Pairs of solutions of constant sign for nonlinear periodic equations with unbounded nonlinearity

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Abstract

We consider periodic problems driven by the ordinary scalar p-Laplacian with a Caratheodory nonlinearity. Using variational techniques, coupled with the method of upper and lower solutions, we obtain two nontrivial solutions, with one positive and the other negative.

1 Introduction

In this paper we study the following periodic problem:

$$\begin{cases} -\left(|x'(t)|^{p-2}x'(t)\right)' = f(t, x(t)) \text{ a.e. on } T \\ x(0) = x(b),\ x'(0) = x'(b),\ 1 < p < \infty. \end{cases} \tag{1}$$

We are looking for multiple solutions of constant sign. Recently, the periodic problem for equations driven by the ordinary p-Laplacian has been studied by various researchers. We refer to the works of Del Pino-Manasevich-Murua [3], Fabry-Fayyad [5], Guo [6], Dang-Opperheimer [2] and Fan-Zhao-Huang [13] (for scalar problems), and Manasevich-Mawhin [8], Mawhin [9,10] Papageorgiou-Yannakakis [12] and Mawhin-Ward [14] (for vector problems). In all these works, the approach is degree theoretic or using the theory of nonlinear operators of monotone type (Papageorgiou-Yannakakis [12]). The question of existence of multiple periodic solutions was addressed only by Del Pino-Manasevich-Murua [3]. In their work the

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right hand side nonlinearity $f(t,x)$ is jointly continuous, and they assume that asymptotically there is no interaction between f and the Fucik spectrum of the scalar ordinary p-Laplacian.

Here in many respects, we go beyond the aforementioned work of Del Pino-Manasevich-Murua [3]. We establish the existence of at least two nontrivial solutions of constant sign. One is strictly positive and the other negative. The nonlinearity $f(t,x)$ is Caratheodory and in general unbounded. Our approach is variational, coupled with the method of upper and lower solutions.

2 Positive Solutions

In this section we prove the existence of a negative solution. For this purpose, we introduce the following hypotheses on the nonlinearity $f(t,x)$.

$\mathbf{H}(f)_1$: $f : T \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(t,0) \leq 0$ a.e. on T and

(i) $t \to f(t,x)$ is measurable for all $x \in \mathbb{R}$;

(ii) $x \to f(t,x)$ is continuous for almost all $t \in T$;

(iii) $|f(t,x)| \leq a(t) + c|x|^{s-1}$ for a.e. $t \in T$ and all $x \in \mathbb{R}$, with some $c > 0$, and $a \in L^{s'}(T)$ such that $\frac{1}{s} + \frac{1}{s'} = 1, 1 \leq s < \infty$;

(iv) $\lim_{x \to -\infty} \frac{pF(t,x)}{|x|^p} = 0$ uniformly for a.e. $t \in T$ with the potential $F(t,x) = \int_0^x f(t,r)dr$ and there is $M > 0$ such that $f(t,x) \leq 0$ or $f(t,x) \geq 0$ for a.e. $t \in T$ and all $x \leq -M$;

(v) $\lim_{x \to -\infty}(xf(t,x) - pF(t,x)) = \infty$ uniformly for almost all $t \in T$;

(vi) $F(t,\eta) > 0$ a.e. on T for some $\eta < 0$.

Remark: Hypothesis $H(iv)$ implies that asymptotically at $-\infty$, the potential function F interacts with the first part of the spectrum of the negative ordinary scalar p-Laplacian with periodic boundary conditions.

Let $W^{1,p}_{per}(T) = \{x \in W^{1,p}(T) : x(0) = x(b)\}$ and let $\varphi_1 : W^{1,p}_{per}(T) \to \mathbb{R}$ be defined by

$$\varphi_1(x) = \frac{1}{p}\|x'\|_p^p - \int_0^b F(t,x(t))dt$$

and $\varphi_2 : W^{1,p}_{per}(T) \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be defined by

$$\varphi_2(x) = \begin{cases} 0 & \text{if } x \in C \\ \infty & \text{if } x \notin C \end{cases}$$

where $C = \left\{x \in W^{1,p}_{per}(T) : x(t) \leq 0 \text{ for all } t \in T\right\}$. We know that $\varphi_1 \in C^1(W^{1,p}_{per}(T))$, and φ_2 is lower semicontinuous and convex (hence also weakly lower semicontinuous), i.e. $\varphi_2 \in \Gamma_0(W^{1,p}_{per}(T))$. Set $\varphi = \varphi_1 + \varphi_2$.

