Fibonacci numbers and sets with the property $D(4)$

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Abstract

It is proved that if k and d are positive integers such that the product of any two distinct elements of the set

$$\{F_{2k},\, 5F_{2k},\, 4F_{2k+2},\, d\}$$

increased by 4 is a perfect square, than $d = 4L_{2k}F_{4k+2}$. This is a generalization of the results of Kedlaya, Mohanty and Ramasamy for $k = 1$.

1 Introduction

Let n be a given nonzero integer. A set of m positive integers $\{a_1, a_2, \ldots, a_m\}$ is called *a $D(n)$-m-tuple* (or *a Diophantine m-tuple with the property $D(n)$*) if $a_i a_j + n$ is a perfect square for all $1 \le i < j \le m$.

Diophantus himself found the $D(256)$-quadruple $\{1, 33, 68, 105\}$, while the first $D(1)$-quadruple, $\{1, 3, 8, 120\}$, was found by Fermat (see [5, 6]). Using the theory on linear forms in logarithms of algebraic numbers and a reduction method based on continued fractions, Baker and Davenport [1] proved that this Fermat's set cannot be extended to a $D(1)$-quintuple. The same result was proved by Kanagasabapathy and Ponnudurai [18] using the quadratic reciprocity law.

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There are several formulas for Diophantine quadruples with elements given in terms of Fibonacci and Lucas numbers, defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{k+2} = F_{k+1} + F_k,$$
$$L_0 = 2, \quad L_1 = 1, \quad L_{k+2} = L_{k+1} + L_k.$$

The numbers $1, 3, 8$ in Fermat's set can be viewed as three consecutive Fibonacci numbers with even subscripts. In 1977, Hoggatt and Bergum [17] proved that for any positive integer k, the set

$$\{F_{2k}, F_{2k+2}, F_{2k+4}, 4F_{2k+1}F_{2k+2}F_{2k+3}\}$$

is a $D(1)$-quadruple. They also conjectured that the fourth element of this set is unique. This conjecture was proved in [9].

A famous conjecture is that there does not exist a $D(1)$-quintuple. The first author proved recently that there does not exist a $D(1)$-sextuple and that there are only finitely many, effectively computable, $D(1)$-quintuples (see [10, 12]).

The question is what can be said about the size of sets with the property $D(n)$ for $n \neq 1$. Let us mention that Gibbs [15] found several examples of Diophantine sextuples, e.g. $\{3267, 11011, 17680, 87120, 234256, 1683715\}$ is a $D(255104784)$-sextuple.

Considering congruences modulo 4, it is easy to prove that if $n \equiv 2 \pmod 4$, then there does not exist a $D(n)$-quadruple (see [4, 16, 21]). On the other hand, if $n \not\equiv 2 \pmod 4$ and $n \not\in \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exists at least one $D(n)$-quadruple (see [7]). These results were generalized to Gaussian integers in [8].

In [11] and [13], bounds for the size of sets with property $D(n)$, for arbitrary nonzero integer n, were given.

In the present paper we consider the sets with property $D(4)$. The first result on nonextendability of $D(4)$-m-tuples was proved by Mohanty and the second author [20]. They proved that $D(4)$-quadruple $\{1, 5, 12, 96\}$ cannot be extended to a $D(4)$-quintuple. Later, Kedlaya [19] proved that if $\{1, 5, 12, d\}$ is a $D(4)$-quadruple, then d has to be 96.

As a consequence of results on sets with property $D(1)$, we prove that there does not exist a $D(4)$-8-tuple. We formulate much stronger conjecture, that for every $D(4)$-triple $\{a, b, c\}$ there exists a unique positive integer d, such that $d > \max(a, b, c)$ and $\{a, b, c, d\}$ is a $D(4)$-quadruple. We will prove this conjecture for a parametric family of $D(4)$-quadruples

$$\{F_{2k}, 5F_{2k}, 4F_{2k+2}, 4L_{2k}F_{4k+2}\}.$$

Since for $k = 1$ this set becomes $\{1, 5, 12, 96\}$, our result generalizes results from [19, 20] in the same way as the above mentioned result from [9] generalizes the result of Baker and Davenport on the Fermat's set.

