2 + I_3 + iI_4. \end{aligned}$$

Let us look at I_1.

It is possible to apply the integration by parts formula (Theorem 2.B.) here, since $F^{t,\alpha}$ is a continuous function, $F^{t,\alpha} \in V_p$ and $g(-x) \in \phi BV$. We have:

$$\int_0^a g(-x)\, dF^{t,\alpha}(x) = g(-a) F^{t,\alpha}(a) - \int_0^a F^{t,\alpha}(x)\, dg(-x).$$

Applying Theorem 2.A. we obtain:

$$\left| \int_0^a \left(F^{t,\alpha}(x) - F^{t,\alpha}(0)\right) dg(-x) \right| \le k \sum_n \phi^{-1}\left(\frac{V}{n}\right) \cdot V_p \cdot \frac{1}{n^p}, \text{ where}$$

$V = V_\phi(g, \mathbb{R}^+)$, $V_p = V_p(F^{t,\alpha}, \mathbb{R}^+)$. Letting $a \to \infty$ and applying Lemma 2.1. and Lemma 2.2. we end up with

$$I_1 = \int_0^\infty g(-x)\, dF^{t,\alpha}(x) = O\left(\left(\frac{1}{t}\right)^{1-\frac{1}{p}}\right).$$

Integrals I_2, I_3, and I_4 are estimated similarly. This proves the Lemma.

Let the function ϕ satisfy conditions of the previous Lemma.

Lemma 4.2. *Let F be an M times differentiable function such that*

1. $F^{(M)}(x)\, e^{\left(\frac{\sigma_0}{2}+a'\right)|x|} \in \phi BV(\mathbb{R}) \cap L^1(\mathbb{R})$
2. $F^{(j)}(x)\, e^{\left(\frac{\sigma_0}{2}+a'\right)|x|} \ll C$, *for* $j \in \{0, ..., M-1\}$.

Then, for $s = \sigma + it$ and $a' > a > 0$ we have the estimate

$$M_{\frac{\sigma_0}{2}} f(s) = O\left(\left(\frac{1}{|t|}\right)^{M+1-\frac{1}{p}}\right) \textit{ uniformly in } -a \le \sigma \le \sigma_0 + a.$$

Proof. Lemma 4.1. implies that

$$\int_{-\infty}^{\infty} F^{(M)}(x)\, e^{-\left(\sigma-\frac{\sigma_0}{2}\right)x} e^{-itx} dx = O\left(\left(\frac{1}{|t|}\right)^{1-\frac{1}{p}}\right) \text{ uniformly in } -a \le \sigma \le \sigma_0 + a.$$

On the other hand, for arbitrary $b < 0 < c$ we have

$$\int_b^c F^{(M)}(x)\, e^{-\left(\sigma-\frac{\sigma_0}{2}+it\right)x} dx = \int_b^c e^{-\left(\sigma-\frac{\sigma_0}{2}+it\right)x} dF^{(M-1)}(x).$$

Application of the classical integration by parts theorem yields

$$\begin{aligned}\int_b^c F^{(M)}(x)\, e^{-\left(\sigma-\frac{\sigma_0}{2}+it\right)x} dx &= e^{-\left(\sigma-\frac{\sigma_0}{2}+it\right)c} F^{(M-1)}(c) - e^{-\left(\sigma-\frac{\sigma_0}{2}+it\right)b} F^{(M-1)}(b) \\ &\quad + \left(\sigma - \frac{\sigma_0}{2} + it\right) \int_b^c F^{(M-1)}(x)\, e^{-\left(\sigma-\frac{\sigma_0}{2}+it\right)x} dx.\end{aligned}$$

Since

$$e^{-\left(\sigma-\frac{\sigma_0}{2}+it\right)c} F^{(M-1)}(c) = F^{(M-1)}(c)\, e^{\left(\frac{\sigma_0}{2}+a'\right)c} e^{-(\sigma+a')c} e^{itc}, \text{and}$$

$$(\sigma + a') > 0, \quad F^{(M-1)}(c)\, e^{\left(\frac{\sigma_0}{2}+a'\right)c} \ll C,$$

letting $c \to +\infty$ we get that

$$e^{-\left(\sigma-\frac{\sigma_0}{2}+it\right)c} F^{(M-1)}(c) \to 0.$$

Similarly, we obtain that

$$e^{-\left(\sigma-\frac{\sigma_0}{2}+it\right)b} F^{(M-1)}(b) \to 0 \text{ when } b \to -\infty.$$

Applying this to the integration by parts formula we get that

$$\int_{-\infty}^{+\infty} F^{(M)}(x)\, e^{-\left(\sigma-\frac{\sigma_0}{2}+it\right)x} dx = -\left(\sigma - \frac{\sigma_0}{2} + it\right) \int_{-\infty}^{+\infty} F^{(M-1)}(x)\, e^{-\left(\sigma-\frac{\sigma_0}{2}+it\right)x} dx.$$

Repeating the same procedure $(M-1)$ times, we end up with

$$O\left(\left(\frac{1}{|t|}\right)^{1-\frac{1}{p}}\right)=\left|\sigma-\frac{\sigma_0}{2}+it\right|^{M}\left|\int\limits_{-\infty}^{+\infty}F\left(x\right)e^{-\left(\sigma-\frac{\sigma_0}{2}+it\right)x}dx\right|.$$

This implies that

$$M_{\frac{\sigma_0}{2}}f\left(s\right)=O\left(\left(\frac{1}{|t|}\right)^{M+1-\frac{1}{p}}\right)\text{ uniformly in }-a\leq\sigma\leq\sigma_0+a.$$

The direct consequence of Lemma 4.2., the definition of a reduced order of a function of regularized product type and the theorem on the asymptotic behaviour of a function of a RHS type is the following

Theorem 4.1. *Let Φ be of a regularized product type of the reduced order (M,m) and let F be a function that satisfies conditions of Lemma 4.2.(for the same M that appears in the assumption about the order of Φ). Let the function f be defined on $\mathbb{R}^+$ by the relation $f(u)=F(-\log u)$. Then, there exists a sequence $\{T_n\}$ tending to infinity such that*

$$\frac{\Phi'}{\Phi}\left(s\right)M_{\frac{\sigma_0}{2}}f\left(s\right)\to 0\text{ when }n\to\infty,$$

for any s on the horizontal line segment defined by $s=\sigma\pm iT_n$ with $-a\leq\sigma\leq\sigma_0+a$.