Proposition 1. *If hypothesis $H(f)_1$ holds, then problem (1) has a nontrivial solution $x \in C^1(T)$ such that $x(t) \le 0$ for all $t \in T$.*

Proof. By virtue of $H(f)_1(v)$, given $\beta > 0$ we can find $M_\beta > 0$ such that for almost all $t \in T$ and all $x \le -M_\beta$ we have

$$xf(t,x) - pF(t,x) \ge \beta.$$

Then for almost all $t \in T$ and all $x \le -M_\beta$ we have

$$\begin{aligned}\frac{d}{dt}\frac{F(t,x)}{|x|^p} &= \frac{|x|^p f(t,x) - p|x|^{p-2}xF(t,x)}{|x|^{2p}} \\ &= \frac{|x|^{p-1}(pF(t,x) - xf(t,x))}{|x|^{2p}} \\ &= \frac{pF(t,x) - xf(t,x)}{|x|^{1+p}} \\ &\le -\frac{\beta}{|x|^{p+1}} \\ &= (-1)^p \frac{\beta}{x^{p+1}}.\end{aligned}$$

Let $z, y \in (-\infty, -M_\beta]$ with $z \le y$. Integrating on the interval $[z, y]$ we obtain

$$\frac{F(t,y)}{|y|^p} - \frac{F(t,z)}{|z|^p} \le (-1)^p \frac{\beta}{p}\left(\frac{1}{z^p} - \frac{1}{y^p}\right),$$

so,

$$\frac{F(t,y)}{|y|^p} - \frac{F(t,z)}{|z|^p} \le \frac{\beta}{p}\left(\frac{1}{|z|^p} - \frac{1}{|y|^p}\right).$$

Let $z \to -\infty$. Because of the $H(f)_1(iv)$, we obtain $\frac{F(t,y)}{|y|^p} \le -\frac{\beta}{p}\frac{1}{|y|^p}$. Therefore, for almost all $t \in T$ and all $y \le -M_\beta$,

$$F(t,y) \le -\frac{\beta}{p}. \tag{2}$$

Since $\beta > 0$ is arbitrary, it follows that $F(t,y) \to -\infty$ uniformly for a.e. $t \in T$ as $y \to -\infty$.

Now we will show that φ is coercive. Suppose this not true. Then we could find $\{x_n\}_{n\ge1} \subset W^{1,p}_{per}(T)$ such that $\|x_n\| \to \infty$, and $\varphi(x_n) \le M_1$ for some $M_1 > 0$ and all $n \ge 1$. Let $y_n = \frac{x_n}{\|x_n\|}$. By passing to a subsequence if necessary, we may assume that $y_n \xrightarrow{w} y$ in $W^{1,p}_{per}(T)$, and $y_n \to y$ in $C(T)$. We recall that $W^{1,p}(T)$ is embedded compactly in $C(T)$. We have that

$$\frac{\varphi(x_n)}{\|x_n\|^p} = \frac{1}{p}\|y_n'\|_p^p - \int_0^b \frac{F(t, x_n(t))}{\|x_n\|^p}dt \le \frac{M_1}{\|x_n\|^p}. \tag{3}$$

Note that

$$\int_0^b \frac{F(t,x_n(t))}{\|x_n\|^p}dt = \int_{\{x_n\le -M_\beta\}} \frac{F(t,x_n(t))}{\|x_n\|^p}dt + \int_{\{-M_\beta<x_n\le 0\}} \frac{F(t,x_n(t))}{\|x_n\|^p}dt. \tag{4}$$