The main tools used in the proof of our main result (Theorem 1) are the congruence method, introduced by Dujella and Pethő in [14], and the theorem of Bennett on simultaneous approximations of quadratic irrationals [3]. The special form of our triples $\{a, b, c\}$, the property that $b = 5a$, makes our problem very suitable for application of Bennett's result. This was the additional motivation for consideration of this particular family of quadruples.

2 Sets with the property $D(4)$

Lemma 1. *There does not exist a D(4)-triple consisting of three odd integers.*

Proof. Assume that $\{a_1, a_2, a_3\}$ is a $D(4)$-triple with odd elements. From $a_1a_2 + 4 \equiv 1 \pmod 8$ it follows $a_1a_2 \equiv 5 \pmod 8$, and analogously $a_1a_3 \equiv 5 \pmod 8$, $a_2a_3 \equiv 5 \pmod 8$. Multiplying these three congruences we obtain

$$(a_1a_2a_3)^2 \equiv 125 \equiv 5 \pmod 8,$$

a contradiction. ■

From Lemma 1 and the main results of [12] we obtain immediately the following result.

Corollary 1. *There does not exist a $D(4)$-8-tuple. There are only finitely many $D(4)$-7-tuples.*

But, we believe that much stronger statement is valid.

Conjecture 1. *There does not exist a $D(4)$-quintuple. Moreover, if $\{a, b, c, d\}$ is a $D(4)$-quadruple with $a < b < c < d$, then*

$$d = a + b + c + \frac{1}{2}(abc + rst), \tag{1}$$

where r, s, t are positive integers defined by

$$ab + 4 = r^2, \quad ac + 4 = s^2, \quad bc + 4 = t^2.$$

It is easy to check that the number d, defined by (1), really extends given $D(4)$-triple $\{a, b, c\}$. First of all, d is a positive integer. Furthermore,

$$ad + 4 = \left(\frac{at + rs}{2}\right)^2, \quad bd + 4 = \left(\frac{bs + rt}{2}\right)^2, \quad cd + 4 = \left(\frac{cr + st}{2}\right)^2.$$

The purpose of the present paper is to prove Conjecture 1 for an infinite family of triples, given in terms of Fibonacci numbers.

3 A parametric family of $D(4)$-quadruples

Let us consider the quadruple $\{F_{2k}, 5F_{2k}, 4F_{2k+2}, 4L_{2k}F_{4k+2}\}$. It holds:

$$\begin{aligned}
F_{2k} \cdot 5F_{2k} + 4 &= L_{2k}^2, \\
F_{2k} \cdot 4F_{2k+2} + 4 &= (2F_{2k+1})^2, \\
F_{2k} \cdot 4L_{2k}F_{4k+2} + 4 &= (2F_{4k+2})^2, \\
5F_{2k} \cdot 4F_{2k+2} + 4 &= (2L_{2k+1})^2, \\
5F_{2k} \cdot 4L_{2k}F_{4k+2} + 4 &= (2L_{4k+1})^2, \\
4F_{2k+2} \cdot 4L_{2k}F_{4k+2} + 4 &= (4F_{2k+2} + 2)^2.
\end{aligned}$$

Therefore $\{F_{2k}, 5F_{2k}, 4F_{2k+2}, 4L_{2k}F_{4k+2}\}$ is a $D(4)$-quadruple. It has the form from Conjecture 1. Indeed, in this case $c = a + b + 2r$ and

$$a + b + c + abc/2 + rst/2 = rst = L_{2k} \cdot 2F_{2k+1} \cdot 2L_{2k+1} = 4L_{2k}F_{4k+2}.$$

Hence, the following theorem is a special case of Conjecture 1.

Theorem 1. *Let k be a positive integer. If the set $\{F_{2k}, 5F_{2k}, 4F_{2k+2}, d\}$ is a $D(4)$-quadruple, then $d = 4L_{2k}F_{4k+2}$.*

Theorem 1 for $k = 1$, i.e. Conjecture 1 for the triple $\{1, 5, 12\}$, was proved by Kedlaya [19]. He also proved Conjecture 1 for the triple $\{1, 5, 96\}$. Previously, Mohanty and Ramasamy [20] proved that the $D(4)$-quadruple $\{1, 5, 12, 96\}$ cannot be extended to a $D(4)$-quintuple.