5 Growth conditions on the test function

Let f and F be measurable functions related by $F(x)=f(e^{-x})$, so $f(u)=F(-\log u)$, $u\in\mathbb{R}^+$.

In what follows we will assume that ϕ is a continuous, convex function on $[0,\infty)$ strictly increasing from 0 to ∞. Let ϕ satisfy conditions (0_1), (∞_1) and the condition $\sum\limits_n\phi^{-1}\left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^{\frac{1}{p}}<\infty$, for some $p>1$. In [2] we introduced

Condition 1. ($\phi-$condition) $F\in\phi BV(\mathbb{R})\cap L^1(\mathbb{R})$.

Condition 2. ($\alpha-$condition) $F(x)=F(0)+O\left(|\log|x||^{-\alpha}\right)$, $(x\to 0)$ for some $\alpha>1$,

and proved that these were sufficient to ensure the existence of the Weil functional. We will state here the corollary of the theorem (general Parseval formula) proved in [2, Th. 6.1.]. This corollary will be used in the evaluation of the Weil functional.

Corollary 5.A. *Let (L,A) be a pair of sequences of complex numbers such that that the corresponding theta function satisfies conditions AS1-AS3, and M be the integer such that $-1\leq M+\mathrm{Re}\,(p_0)<0$. Let f be an M times differentiable function such that $f^{(M)}$ satisfies the $\phi-$condition and $\alpha-$condition for some $\alpha>M+2$. Suppose also that $f^{(j)}\in HBV(\mathbb{R})\cap L^1(\mathbb{R})$ for $j\in\{0,...,M-1\}$. Then, for any*

$w \in \mathbb{C}$ such that $\mathrm{Re}(w) > 0$ and $\mathrm{Re}(w) > \max_k (-\mathrm{Re}(\lambda_k))$ and $a \in \mathbb{R}^+$, we have

$$\lim_{A\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \widehat{f}(t)\, I_w(a+it)\, dt =$$

$$= \int_0^\infty \left(\theta_a(x) f^-(x) - \sum_{k+\mathrm{Re}(p_0)<0} c_k(a,x) f^{(k)}(0) \right) e^{-wx} dx.$$

The function I_w and the coefficients $c_k(a,x)$ are defined in the generalization of the Gauss formula, proved by J. Jorgenson and S. Lang in [6, Th. 4.1., p. 49].

The third condition we will impose on F is concerned with the evaluation of the integral of the functions $M_{\frac{\sigma_0}{2}} f(s) \frac{\Phi'}{\Phi}(s)$ and $M_{\frac{\sigma_0}{2}} f(s) \frac{Z'}{Z}(s)$ along the horizontal line segments $\sigma \pm iT_n$ for $-a \leq \sigma \leq \sigma_0 + a$. We will assume the following

Condition 3. (exponential $\phi-$condition) There exists $a' > 0$ such that $F(x)\, e^{\left(a'+\frac{\sigma_0}{2}\right)|x|} \in \phi BV(\mathbb{R})$.

In the following sections we will prove that under **Conditions 1-3.** posed on the test function F, the explicit formula holds for the functions in the fundamental class.

6 The explicit formula

Let $\left(Z, \widetilde{Z}, \Phi\right)$ be in the fundamental class. Let $a > 0$ be such that for σ_0' from the definition of the fundamental class we have $\sigma_0' < \sigma_0 + a$, and such that $Z, \widetilde{Z}$ and Φ do not have a zero or a pole on the line $\mathrm{Re}(s) = -a$ and $\mathrm{Re}(s) = \sigma_0 + a$.We will assume that Φ is of regularized product type of a reduced order (M, m). Then, by generalization of Cramer's theorem, proved in [5] functions Z and $\widetilde{Z}$ are of regularized product type of the reduced order $(M, m+1)$ or (M, m).

We will denote by:

- R_a the infinite rectangle bounded by the lines $\mathrm{Re}(s) = -a$ and $\mathrm{Re}(s) = \sigma_0 + a$.
- $R_a(T)$ the finite rectangle bounded by the lines $\mathrm{Re}(s) = -a$, $\mathrm{Re}(s) = \sigma_0 + a$ and horizontal lines $\mathrm{Im}(s) = \pm T$.
- $\{\rho\}$ the set of zeros and poles of Z in the full strip R_a.
- $\{\kappa\}$ the set of zeros and poles of Φ in the half strip $R_{a,\frac{\sigma_0}{2}}$, $-a \leq \sigma \leq \frac{\sigma_0}{2}$.

We will assume that Φ has finitely many zeros and poles in the half strip $R_{a,\frac{\sigma_0}{2}}$ and that it has no zeros or poles on the line $\mathrm{Re}(s) = \frac{\sigma_0}{2}$.