By the hypothesis $H(f)_1(iii)$, we can find $a_1 \in L^1(T)$ such that

$$\int_{\{-M_\beta<x_n\le 0\}} \frac{F(t,x_n(t))}{\|x_n\|^p}dt \le \int_0^b \frac{a_1(t)}{\|x_n\|^p}dt \to 0 \qquad \text{as } n\to\infty. \tag{5}$$

Also from (2) we have that

$$\int_{\{x_n\le -M_\beta\}} \frac{F(t,x_n(t))}{\|x_n\|^p}dt \le \frac{1}{\|x_n\|^p}(-\frac{\beta}{p})\lambda(\{x_n\le -\beta\}) \le \frac{\beta b}{p}\frac{1}{\|x_n\|^p},$$

where λ denotes the Lebesgue measure on $\mathbb{R}$

Therefore we obtain

$$\limsup_{n\to\infty} \int_{\{x_n\le -M_\beta\}} \frac{F(t,x_n(t))}{\|x_n\|^p}dt \le 0. \tag{6}$$

So, returning to (4), and using (5) and (6), we obtain

$$\limsup_{n\to\infty} \int_0^b \frac{F(t,x_n(t))}{\|x_n\|^p}dt \le 0. \tag{7}$$

Therefore if we pass to the limit in (3), and use (7) and the weak lower semicontinuity of the norm in a Banach space, we obtain

$$\|y'\|_p = 0, \ \text{i.e., } y=\xi\in\mathbb{R}.$$

If $\xi = 0$, then we have $\|y_n'\|_p \to 0$ and so $y_n \to 0$ in $W^{1,p}_{per}(T)$, a contradiction to the fact that $\|y_n\| = 1$ for all $n \ge 1$. Therefore, $\xi \ne 0$. Thus for any $t \in T$, we have $x_n(t) \to -\infty$ as $n \to \infty$. We claim that this convergence is uniform in $t \in T$. Indeed, let $\delta > 0$ be such that $\delta < |\xi|$. Since $y_n \to \xi$ in $C(T)$, we can find $n_0 \ge 1$ such that for all $n \ge n_0$ and $t \in T$, we have $|y_n(t) - \xi| < \delta$. Therefore,

$$|y_n(t)| \ge |\xi| - \delta = \delta_1 > 0.$$

Since by hypothesis $\|x_n\| \to \infty$, given $\beta_1 > 0$ we can find $n_1 \ge 1$ such that for all $n \ge n_1$ we have

$$\|x_n\| \ge \beta_1 > 0.$$

Let $n_2 = \max\{n_0, n_1\}$. Then for all $t \in T$ and all $n \ge n_2$ we have

$$\frac{|x_n(t)|}{\beta_1} \ge \frac{|x_n(t)|}{\|x_n\|} = |y_n(t)| \ge \delta_1 > 0.$$

Therefore, $|x_n(t)| \ge \delta_1\beta_1$.

Since $\beta_1 > 0$ is arbitrary and $\delta_1 > 0$, we can conclude that $x_n(t) \to -\infty$ uniformly in $t \in T$. Recall that $F(t, y) \to -\infty$ uniformly for almost all $t \in T$ as $y \to -\infty$, see

(2). So, given $\beta_2 > 0$ we can find $n_3 \geq 1$ such that $F(t, x_n(t)) \leq -\beta_2$ for almost all $t \in T$ and all $n \geq n_3$. Then from the choice of the sequence of $\{x_n\}_{n\geq 1} \subset W^{1,p}_{per}(T)$, for all $n \geq n_3$ we have $\varphi(x_n) \leq M_1$. Thus,

$$-\int_0^b F(t, x_n(t))dt \leq M_1.$$

So, $b\beta_2 \leq M_1$.

Because $\beta_2 > 0$ is arbitrary, this last inequality leads to a contradiction. Thus, we have proved the claim that φ is coercive.