4 Systems of Pellian equations

Let $\{a, b, c\}$, where $0 < a < b < c$, be a $D(4)$-triple and let the positive integers r, s, t be defined by

$$ab + 4 = r^2, \quad ac + 4 = s^2, \quad bc + 4 = t^2.$$

Assume that $d > c$ is a positive integer such that $\{a, b, c, d\}$ is a $D(4)$-quadruple. We have

$$ad + 4 = x^2, \quad bd + 4 = y^2, \quad cd + 4 = z^2, \tag{2}$$

for some positive integers x, y, z. Eliminating d from (2) we obtain the following system of Pellian equations

$$az^2 - cx^2 = 4(a - c), \tag{3}$$
$$bz^2 - cy^2 = 4(b - c). \tag{4}$$

We will now describe the sets of solutions of equations (3) and (4). We will follow the argumentation of Stolt [22, Theorem 2].

Lemma 2. *There exist positive integers i_0, j_0 and integers $z_0^{(i)}$, $x_0^{(i)}$, $z_1^{(j)}$, $y_1^{(j)}$, $i = 1, \ldots, i_0$, $j = 1, \ldots, j_0$, with the following properties:*

(i) $(z_0^{(i)}, x_0^{(i)})$ and $(z_1^{(j)}, y_1^{(j)})$ are solutions of (3) and (4), respectively.

(ii) $z_0^{(i)}$, $x_0^{(i)}$, $z_1^{(j)}$, $y_1^{(j)}$ satisfy the following inequalities

$$1 \le x_0^{(i)} \le \sqrt{\frac{a(c-a)}{s-2}}, \tag{5}$$

$$|z_0^{(i)}| \le \sqrt{\frac{(s-2)(c-a)}{a}}, \tag{6}$$

$$1 \le y_1^{(j)} \le \sqrt{\frac{b(c-b)}{t-2}}, \tag{7}$$

$$|z_1^{(j)}| \le \sqrt{\frac{(t-2)(c-b)}{b}}. \tag{8}$$

(iii) If (z, x) and (z, y) are positive integer solutions of (3) and (4) respectively, then there exist $i \in \{1, \dots, i_0\}$, $j \in \{1, \dots, j_0\}$ and integers $m, n \geq 0$ such that

$$z\sqrt{a} + x\sqrt{c} = (z_0^{(i)}\sqrt{a} + x_0^{(i)}\sqrt{c})\Big(\frac{s + \sqrt{ac}}{2}\Big)^m, \tag{9}$$

$$z\sqrt{b} + y\sqrt{c} = (z_1^{(j)}\sqrt{b} + y_1^{(j)}\sqrt{c})\Big(\frac{t + \sqrt{bc}}{2}\Big)^n. \tag{10}$$

Proof. It is clear that it suffices to prove the statement of the lemma for equation (3). Let (z, x) be a solution of (3) in positive integers. Consider all pairs (z^*, x^*) of integers of the form

$$z^*\sqrt{a} + x^*\sqrt{c} = (z\sqrt{a} + x\sqrt{c})\Big(\frac{s + \sqrt{ac}}{2}\Big)^m, \quad m \in \mathbb{Z}.$$

Since $(zs - xc)(zx + xc) = 4(z^2 + c(a - c))$, we conclude that (z^*, x^*) is an integer solution of (3). Also, from $z^*\sqrt{a} + x^*\sqrt{c} > 0$ and $|x^*\sqrt{c}| > |z^*\sqrt{a}|$ it follows that x^* is a positive integer. Among all pairs (z^*, x^*), we choose a pair with the property that x^* is minimal, and we denote that pair by (z_0, x_0). Define integers z' and x' by

$$z'\sqrt{a} + x'\sqrt{c} = (z_0\sqrt{a} + x_0\sqrt{c})\Big(\frac{s - \varepsilon\sqrt{ac}}{2}\Big),$$