If T is chosen so that functions $Z, \widetilde{Z}$ and Φ have no zeros or poles on the horizontal lines that border $R_a(T)$, it is possible to form the following sums:

$$S_{Z,a}(f,T) = \sum_{\rho \in R_a(T)} ord(\rho) \cdot M_{\frac{\sigma_0}{2}} f(\rho) \tag{1}$$

and

$$S_{\Phi,a,\frac{\sigma_0}{2}}(f,T)=\sum_{\kappa\in R_{a,\frac{\sigma_0}{2}}(T)} ord\,(\kappa)\cdot M_{\frac{\sigma_0}{2}}f(\kappa)\,, \tag{2}$$

where $ord\,(\rho)$ denotes the order of ρ as a zero resp. minus the order of ρ as a pole.

Letting $T\to\infty$ in the second sum we will get the sum

$$S_{\Phi,a,\frac{\sigma_0}{2}}(f)=\lim_{T\to\infty}\sum_{\kappa\in R_{a,\frac{\sigma_0}{2}}(T)} ord\,(\kappa)\cdot M_{\frac{\sigma_0}{2}}f(\kappa)\,.$$

that is finite. The limit of the first sum need not be finite. We will be interested in conditions on the test function f (or, equivalently, on F) that ensure the existence of the limit

$$S_{Z,a}(f)=\lim_{T\to\infty}\sum_{\rho\in R_a(T)} ord\,(\rho)\cdot M_{\frac{\sigma_0}{2}}f(\rho)\,.$$

On a test function F we will impose the following conditions:

C I. F is M times differentiable, $F^{(M)}$ satisfies $\phi-$condition and $F^{(j)}\in L^1(\mathbb{R})\cap HBV(\mathbb{R})$ for $j=\overline{0,(M-1)}$.

C II. $F^{(M)}(x)$ satisfies $\alpha-$condition, for some $\alpha>M+2$.

C III. a) There exists $a'>a$ such that $F^{(M)}(x)\,e^{\left(a'+\frac{\sigma_0}{2}\right)|x|}\in\phi BV(\mathbb{R})\cap L^1(\mathbb{R})$,

b) $F^{(j)}(x)\,e^{\left(a'+\frac{\sigma_0}{2}\right)|x|}\ll C$, $j=\overline{1,M-1}$ (if $M\geq 2$).

c) $F(x)\,e^{\left(a'+\frac{\sigma_0}{2}\right)|x|}\in HBV(\mathbb{R})\cap L^1(\mathbb{R})$

Additional condition (Δ_2) on ϕ would simplify C I in an obvious way, which we omit here.

It goes without saying that in case $M=0$ requirements involving $j=\overline{0,M-1}$ are non-existent.

Growth conditions C I and C II are related to the evaluation of the Weil's functional

$$W_\Phi(F)=\lim_{n\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-T_n}^{T_n}\widehat{F}(t)\,\frac{\Phi'}{\Phi}\left(\frac{\sigma_0}{2}+it\right)dt,$$

where numbers T_n are chosen as in Theorem 4.1.

The main result of this paper is the following

Theorem 6.1. (The explicit formula) *Let $\left(Z,\widetilde{Z},\Phi\right)$ be in the fundamental class and assume that Φ is of reduced order (M,m) with finitely many zeros and poles in the half strip $R_{a,\frac{\sigma_0}{2}}$ and no zeros or poles on the line* $\operatorname{Re}(s)=\frac{\sigma_0}{2}$. *Then for any function F that satisfies conditions* C I-C III *functionals $S_{Z,a}$ and W_Φ are well defined and the explicit formula, i.e. the formula*

$$S_{Z,a}(f)+S_{\Phi,a,\frac{\sigma_0}{2}}(f)=\sum_q\frac{-c(q)\log q}{q^{\frac{\sigma_0}{2}}}f(q)+\sum_{\widetilde{q}}\frac{-c(\widetilde{q})\log\widetilde{q}}{\widetilde{q}^{\frac{\sigma_0}{2}}}f\left(\frac{1}{\widetilde{q}}\right)+W_\Phi(F)$$

holds.

Sums over q and $\widetilde{q}$ in the formula above correspond to the representation of functions $\log Z$ and $\log\widetilde{Z}$ as a Dirichlet series.

The proof of the theorem consists of two major parts. In the first part, integrating over the boundary of the rectangle $R_a(T_n)$ and using the residue theorem, we will express functionals $S_{Z,a}(f,T_n)$, $S_{\Phi,a,\frac{\sigma_0}{2}}(f,T_n)$ in the form of sums over q and $\widetilde{q}$ plus the Weil's functional. In the second part we will apply a general Parseval formula [2, Theorem 6.1] to finish the proof of the theorem.

7 Evaluation of sums

Let $B_a(T_n)$ denote the boundary of the rectangle $R_a(T_n)$ defined above. By the residue theorem,

$$\sum_{\rho\in R_a(T_n)} ord(\rho)\cdot M_{\frac{\sigma_0}{2}}f(\rho) = \frac{1}{2\pi i}\int\limits_{B_a(T_n)} M_{\frac{\sigma_0}{2}}f(s)\frac{Z'}{Z}(s)\,ds =$$
$$= \frac{1}{2\pi i}\left[\int\limits_{-a-iT_n}^{\sigma_0+a-iT_n} + \int\limits_{\sigma_0+a-iT_n}^{\sigma_0+a+iT_n} + \int\limits_{\sigma_0+a+iT_n}^{-a+iT_n} + \int\limits_{-a+iT_n}^{-a-iT_n}\right] = I_1+I_2+I_3+I_4.$$