Since φ is coercive, it is bounded below. Moreover, it is also lower semicontinuous. Since $W^{1,p}_{per}(T)$ is reflexive, by the Weierstrass theorem it follows that we can find $x \in W^{1,p}_{per}(T)$ such that

$$m = \inf \varphi = \varphi(x).$$

Evidently, $x \in C$. Moreover, by hypothesis $H(f)_1(vi)$ we can find $\eta < 0$ such that $F(t, \eta) > 0$ a.e. on T, and so $\varphi(\eta) < 0$. Therefore, $m = \varphi(x) < 0 = \varphi(0)$, which implies that $x \neq 0$

By the Ekeland's variational principle, see Mawhin-Willem [11,p.75], we can find $\{x_n\}_{n\geq 1} \subset C$, a minimizing sequence for φ, i.e. $\varphi(x_n) \downarrow m = \inf \varphi = \varphi(x)$, such that

$$-\frac{1}{n}\|u - x_n\| \leq \varphi(u) - \varphi(x_n) \quad \text{for all} \quad u \in W^{1,p}_{per}(T).$$

Let $u = (1 - \lambda)x_n + \lambda v$, with some $\lambda \in (0, 1)$ and $v \in W^{1,p}_{per}(T)$. Since φ_2 is convex, we obtain

$$-\frac{\lambda}{n}\|v - x_n\| \leq \varphi_1(x_n + \lambda(v - x_n)) - \varphi_1(x_n) + \lambda(\varphi_2(v) - \varphi_2(x_n)).$$

Therefore, for all $v \in W^{1,p}_{per}(T)$ we have

$$-\frac{1}{n}\|v - x_n\| \leq \langle \varphi_1'(x_n), v - x_n\rangle + \varphi_2(v) - \varphi_2(x_n). \tag{8}$$

Since $\{\varphi_1(x_n) = \varphi(x_n)\}_{n\geq 1}$ is bounded and φ is coercive, it follows that the sequence $\{x_n\} \subset C$ is bounded. So, we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}_{per}(T)$ and $x_n \to x$ in $C(T)$. In (8) let $v = x \in C$ and note that $\varphi_1'(x_n) = A(x_n) - N_f(x_n)$ with $A : W^{1,p}_{per}(T) \to W^{1,p}_{per}(T)^*$ being the nonlinear operator defined by

$$\langle A(x), y\rangle = \int_0^b |x'(t)|^{p-2}x'(t)y'(t)dt$$

for all $x, y \in W^{1,p}_{per}(T)$. Here $\langle\cdot, \cdot\rangle$ denotes the duality brackets for the pair

$$(W^{1,p}_{per}(T), W^{1,p}_{per}(T)^*)$$

and $N_f : L^s(T) \to L^{s'}(T)$ is the Nemitskii operator corresponding to the function f, i.e., $N_f(x)(\cdot) = f(\cdot, x(\cdot))$. Then, from (8) with $v = y \in C$,

$$\langle A(x_n), x_n - x\rangle - \int_0^b f(t, x_n(t))(x_n - x)(t)dt \leq \frac{1}{n}\|x_n - x\|.$$

Observe that $\int_0^b f(t,x_n(t))(x_n-x)(t)dt \to 0$ and $\frac{1}{n}\|x_n - x\| \to 0$, as $n \to \infty$. So

$$\limsup \langle A(x_n), x_n - x\rangle \le 0.$$

It is easy to check that A is demicontinuous and monotone, hence it is maximal monotone. Therefore, it is generalized pseudomonotone, see Hu-Papageorgiou [7, p.365]. So

$$\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle.$$

Thus, $\|x_n'\|_p \to \|x'\|_p$.