where $\varepsilon = 1$ if $z_0 \geq 0$, and $\varepsilon = -1$ if $z_0 < 0$. From the minimality of x_0 we conclude that $x' = \frac{1}{2}(sx_0 - \varepsilon a z_0) \geq x_0$ and this leads to $a|z_0| \leq (s - 2)x_0$. Squaring this inequality we obtain

$$x_0^2 \leq \frac{a(c - a)}{s - 2}.$$

Now we have

$$z_0^2 = \frac{1}{a}(cx_0^2 + 4(a - c)) \leq \frac{1}{a}\Big(\frac{ac(c - a)}{s - 2} + 4(a - c)\Big) = \frac{(s - 2)(c - a)}{a}. \tag{11}$$

Hence, we have proved that there exists a solution (z_0, x_0) of (3) which satisfies (5) and (6) (and accordingly belongs to a finite set of solutions) and an integer $m \in \mathbb{Z}$ such that

$$z\sqrt{a} + x\sqrt{c} = (z_0\sqrt{a} + x_0\sqrt{c})\Big(\frac{s + \sqrt{ac}}{2}\Big)^m.$$

It remains to show that $m \geq 0$. Suppose that $m < 0$. Then $\Big(\frac{s+\sqrt{ac}}{2}\Big)^m = \frac{\alpha-\beta\sqrt{ac}}{2}$, where α, β are positive integers satisfying $\alpha^2 - ac\beta^2 = 4$. We have $z = \frac{1}{2}(\alpha z_0 - \beta c x_0)$ and from the condition $z > 0$ we obtain $z_0^2 > 4\beta^2 c(c - a) \geq 4c(c - a)$ which clearly contradicts (11). ∎

From (3) we conclude that $z = v_m^{(i)}$ for some index i and integer $m \geq 0$, where

$$v_0^{(i)} = z_0^{(i)}, \quad v_1^{(i)} = \frac{1}{2}(sz_0^{(i)} + cx_0^{(i)}), \quad v_{m+2}^{(i)} = sv_{m+1}^{(i)} - v_m^{(i)}, \tag{12}$$

and from (4) we conclude that $z = w_n^{(j)}$ for some index j and integer $n \geq 0$, where

$$w_0^{(j)} = z_1^{(j)}, \quad w_1^{(j)} = \frac{1}{2}(tz_1^{(j)} + cy_1^{(j)}), \quad w_{n+2}^{(j)} = tw_{n+1}^{(j)} - w_n^{(j)}. \tag{13}$$

It follows easily by induction that $v_{2m}^{(i)} \equiv v_0^{(i)} \pmod c$, $v_{2m+1}^{(i)} \equiv v_1^{(i)} \pmod c$, $w_{2n}^{(j)} \equiv w_0^{(j)} \pmod c$, $w_{2m}^{(j)} \equiv w_1^{(j)} \pmod c$.

From (2), it follows $z^2 \equiv 4 \pmod c$. Hence, the initial values satisfy $(z_0^{(i)})^2 \equiv (z_1^{(j)})^2 \equiv 4 \pmod c$.

Let us now consider the case $\{a, b, c\} = \{F_{2k}, 5F_{2k}, 4F_{2k+2}\}$. Note that in this case $b = 5a$ and $10a < c \leq 12a$. Therefore, Lemma 2 implies

$$(z_0^{(i)})^2 < \frac{(s-2)(c-a)}{a} < (c-a)\sqrt{\frac{c}{a}} < 3.18\, c,$$
$$(z_1^{(j)})^2 < \frac{(t-2)(c-b)}{b} < (c-b)\sqrt{\frac{c}{b}} < 0.91\, c.$$

Thus, we have $z_1^2 = 4$ and $z_0^2 = 4$, $c + 4$, $2c + 4$ or $3c + 4$. We omitted the superscripts (i) and (j), and we will continue to do so.

We have to consider four cases depending on parities of m and n in $v_m = w_n$.

1) If m and n are both even, then we have $z_0 \equiv z_1 \pmod c$. Hence, $z_0 = z_1 = \pm 2$.