Since the function Z is of reduced order at most $(M, m+1)$, applying Theorem 4.1. we have that $M_{\frac{\sigma_0}{2}}f(s)\frac{Z'}{Z}(s)\to 0$, $n\to\infty$ uniformly in s, for s on the lines $\sigma\pm iT_n$, $-a\le\sigma\le\sigma_0+a$. Therefore, $I_1\to 0$, and $I_3\to 0$ when $n\to\infty$. Using the functional equation we obtain that

$$\sum_{\rho\in R_a(T_n)} ord(\rho)\cdot M_{\frac{\sigma_0}{2}}f(\rho)+o(1) = \frac{1}{2\pi i}\int\limits_{\sigma_0+a-iT_n}^{\sigma_0+a+iT_n} M_{\frac{\sigma_0}{2}}f(s)\frac{Z'}{Z}(s)\,ds+$$
$$\frac{1}{2\pi i}\int\limits_{-a+iT_n}^{-a-iT_n} M_{\frac{\sigma_0}{2}}f(s)\left(-\frac{\widetilde{Z}'}{\widetilde{Z}}(\sigma_0-s)\right)ds + +\frac{1}{2\pi i}\int\limits_{-a+iT_n}^{-a-iT_n} M_{\frac{\sigma_0}{2}}f(s)\left(-\frac{\Phi'}{\Phi}(s)\right)ds$$
$$= J_1+J_2+J_3.$$

In this section we will deal with J_1 and J_2 and in the next section we will treat J_3. For simplicity, we will evaluate J_1. For $s=\sigma_0+a+it$, $-T_n\le t\le T$, it is possible to interchange the sum and the derivative in the Euler sum for $\log Z(s)$, since the sum converges uniformly for $\operatorname{Re}(s)=\sigma_0+a>\sigma_0'$. After the change of variables $s=\sigma_0+a+it$, $-T_n\le t\le T_n$ and $x=y-\log q$ we get

$$\begin{aligned} J_1 &= \frac{-1}{2\pi}\int\limits_{-T_n}^{T_n}\left(\int\limits_{-\infty}^{\infty}F(x)\,e^{-\left(\sigma_0-\frac{\sigma_0}{2}+a+it\right)x}dx\right)\cdot\sum_q\frac{c(q)\log q}{q^{\sigma_0+a+it}}dt = \\ &= \frac{-1}{2\pi}\int\limits_{-T_n}^{T_n}dt\sum_q\int\limits_{-\infty}^{\infty}F(y-\log q)\,e^{-\left(\frac{\sigma_0}{2}+a+it\right)y}q^{-\frac{\sigma_0}{2}}c(q)\log q\,dy.\end{aligned}$$

Let us consider the series

$$\sum_q F(y-\log q)\,e^{-\left(\frac{\sigma_0}{2}+a+it\right)y}q^{-\frac{\sigma_0}{2}}c(q)\log q = \sum_q B_q(y)\,e^{-ity}. \qquad (3)$$

Using the condition C III it the same way as in [7] we obtain that the series (3) converges uniformly. This enables us to interchange the integral and the sum in J_1, so it becomes:

$$J_1 = -\frac{1}{2\pi}\int\limits_{-T_n}^{T_n}dt\int\limits_{-\infty}^{\infty}\left(\sum_q B_q(y)\right)e^{-ity}dy.$$

Let us put $B(y)=\sum_q B_q(y)$. By the assumptions (C III) of the theorem, it is obvious that $B_q\in L^1(\mathbb{R})$ for all q, so $B\in L^1(\mathbb{R})$.

Let us write the function B_q in the following way

$$B_q(y+\log q)=\begin{cases} F(y)\,e^{-\left(\frac{\sigma_0}{2}+a\right)y}R(q), & \text{for } y\geq 0\\ F(y)\,e^{\left(\frac{\sigma_0}{2}+a'\right)|y|}e^{-(a'-a)|y|}R(q), & \text{for } y<0\end{cases}$$

where $R(q)$ denotes all factors that depend on q only.

The assumption C III, c) implies that $F(y)\in HBV(\mathbb{R})$ since the function $e^{-\left(\frac{\sigma_0}{2}+a'\right)|y|}$ is of bounded variation. For the same reason, $F(y)\,e^{-\left(\frac{\sigma_0}{2}+a\right)y}\in HBV(\mathbb{R}^+)$ and $F(y)\,e^{\left(\frac{\sigma_0}{2}+a'\right)|y|}e^{-(a'-a)|y|}\in HBV(\mathbb{R}^-)$, since $a'-a>0$.

Now, we have that $B_q(y+\log q)\in HBV(\mathbb{R})$ and therefore $B_q\in HBV(\mathbb{R})$.

Let I_j be non-overlapping subintervals of $\mathbb{R}$. We have:

$$\sum_j\frac{|B(I_j)|}{j}=\sum_j\frac{1}{j}\left|\sum_q B_q(I_j)\right|\leq\sum_q\sum_j\frac{|B_q(I_j)|}{j}.$$

Taking the supremum over all such I_j,we obtain that $V_H(B)\leq\sum_q V_H(B_q)$ where $V_H(f)$ denotes the harmonic variation of f on $\mathbb{R}$.

Let us evaluate $V_H(B_q)$. By the definition,

$$B_q(y)=F(y-\log q)\,e^{-\left(\frac{\sigma_0}{2}+a\right)(y-\log q)}\cdot q^{-(\sigma_0+a)}c(q)\log q=G_q(y)\,\frac{c(q)\log q}{q^{\sigma_0+a}},$$

so

$$V_H(B_q(y))\leq\frac{|c(q)\log q|}{q^{\sigma_0+a}}V_H(G_q(y)).$$

Now, we will use the fact that $F(y)\,e^{-\left(\frac{\sigma_0}{2}+a\right)y}\in HBV(\mathbb{R})$ (this can be established in an analogous way as above) to get

$$V_H(G_q(y))=V_H(G_q(y+\log q))=V_H\left(F(y)\,e^{-\left(\frac{\sigma_0}{2}+a\right)y}\right)=V<\infty.$$

This, together with the convergence assumptions on the Euler sum, implies that

$$V_H(B)\leq\sum_q\frac{|c(q)\log q|}{q^{\sigma_0+a}}\cdot V\leq K,\ \text{so}$$

$B\in HBV(\mathbb{R})$.