Recall that $x_n' \xrightarrow{w} x'$ in $L^p(T)$ and because $L^p(T)$ is uniformly convex, we have that $x_n' \to x'$ in $L^p(T)$ due to the Kadec-Klee property of the Banach space $L^p(T)$, see Hu-Papageorgiou [7, p.28]. Therefore, $x_n \to x$ in $W^{1,p}_{per}(T)$. Returning to (8) and passing to the limit, we obtain for all $v \in W^{1,p}_{per}(T)$

$$0 \le \langle \varphi_1'(x), v - x\rangle + \varphi_2(v) - \varphi_2(x).$$

Hence we have $-\varphi_1'(x) \in \partial\varphi_2(x) = N_C(x)$, where $\partial\varphi_2(x)$ denotes the convex subdifferential of φ_2 at x which is equal to the normal cone to C at x, see Hu-Papageorgiou [7, p.345]. So, we have

$$0 \le \langle \varphi_1'(x), v - x\rangle \quad \text{for all} \quad v \in C,$$

thus $0 \le \langle A(x) - N_f(x), v - x\rangle$ for all $v \in C$.

Assume that the first alternative of the last part of hypothesis $H(f)_1(iv)$ holds, namely that $f(t,y) \ge 0$ for almost all $t \in T$ and all $y \le -M$. Let $h \in W^{1,p}_{per}(T)$, $\varepsilon > 0$, and set $v = -(\varepsilon h - x)^+ = -(\varepsilon h - x) - (\varepsilon h - x)^- \in W^{1,p}_{per}(T)$, see Evans-Gariepy [4, p.130]. Here for $g \in L^p(T)$, $g^+ = \max\{g, 0\}$ and $g^- = \max\{-g, 0\}$. We have $v - x = -\varepsilon h - (\varepsilon h - x)^-$. If $x^* = A(x) - N_f(x)$, we have $0 \le \langle x^*, v - x\rangle$. Therefore,

$$-\varepsilon\langle x^*, h\rangle \ge \langle x^*, (\varepsilon h - x)^-\rangle = \langle A(x), (\varepsilon h - x)^-\rangle - \int_0^b f(t,x)(\varepsilon h - x)^- dt.$$

Set $T_-^\varepsilon = \{t \in T : (\varepsilon h - x)(t) < 0\}$. We know that

$$\left[(\varepsilon h - x)^-\right]'(t) = \begin{cases} 0 & \text{a.e. on } \ (T_-^\varepsilon)^c \\ -(\varepsilon h - x)'(t) & \text{a.e. on } \ T_-^\varepsilon, \end{cases}$$

see Evans-Gariepy [4, p.130]. Therefore,

$$\begin{aligned} \langle A(x), (\varepsilon h - x)^-\rangle &= \int_0^b |x'|^{p-2}x'[(\varepsilon h - x)^-]'dt \\ &= -\int_{T_-^\varepsilon} |x'|^{p-2}x'(\varepsilon h - x)'dt \\ &\ge -\varepsilon \int_{T_-^\varepsilon} |x'|^{p-2}x'h'dt. \end{aligned}$$

Also we have

$$
\begin{aligned}
-\int_0^b f(t,x)(\varepsilon h-x)^- dt &= \int_{T_-^\varepsilon} f(t,x)(\varepsilon h - x)dt \\
&= \int_{T_-^\varepsilon\cap\{x\le -M\}} f(t,x)(\varepsilon h-x)dt \\
&\quad + \int_{T_-^\varepsilon\cap\{x> -M\}} f(t,x)(\varepsilon h-x)dt.
\end{aligned}
$$

By assumption, we have $f(t,x(t))\ge 0$ a.e. on $T_-^\varepsilon\cap\{x(t)\le -M\}$ and $x(t)\le 0$ for all $t\in T$. So,

$$
-\int_{T_-^\varepsilon\cap\{x\le -M\}} f(t,x)x dt \ge 0.
$$

Therefore, we obtain

$$
\int_{T_-^\varepsilon\cap\{x\le -M\}} f(t,x)(\varepsilon h-x)dt \ge \varepsilon \int_{T_-^\varepsilon\cap\{x\le -M\}} f(t,x)hdt.
$$