2) If m is odd and n is even, then we have $\frac{1}{2}(sz_0 \pm cx_0) \equiv z_1 \pmod c$. Since $|(sz_0 + cx_0)(sz_0 - cx_0)| = 4c(c-a) - 4z_0^2 < 4c^2$, we have $\frac{1}{2}(sz_0 - \varepsilon cx_0) = z_1$, where $\varepsilon \in \{-1, 1\}$ and $\varepsilon z_0 > 0$. But, $|sz_0 + \varepsilon cx_0| < 2cx_0 < 2c\sqrt{2.75\sqrt{ac}} < 2c\sqrt{c}$, and $|sz_0 - \varepsilon cx_0| > 2.5c^2/2c\sqrt{c} \geq 1.25\sqrt{c} > 4 = |z_1|$, a contradiction.

3) If m is even and n is odd, then we have $\frac{1}{2}(tz_1 + cy_1) \equiv z_0 \pmod c$. Hence $z_0 \equiv \pm t \pmod c$. It implies $|z_0| = t$ or $|z_0| = c - t$. But $t > 0.4c$ and $c - t > 0.3c$, and we obtained a contradiction with Lemma 2 (for $k \geq 3$). For $k = 1$ and $k = 2$ we can check directly that this case is impossible.

4) If m and n are both odd, then we have $\frac{1}{2}(sz_0 \pm cx_0) \equiv \frac{1}{2}(tz_1 \pm cy_1) \pmod c$. Hence $|cx_0 - s|z_0|| = 2t$ or $2c - 2t$. Assume first that $|z_0| \neq 2$. Then $|z_0| > \sqrt{c}$ and $|cx_0 + |sz_0|| > 2s|z_0| > 2c\sqrt{a}$. It implies $|cx_0 - s|z_0|| < \frac{2c}{\sqrt{a}} < 6.93\sqrt{c}$. As in **3)**, this leads to a contradiction (for $k \geq 4$, while the cases $k = 1$, $k = 2$ and $k = 3$ can be checked directly). Therefore, it remains to consider the case $|z_0| = 2$. Then $x_0 = 2$ and $cx_0 - s|z_0| = 2t$. However, in this case we have $v_m \equiv v_1 \pmod{2c}$, $w_n \equiv w_1 \pmod{2c}$ for odd m and n. It implies $t \pm s \equiv 0 \pmod{2c}$, which is impossible since $s + t = c$, and $0 < t - s < c$.

Hence, we proved

Proposition 1. *Let $\{F_{2k}, 5F_{2k}, 4F_{2k+2}, d\}$ be a $D(4)$-quadruple and $4F_{2k+2}d+1 = z^2$. Then there exist positive integers m and n such that*

$$z = v_{2m} = w_{2n},$$

where the binary recursive sequences $\{v_m\}$ and $\{w_n\}$ are defined by (12) and (13) with $z_0 = z_1 = \pm 2$ and $x_0 = y_1 = 2$.

5 Lower bound for solutions

In the previous section we proved that $v_m = w_n$ implies that m and n are both even. In this section we will derive a lower bound for m and n satisfying the equation $v_{2m} = w_{2n}$. Our main tool will be congruence consideration modulo c^2. The following lemma can be easily proved by induction.

Lemma 3.

$$v_{2m} \equiv z_0 + \frac{1}{2}c(az_0m^2 + sx_0m) \pmod{c^2},$$
$$w_{2n} \equiv z_1 + \frac{1}{2}c(bz_1n^2 + ty_1n) \pmod{c^2}.$$

Since in our situation $|z_0| = |z_1| = x_0 = y_1 = 2$, the equation $v_{2m} = w_{2n}$ and Lemma 3 imply

$$\pm am^2 + sm \equiv \pm bn^2 + tn \pmod{c}.$$

Inserting our concrete values for a, b, c, we obtain

$$F_{2k}(\pm 5n^2 + 2n \mp m^2) \equiv 2mF_{2k+1} \equiv -2mF_{2k} \pmod{F_{2k+2}}$$

and, since F_{2k} and F_{2k+2} are relatively prime,

$$\pm 5n^2 + 2n \equiv \pm m^2 - 2m \pmod{F_{2k+2}}. \tag{14}$$

Assume that $6n^2 \leq F_{2k+2}$. Then we may replace $\equiv$ by $=$ in (14).