The function B satisfies the assumptions of Theorem 2.C. Therefore, we have:

$$\lim_{n\to\infty}J_1=\lim_{n\to\infty}\frac{-1}{\sqrt{2\pi}}\int_{-T_n}^{T_n}\widehat{B}(t)\,dt=-B(0)=-\sum_q\frac{c(q)\log q}{q^{\frac{\sigma_0}{2}}}f(q).$$

It is possible to carry out similar arguments for J_2, now putting

$$B_{\widetilde{q}}(y)=F(y+\log\widetilde{q})\,e^{\left(\frac{\sigma_0}{2}+a\right)y}\widetilde{q}^{-\frac{\sigma_0}{2}}c(\widetilde{q})\log\widetilde{q}.$$

Finally we get that

$$\lim_{n\to\infty} J_2 = -\widetilde{B}(0) = -\sum_{\widetilde{q}} \frac{c(\widetilde{q})\log \widetilde{q}}{\widetilde{q}^{\frac{\sigma_0}{2}}} f\left(\frac{1}{\widetilde{q}}\right).$$

At this point, we have proved that

$$S_{Z,a}(f) = \sum_{q} \frac{-c(q)\log q}{q^{\frac{\sigma_0}{2}}} f(q) + \sum_{\widetilde{q}} \frac{-c(\widetilde{q})\log \widetilde{q}}{\widetilde{q}^{\frac{\sigma_0}{2}}} f\left(\frac{1}{\widetilde{q}}\right) + \tag{4}$$
$$+\lim_{n\to\infty} \frac{1}{2\pi i} \int_{-a+iT_n}^{-a-iT_n} M_{\frac{\sigma_0}{2}} f(s) \left(-\frac{\Phi'}{\Phi}(s)\right) ds.$$

8 Weil's functional

As promised, in this section we will evaluate the last summand on the right hand side of (4). In this integral we will move path of integration to the right, in the following way:

$$S_{\Phi,a,\frac{\sigma_0}{2}}(f,T_n) = \frac{1}{2\pi i} \int_{-a+iT_n}^{-a-iT_n} M_{\frac{\sigma_0}{2}} f(s) \frac{\Phi'}{\Phi}(s)\, ds + \frac{1}{2\pi i} \int_{-a-iT_n}^{\frac{\sigma_0}{2}-iT_n} M_{\frac{\sigma_0}{2}} f(s) \frac{\Phi'}{\Phi}(s)\, ds$$
$$+\frac{1}{2\pi i} \int_{\frac{\sigma_0}{2}-iT_n}^{\frac{\sigma_0}{2}+iT_n} M_{\frac{\sigma_0}{2}} f(s) \frac{\Phi'}{\Phi}(s)\, ds + \frac{1}{2\pi i} \int_{\frac{\sigma_0}{2}+iT_n}^{-a+iT_n} M_{\frac{\sigma_0}{2}} f(s) \frac{\Phi'}{\Phi}(s)\, ds.$$

Since Φ is of regularized product type of reduced order (M,m), by Theorem 4.1. $M_{\frac{\sigma_0}{2}} f(s) \frac{\Phi'}{\Phi}(s) \to 0$ for $s = \sigma \pm iT_n,\ -a \le \sigma \le \frac{\sigma_0}{2}$. Letting $n \to \infty$ in the last equality and using (4), we obtain (after a change of variable)

$$S_{Z,a}(f) + S_{\Phi,a,\frac{\sigma_0}{2}}(f) = \sum_{q} \frac{-c(q)\log q}{q^{\frac{\sigma_0}{2}}} f(q) + \sum_{\widetilde{q}} \frac{-c(\widetilde{q})\log \widetilde{q}}{\widetilde{q}^{\frac{\sigma_0}{2}}} f\left(\frac{1}{\widetilde{q}}\right) +$$
$$+\frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \int_{-T_n}^{T_n} \widehat{F}(t) \frac{\Phi'}{\Phi}\left(\frac{\sigma_0}{2} + it\right) dt.$$

It is left to prove that $W_\Phi(F) = \frac{1}{\sqrt{2\pi}} \lim\limits_{n\to\infty} \int_{-T_n}^{T_n} \widehat{F}(t) \frac{\Phi'}{\Phi}\left(\frac{\sigma_0}{2} + it\right) dt$ is well defined, and to evaluate it.

Φ is of a regularized product type of reduced order (M,m), so

$$\Phi(z) = Q(z)\, e^{P(z)} \prod_{j=1}^{n} \left[D_j(\alpha_j z + \beta_j)\right]^{k_j},$$

in the notation of [7, p. 36].

By the linearity of the integral, it is easy to see that it is enough to consider the following three cases:

1. $\Phi(z) = Q(z)$, for some rational function Q.

The expression $\frac{Q'}{Q}(z)$ for $z = \frac{\sigma_0}{2} + it$ can be written in the form $\frac{Q'}{Q}\left(\frac{\sigma_0}{2} + it\right) = \sum_\beta \frac{A_\beta}{\frac{\sigma_0}{2}+it-\beta}$, where the sum on the right is taken over all zeros and poles of $Q(z)$. By the assumption on Φ, we have $\mathrm{Re}(\beta) \neq \frac{\sigma_0}{2}$ and the last expression becomes

$$\frac{Q'}{Q}\left(\frac{\sigma_0}{2} + it\right) = \sum_\alpha \frac{A_\alpha}{t+\alpha},$$

for some complex α, such that $\mathrm{Im}(\alpha) \neq 0$.