Also, by hypothesis $H(f)_1(iii)$ we see that $|f(t,x(t))|\le \xi_1(t)$ for a.e. $t\in T_-^\varepsilon\cap\{x(t)>-M\}$ and some $\xi_1\in L^{s'}(T)_+$. So, a.e. on $T_-^\varepsilon\cap\{x(t)>-M\}$ we have

$$
f(t,x(t))(\varepsilon h-x)(t)\ge \xi_1(t)(\varepsilon h-x)(t).
$$

Therefore, if $\hat{T}_-^\varepsilon = T_-^\varepsilon\cap\{x<0\}$, then

$$
\int_{T_-^\varepsilon\cap\{x>-M\}} f(t,x)(\varepsilon h-x)dt \ge \varepsilon\int_{T_-^\varepsilon\cap\{x\le -M\}} f(t,x)hdt + \int_{\hat{T}_-^\varepsilon\cap\{x>-M\}} \xi_1(\varepsilon h-x)dt.
$$

Thus we finally obtain

$$
-\langle x^*,h\rangle \ge -\int_{T_-^\varepsilon} |x'|^{p-2}x'h'dt + \int_{T_-^\varepsilon\cap\{x\le -M\}} f(t,x)hdt + \frac{1}{\varepsilon}\int_{\hat{T}_-^\varepsilon\cap\{x>-M\}} \xi_1(\varepsilon h-x)dt.
$$

Note that since $x(t)\le 0$ on T, we have $T_-^\varepsilon\to T_0=\{x=0\}$ as $\varepsilon\downarrow 0$ and $\lambda(T_-^\varepsilon\cap\{x\le -M\})\to 0$ as $\varepsilon\downarrow 0$. So, from the last inequality we obtain

$$
0\ge \langle x^*,h\rangle \quad \text{for all} \quad h\in W^{1,p}_{per}(T),
$$

which implies that $x^*=A(x)-N_f(x)=0$ and therefore,

$$
A(x)=N_f(x).
$$

Now assume that the second option in the last part of hypothesis $H(f)_1(iv)$ holds, namely $f(t,y)\le 0$ for almost all $t\in T$ and all $y\le -M$. In this case we have

$$\begin{aligned}
-\int_0^b f(t,x)(\varepsilon h - x)^- dt &= \int_{T_-^\varepsilon} f(t,x)(\varepsilon h - x)dt \\
&= \int_{T_-^\varepsilon \cap \{x \le -M\}} f(t,x)(\varepsilon h - x)dt \\
&\quad + \int_{T_-^\varepsilon \cap \{x > -M\}} f(t,x)(\varepsilon h - x)dt \\
&\ge \int_{T_-^\varepsilon \cap \{x > -M\}} f(t,x)(\varepsilon h - x)dt \\
&\ge \int_{\hat{T}_-^\varepsilon \cap \{x > -M\}} \xi_1(\varepsilon h - x)dt.
\end{aligned}$$

Therefore,

$$-\langle x^*, h\rangle \ge -\int_{T_-^\varepsilon} |x'|^{p-2}x'h'dt + \frac{1}{\varepsilon}\int_{\hat{T}_-^\varepsilon \cap \{x > -M\}} \xi_1(\varepsilon h - x)dt.$$

Again, let $\varepsilon \downarrow 0$ to see $0 \ge \langle x^*, h\rangle$ for all $h \in W^{1,p}_{per}(T)$. Thus,

$$x^* = A(x) - N_f(x) = 0.$$

Finally, $A(x) = N_f(x)$.

So, in both cases we have proved that $A(x) = N_f(x)$. From the representation theorem for the elements of $W^{-1,q}(T) = W_0^{1,p}(T)^*$, with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$(|x'|^{p-2}x')' \in W^{-1,q}(T),$$

see Adams [1, p.50]. Let $\langle \cdot, \cdot \rangle_0$ denote the brackets for the pair $(W_0^{1,p}(T), W^{-1,q}(T))$. For each $v \in C_0^1(T) = \{v \in C^1(T) : v(0) = v(b) = 0\}$ we have

$$\langle A(x), v\rangle_0 = \int_0^b f(t, x(t))v(t)dt,$$

hence integration by parts leads to $\langle -(|x'|^{p-2}x')', v\rangle_0 = \int_0^b f(t,x(t))v(t)dt$.