This implies

$$(5n \pm 1)^2 - 5(m \mp 1)^2 = -4. \tag{15}$$

It follows easily by induction that for a positive integer n it holds $v_{2n} > w_n$. Hence, $v_{2m} = w_{2n}$ implies $m \leq 2n - 1$. Inserting this in (15), we obtain $n = 0$ for "$+$" sign, and $n = 0$ or $n = 1$ for "$-$" sign. If $n = 0$, then $d = 0$. If $n = 1$ and $z_0 = z_1 = -2$, then $z = v_2 = w_2 = 4F_{2k+2} + 2$ and $d = 4L_{2k}F_{4k+2}$.

Hence we proved

Lemma 4. *If $\{F_{2k}, 5F_{2k}, 4F_{2k+2}, d\}$ is a $D(4)$-quadruple and $d \neq 4L_{2k}F_{4k+2}$, then $4F_{2k+2}d + 1 = z^2$, where $z = v_{2m} = w_{2n}$ and*

$$n > \sqrt{\frac{F_{2k+2}}{6}}.$$

6 Simultaneous Diophantine approximations

In this section we will derive an upper bound for solutions of the system (3) and (4), using a theorem of Bennett on simultaneous Diophantine approximations of square roots of two rationals which are very close to 1.

Let us mention that Bennett used this theorem in the proof of the fact that systems of simultaneous Pell equations of the form

$$x^2 - Az^2 = 1, \qquad y^2 - Bz^2 = 1,$$

where A and B are distinct positive integers, possess at most three solutions (x, y, z) in positive integers.

Lemma 5 ([3]). *If a_i, p_i, q and N are integers for $0 \le i \le 2$, with $a_0 < a_1 < a_2$, $a_j = 0$ for some $0 \le j \le 2$, $q \ge 1$ and $N > M^9$, where*

$$M = \max_{0 \le i \le 2} \{|a_i|\},$$

then we have

$$\max_{0 \le i \le 2} \left\{ \left| \sqrt{1 + \frac{a_i}{N}} - \frac{p_i}{q} \right| \right\} > (130N\gamma)^{-1} q^{-\lambda}$$

where

$$\lambda = 1 + \frac{\log(33N\gamma)}{\log\left(1.7N^2 \prod_{0 \le i < j \le 2} (a_i - a_j)^{-2}\right)}$$

and

$$\gamma = \begin{cases} \frac{(a_2-a_0)^2(a_2-a_1)^2}{2a_2-a_0-a_1} & \textit{if } a_2 - a_1 \ge a_1 - a_0, \\ \frac{(a_2-a_0)^2(a_1-a_0)^2}{a_1+a_2-2a_0} & \textit{if } a_2 - a_1 < a_1 - a_0. \end{cases}$$

We will apply Lemma 5 to the numbers

$$\theta_1 = \frac{s}{a}\sqrt{\frac{a}{c}} \quad \text{and} \quad \theta_2 = \frac{t}{b}\sqrt{\frac{b}{c}}\,.$$

Note that in our case $b = 5a$ and c is divisible by 4, say $c = 4c'$. It holds

$$\begin{aligned} \theta_1 &= \sqrt{1 + \frac{4}{ac}} = \sqrt{1 + \frac{5}{bc'}}\,, \\ \theta_2 &= \sqrt{1 + \frac{4}{bc}} = \sqrt{1 + \frac{1}{bc'}}\,. \end{aligned}$$

Lemma 6.

$$\max\left(\left|\theta_1 - \frac{sx}{az}\right|, \left|\theta_2 - \frac{ty}{bz}\right| \right) < \frac{2c}{a} \cdot z^{-2}.$$

Proof. We have

$$\left|\theta_1 - \frac{sx}{az}\right| = \left|\frac{s}{a}\sqrt{\frac{a}{c}} - \frac{sx}{az}\right| = \frac{s}{az\sqrt{c}}|z\sqrt{a} - x\sqrt{c}| = \frac{s}{az\sqrt{c}} \cdot \frac{4(c-a)}{z\sqrt{a} + x\sqrt{c}} < \frac{4s(c-a)}{2az^2\sqrt{ac}} < \frac{2c}{a} \cdot z^{-2},$$

and analogously

$$\left|\theta_2 - \frac{ty}{bz}\right| < \frac{2c}{b} \cdot z^{-2} < \frac{2c}{a} \cdot z^{-2}.$$