Now, we see that it is enough to consider the case $\Phi\left(\frac{\sigma_0}{2} + it\right) = \frac{1}{t+\alpha}$, $\mathrm{Im}(\alpha) \neq 0$. This case is treated in the following lemma.

Lemma 8.1. *Assume that* $g \in HBV(\mathbb{R}) \cap L^1(\mathbb{R})$. *Then*

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \widehat{g}(t) \frac{1}{t+\alpha} dt = \begin{cases} -i\int_0^{+\infty} g(x) e^{i\alpha x} dx, & \mathrm{Im}\ \alpha > 0 \\ i\int_0^{+\infty} g(-x) e^{-i\alpha x} dx, & \mathrm{Im}\ \alpha < 0 \end{cases}.$$

Proof. We will prove the statement in the case $\mathrm{Im}\alpha > 0$. The case $\mathrm{Im}\alpha < 0$ is treated similarly. Since, for $\mathrm{Im}\alpha > 0$

$$\int_{-T}^{T} \widehat{g}(t) \frac{1}{t+\alpha} dt = \int_{-T}^{T} \widehat{g}(t) \int_0^{+\infty} -ie^{i(t+\alpha)x} dx dt$$

and $\frac{\widehat{g}(t)}{t+\alpha} \in L^1[-T,T]$, we can apply Fubini's theorem to get

$$\frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \widehat{g}(t) \frac{1}{t+\alpha} dt = \int_0^{+\infty} -ie^{i\alpha x} g_T(x) dx.$$

The uniform boundedness of $\|g_T\|_\infty$ and the fact that $e^{i\alpha x} \in L^1(0,\infty)$ enables us to apply the Lebesgue dominated convergence theorem. This, together with the Theorem 2.C gives us:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \widehat{g}(t) \frac{1}{t+\alpha} dt = \int_0^{+\infty} -ie^{i\alpha x} \lim_{T\to\infty} g_T(x) dx = -i\int_0^{+\infty} g(x) e^{i\alpha x} dx.$$

This finishes the proof of the lemma.

2. $\Phi(z) = e^{P(z)}$, so $\frac{\Phi'}{\Phi}(z) = P'(z)$, $\deg P'(z) \leq M$.

In this case, the evaluation of the Weil functional reduces to the Fourier inversion formula, i.e. to Theorem 2.C, as the following lemma demonstrates:

Lemma 8.2. *Assume that* $M-$*times differentiable function* g *is such that*

$$g^{(j)} \in HBV(\mathbb{R}) \cap L^1(\mathbb{R}), \text{ for } j = \overline{0, M}.$$

Then, for all $n \in \{0, 1, ..., M\}$ *we have that*

$$\lim_{T\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \widehat{g}(t) (it)^n e^{itx} dt = g^{(n)}(x).$$

Proof. In the case $n = 0$ the statement of lemma is the same as Theorem 2.C. Let $n \geq 1$. Applying Theorem 2.C to the function $\widehat{g^{(n)}}(t)$we have that

$$\lim_{T\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \widehat{g^{(n)}}(t)\, e^{itx} dt = g^{(n)}(x). \tag{5}$$

On the other hand, for $a < 0 < b$ we have:

$$\int_a^b g^{(n)}(x)\, e^{-itx} dx = \int_a^b e^{-itx} dg^{(n-1)}(x).$$

Applying the classical integration by parts formula,(note that $e^{-itx} \in BV[a,b]$ and $g^{(n-1)}(x)$ is differentiable and so continuous), we get that

$$\int_a^b g^{(n)}(x)\, e^{-itx} dt = g^{(n-1)}(b)\, e^{-itb} - g^{(n-1)}(a)\, e^{-ita} - \int_a^b (-it)\, e^{-itx} g^{(n-1)}(x)\, dx.$$

Letting $a \to -\infty$, $b \to \infty$ and applying Lemma 2.2. we have that

$$\widehat{g^{(n)}}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g^{(n)}(x)\, e^{-itx} dx = \frac{(it)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itx} g^{(n-1)}(x)\, dx.$$

Repeating this $(n-1)$ −times, we end up with

$$\widehat{g^{(n)}}(t) = (it)^n\, \widehat{g}(t).$$

Putting this into (5) gives us the statement.

A direct consequence of the lemma is the following corollary

Corollary 8.1. *If g satisfies conditions of Lemma 8.2. then*

$$\lim_{T\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \widehat{g}(t)\, P'(t)\, dt = P'(-i\partial)\, g(0)$$

for every polynomial P', such that $\deg P' \leq M$.

3. $\Phi(z) = D_L(z)$ for some regularized product D_L associated to the pair (L, A).

For any $n \in \mathbb{N}$ let $L_n = \{\lambda_{n+1}, ...\}$. Then

$$\frac{D'_L}{D_L}(z) = \sum_{k=0}^{n} \frac{a_k}{z + \lambda_k} + \frac{D'_{L_n}}{D_{L_n}}(z).$$

To the sum on the right-hand side we can apply Lemma 8.1. and evaluate Weil's functional. That is why, without loss of generality, we may assume that L is such that

$$\max_{\lambda_k \in L} \{-\mathrm{Re}(\lambda_k)\} < \frac{\sigma_0}{2}.$$

The generalization of Gauss formula, proved in [6] yields, for any $\alpha > 0$,

$$\lim_{n\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-T_n}^{T_n}\widehat{F}(t)\left(\frac{D_L{}'}{D_L}\left(\frac{\sigma_0}{2}+it\right)\right)dt =$$

$$\lim_{n\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-T_n}^{T_n}\widehat{F}(t)\,I_\alpha\left(\frac{\sigma_0}{2}+it-\alpha\right)dt+\lim_{n\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-T_n}^{T_n}\widehat{F}(t)\,S_\alpha\left(\frac{\sigma_0}{2}+it-\alpha\right)dt.$$

where, in the notation of [6] we have

$$I_\alpha\left(\frac{\sigma_0}{2}+it-\alpha\right)=\int_0^\infty\left[\theta_{\left(\frac{\sigma_0}{2}+it-\alpha\right)}(u)-P_0\theta_{\left(\frac{\sigma_0}{2}+it-\alpha\right)}(u)\right]e^{-\alpha u}du$$

and $S_\alpha(z)$ is a polynomial of a degree less or equal to M .

Since S_α is a polynomial of a degree less or equal to M and F satisfies condition C I, Corollary 8.1. can be applied to the second summand on the right-hand side. The first summand can be evaluated using Corollary 5.A. Let us put $S_\alpha\left(\frac{\sigma_0}{2}+it-\alpha\right)=U_{\alpha,\frac{\sigma_0}{2}}(t)$. The evaluation of the Weil's functional in this case can be described in the form of the following lemma.

Lemma 8.3 *Let the function F satisfy conditions C I and C II. Let $\Phi(z)=D_L(z)$ for some regularized product D_L associated to the pair (L,A) such that $\max\limits_{\lambda_k\in L}\{-\mathrm{Re}(\lambda_k)\}<\frac{\sigma_0}{2}$. Then*

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-T_n}^{T_n}\widehat{F}(t)\frac{\Phi'}{\Phi}\left(\frac{\sigma_0}{2}+it\right)dt\\ =\;& U_{\alpha,\frac{\sigma_0}{2}}(-i\partial)F(0)+\int_0^\infty\left[\theta_{\left(\frac{\sigma_0}{2}-\alpha\right)}(x)F^-(x)-P_0\theta_{\left(\frac{\sigma_0}{2}+it-\alpha\right)}(u)\right]e^{-\alpha x}dx,\end{aligned}$$

for all $\alpha > 0$.

Lemma 8.1., Lemma 8.2. and Lemma 8.3. show us that Weil's functional $W_\Phi(F)$ is well defined for the class of functions that satisfy conditions C I - C II and also give us the direct evaluation of $W_\Phi(F)$. This finishes the proof of the explicit formula.

9 Application: Weil's explicit formula for the L-function

In this section we will show how to apply our results and prove the Barner - Weil explicit formula from [3] for a larger class of test functions. We will use the same notation as in [8].

Let k be a complete number field, χ a Hecke character of the ideal classes and $d\chi=\mathbf{N}(\mathcal{D}\mathfrak{f}_\chi)$, where $\mathcal{D}$ denotes the local different and $\mathfrak{f}_\chi$ is the conductor of χ. We put $A=2^{-2r_2}\pi^{-N}d\chi$, where N is the absolute degree of k and r_2 is the number of complex primes v that belong to S_∞, the set of archimedean prime spots of k.

The Hecke $L-$function, $L(s,\chi)$, $s=\sigma+it$, $t\in\mathbb{R}$ is defined for $\sigma>1$ as

$$L(s,\chi)=\prod_{\mathfrak{p}\notin S_\chi}\left(1-\frac{\chi(\mathfrak{p})}{\mathbf{N}\mathfrak{p}^s}\right)^{-1}.$$

Here S_χ denotes the set of all prime ideals $\mathfrak{p}$ of k that are ramified for χ.

In [8] it is proved that the logarithmic derivative of the function $L(s,\chi)$ has an Euler sum

$$-d\log L(s,\chi) = -\frac{L'}{L}(s,\chi) = \sum_{\mathfrak{p},m}(\log \mathbf{N}\mathfrak{p})\chi(\mathfrak{p}^m)(\mathbf{N}\mathfrak{p})^{-ms}$$

that converges absolutely in a half plane $\mathrm{Re}(s) > 1$ and uniformly in any half plane of the form $\mathrm{Re}(s) \geq 1+\varepsilon > 1$ $(\varepsilon > 0)$. The sum on the right hand side is taken over all prime ideals $\mathfrak{p}$ and natural numbers m.

Let

$$\Lambda(s,\chi) = A^{\frac{s}{2}} \prod_{v\in S_\infty} \Gamma\left(\frac{s_v}{2}\right) L(s,\chi),$$

where $s_v = s_v(\chi) = N_v(s + i\varphi_v(\chi)) + |m_v(\chi)|$.

In [8] it is also proved that the function $[s(s-1)]^{\delta_\chi}\Lambda(s,\chi)$, where

$$\delta_\chi = \begin{cases} 1 & \text{if } \chi \text{ is a principal character} \\ 0 & \text{if } \chi \text{ is a non - principal character.} \end{cases}$$

is an entire function of order 1. It is also proved that the function $\Lambda(s,\chi)$ satisfies the functional equation

$$W(\chi)\Lambda(s,\chi) = \Lambda(1-s,\overline{\chi}),$$

where $W(\chi)$ denotes the constant of modulus 1 depending on χ only. We can write the last equation in the form:

$$W(\chi)G(s,\chi)L(s,\chi) = G(1-s,\overline{\chi})L(1-s,\overline{\chi}),$$

for

$$G(s,\chi) = A^{\frac{s}{2}}\prod_{v\in S_\infty}\Gamma\left(\frac{s_v}{2}\right), \quad G(1-s,\overline{\chi}) = A^{\frac{1-s}{2}}\prod_{v\in S_\infty}\Gamma\left(\frac{\widehat{s_v}}{2}\right),$$

and $\widehat{s_v} = \widehat{s_v}(\chi) = N_v(1-s-i\varphi_v(\chi)) + |m_v(\chi)|$.