■

We apply Lemma 5 with $a_0 = 0$, $a_1 = 1$, $a_2 = 5$, $N = bc'$, $M = 5$, $q = bz$, $p_1 = 5sx$, $p_2 = ty$. The condition $N > M^9$ becomes $5F_{2k}F_{2k+2} > 5^9$, which is satisfied for $k \ge 8$. In order to obtain an upper bound comparable with the lower bound from Lemma 4, we now assume that $k \ge 9$, i.e. $a \ge 2584$.

We have

$$\lambda = 1 + \frac{\log(33bc' \cdot \frac{400}{9})}{\log(1.7b^2c'^2 \cdot \frac{1}{400})} = 2 - \lambda_1 \,,$$

where

$$\lambda_1 = \frac{\log\left(\frac{51}{17600000}bc'\right)}{\log(0.00425b^2c'^2)} \,.$$

Lemma 5 and Lemma 6 imply

$$\frac{2c}{az^2} > (130bc' \cdot 4009)^{-1}(bz)^{\lambda_1 - 2}.$$

This implies

$$z^{\lambda_1} < 14445b^2c^2$$

and

$$\log z < \frac{\log\left(52002000a^4\right)\log\left(0.95624a^4\right)}{\log(0.0000434659a^2)} \,. \tag{16}$$

Since $a \geq 2584$, (16) implies

$$\log z < \frac{\log\left(a^{6.2613}\right)\log\left(a^4\right)}{\log(a^{0.7217})} < 34.71\, \log a \,. \tag{17}$$

We have $z = w_{2n}$ for a positive integer n. By Lemma 4, if we assume that $n > 1$ (i.e. $d \neq 4L_{2k}F_{2k+2}$), then $n > \sqrt{\frac{F_{2k+2}}{6}}$. From

$$w_n > 2F_{2k+1}(2L_{2k+1} - 1)^{n-1} > (2a)^n,$$

it follows

$$\log z > 2n \log(2a) > 41.5 \log a, \tag{18}$$

which is in contradiction with (17).

Hence, we proved Theorem 1 for $k \geq 9$.

7 The case $k \leq 8$

In remains to consider the case $k \leq 8$. This can be done by some of standard methods for solving systems of Pellian equation, e.g. by Baker-Davenport method [1]. In the standard way (see e.g. [1] or [10, Lemma 5], we transform the exponential equation $v_m = w_n$ into the following logarithmic inequality:

$$0 < m \log\left(\frac{s + \sqrt{ac}}{2}\right) - n \log\left(\frac{t + \sqrt{bc}}{2}\right) + \log \frac{\sqrt{b}(\sqrt{c} \pm \sqrt{a})}{\sqrt{a}(\sqrt{c} \pm \sqrt{b})} < 4.71\left(\frac{s - \sqrt{ac}}{2}\right)^{2m}.$$

Then we apply Baker's theory of linear forms in logarithms of algebraic numbers (e.g. a theorem of Baker and Wüstholz [2]). This gives us (large) absolute upper bound for m (for $k \leq 8$ we obtained $m < 2 \cdot 10^{19}$). Then we apply Baker-Davenport reduction ([1], see also [14, Lemma 5]), which reduces this large upper bound to $m \leq 19$. The next step of the reduction reduces further this bound to $m \leq 2$. It is easy to check directly that for $k \leq 8$ the only solutions of the equation $v_m = w_n$ which satisfy $m \leq 2$, correspond to trivial solution $d = 0$ or to solution $d = 4L_{2k}F_{4k+2}$, as claimed in Theorem 1.

Remark 1. Another possibility in the case of small k is to apply the Mohanty-Ramasamy method [20], which is an elementary method based on theory of quadratic residues. The method is implemented in MATHEMATICA by Kedlaya [19]. Using Kedlaya's program we were able to solve the cases $k = 1, 2, 3, 5$ and 6.

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