If, for a fixed χ we put $G(s) = G(s,\chi)$; $\widetilde{G}(s) = G(s,\overline{\chi})$; $Z(s) = L(s,\chi)$ and $\widetilde{Z}(s) = L(s,\overline{\chi})$ the functional equation becomes:

$$W(\chi)Z(s)G(s) = \widetilde{G}(1-s)\widetilde{Z}(1-s),$$

or

$$Z(s)\Phi(s) = \widetilde{Z}(1-s).$$

Here,

$$\Phi(s) = W(\chi)A^{s-\frac{1}{2}}\prod_{v\in S_\infty}\frac{\Gamma\left(\frac{s_v}{2}\right)}{\Gamma\left(\frac{\widehat{s_v}}{2}\right)},$$

so the fudge factor Φ of the last equation is of regularized product type of a reduced order $(0,0)$. This follows from the fact that the function $\frac{1}{\Gamma(s)}$ is a regularized product of a reduced order $(0,0)$.

Now, we see that $\left(Z,\widetilde{Z},\Phi\right)$ is in the fundamental class with $\sigma_0 = \sigma_0' = 1$, so, it is possible to apply Theorem 6.1., with $M = 0$ as well as Corollary 6.2 from [2], to obtain the explicit formula for the L-function. Let us note that functions $\Gamma\left(\frac{s_v}{2}\right)$

and $\Gamma\left(\frac{\widehat{s}_v}{2}\right)$ have finitely many zeros and poles in the half strip $R_{a,\frac{1}{2}}$ and no zeros or poles on the line $\mathrm{Re}\,(s) = \frac{1}{2}$, so conditions posed on $Z, \widetilde{Z}$ and Φ are satisfied.

The following corollary is a generalization of Weil's explicit formula from [3].

Corollary 9.1. *Let F satisfy conditions* C I-C III *with $M = 0$, introduced in the Section 6. Then, the following explicit formula holds:*

$$\lim_{T\to\infty} \sum_{|\gamma|<T} ord\,(\rho)\, M_{\frac{1}{2}} f\,(\rho) = \delta_\chi \int_{-\infty}^{\infty} F\,(x) \left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right) dx + F\,(0) \log A - \tag{6}$$

$$-\sum_{\mathfrak{p},n} \frac{\log \mathbf{N}\mathfrak{p}}{\mathbf{N}\mathfrak{p}^{\frac{n}{2}}} \left[\chi\,(\mathfrak{p})^n f\,(\mathbf{N}\mathfrak{p}^n) + \chi\,(\mathfrak{p})^{-n} f\left(\frac{1}{\mathbf{N}\mathfrak{p}^n}\right)\right] +$$

$$+ \sum_{v\in S_\infty} \int_0^\infty \left[\frac{N_v F\,(0)}{x} - \left(F_\chi\,(x) + F_\chi\,(-x)\right) \cdot \frac{e^{\left(\frac{2-|m_v|}{N_v} - \frac{1}{2}\right)x}}{1 - e^{\frac{-2x}{N_v}}}\right] e^{\frac{-2x}{N_v}} dx$$

where the sum on the left is taken over all non-trivial zeros $\rho = \beta + i\gamma$, $0 \le \beta \le 1$ of the L-function and $F_\chi\,(x) = F\,(x)\, e^{-i\varphi_v x}$.

The first integral on the right-hand side of (6) appears in the case when χ is a principal character, since then, the function Λ has simple poles at $s = 0$ and $s = 1$. Otherwise, it is equal to zero.

Let us note also, that the evaluation of the Weil functional on the right - hand side of (6) is a direct consequence of the Corollary 6.1. from [2], and the Fourier inversion theorem, since

$$\frac{\Phi'}{\Phi}\,(s) = \log A + \sum_{v\in S_\infty} \frac{N_v}{2} \left(\frac{\Gamma'}{\Gamma}\left(\frac{s_v}{2}\right) + \frac{\Gamma'}{\Gamma}\left(\frac{\widehat{s}_v}{2}\right)\right).$$

Applying [2, Corollary 6.1.] we have

$$\lim_{T\to\infty} \int_{-T}^{T} \widehat{F}\,(t) \left(\frac{\Gamma'}{\Gamma}\left(\frac{s_v}{2}\right) + \frac{\Gamma'}{\Gamma}\left(\frac{\widehat{s}_v}{2}\right)\right) dt =$$

$$= \int_0^\infty \left(\frac{F\,(0)}{x} - F_\chi\,(-x)\, \frac{\frac{2}{N_v} e^{\left(\frac{2-|m_v|}{N_v} - \frac{1}{2}\right)x}}{1 - e^{\frac{-2x}{N_v}}}\right) e^{\frac{-2x}{N_v}} dx +$$

$$+ \int_0^\infty \left(\frac{F\,(0)}{x} - F_\chi\,(x)\, \frac{\frac{2}{N_v} e^{\left(\frac{2-|m_v|}{N_v} - \frac{1}{2}\right)x}}{1 - e^{\frac{-2x}{N_v}}}\right) e^{\frac{-2x}{N_v}} dx,$$

for the function F that satisfies C I and C II.

Summation over $v \in S_\infty$ and application of the Theorem 2.C. to the term containing $\log A$ gives us the term $F\,(0) \log A$ and the last sum on the right-hand side of (6).

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