Since $C_0^1(T)$ is dense in $W_0^{1,p}(T)$, we obtain

$$\begin{cases} -\left(|x'(t)|^{p-2}x'(t)\right)' = f(t, x(t)) \quad \text{a.e. on } \; T \\ x(0) = x(b). \end{cases} \tag{9}$$

Also, for each $y \in W^{1,p}_{per}(T)$ by Green's identity, using (9), we have

$$|x'(0)|^{p-2}x'(0)y(0) = |x'(b)|^{p-2}x'(b)y(b),$$

so $|x(0)|^{p-2}x'(0) = |x'(b)|^{p-2}x'(b)$ and consequently,

$$x'(0) = x'(b).$$

Therefore, $x \in W^{1,p}_{per}(T)$, with $x \ne 0, x(t) \le 0$ for all $t \in T$, which is a solution of (1). Since $|x'|^{p-2}x' \in W^{1,r'}(T)$, with $r' = \min\{q, r\}$, we have $|x'|^{p-2}x' \in C(T)$ and so, $x' \in C(T)$. Thus, $x \in C^1(T)$. ∎

3 Positive Solutions

In this section we establish the existence of a strictly positive solution for problem (1). Now the hypotheses on $f(t,x)$ are the following:

$\mathbf{H}(f)_2$: $f : T \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) $t \to f(t,x)$ is measurable for all $x \in \mathbb{R}$;

(ii) $x \to f(t,x)$ is continuous for almost all $t \in T$;

(iii) $|f(t,x)| \le a(t) + c|x|^{s-1}$, for almost all $t \in T$ and all $x \in \mathbb{R}$, some $c > 0$ and $a \in L^{s'}(T)$ with $\frac{1}{s} + \frac{1}{s'} = 1$ and $1 \le s < \infty$;

(iv) $f(t,x) \le g(t)$ a.e. $t \in T$ and all $x \ge M_0$ for some $M_0 > 0$ and all $g \in L^1(T)$ with $\int_0^b g(t)dt \le 0$;

(v) $f(t,\eta) \ge 0$ a.e. on T for some $\eta > 0$.

We now recall the definitions of upper and lower solutions for problem (1).

Definition: (a) A function $\psi \in C^1(T)$ with $|\psi'|^{p-2}\psi' \in W^{1,1}(T)$ is called a lower solution for problem (1) if

$$\begin{cases} -(|\psi'(t)|^{p-2}\psi'(t)) \le f(t,\psi(t)) \ \text{ a.e. on } \ T, \\ \psi(0) = \psi(b), \ \psi'(0) \ge \psi'(b). \end{cases}$$

(b) A function $\varphi \in C^1(T)$ with $|\varphi'|^{p-2}\varphi' \in W^{1,1}(T)$ is called an upper solution for problem (1) if

$$\begin{cases} -(|\varphi'(t)|^{p-2}\varphi'(t)) \ge f(t,\varphi(t)) \ \text{ a.e. on } \ T, \\ \varphi(0) = \varphi(b), \ \varphi'(0) \le \varphi'(b). \end{cases}$$

Proposition 2. *If hypothesis $H(f)_2$ holds, then problem (1) has a solution $x \in C^1(T)$ such that $x(t) > 0$ for all $t \in T$.*

Proof. Let $h(t) = g(t) - \bar{g}$ with $\bar{g} = \frac{1}{b}\int_0^b g(t)dt$, and consider the periodic problem

$$\begin{cases} -(|u'(t)|^{p-2}u'(t))' = h(t) \ \text{ a.e. on } \ T \\ u(0) = u(b), \ u'(0) = u'(b). \end{cases} \tag{10}$$

Let $a_0 : \mathbb{R} \to \mathbb{R}$ be the homeomorphism defined by $a_0(x) = |x|^{p-2}x$. For every $\theta \in C(T)$, let $G_0 : \mathbb{R} \to \mathbb{R}$ be the map defined by

$$G_0(\xi) = \int_0^b a_0^{-1}(\xi - \theta(t))dt.$$

From Proposition 2.2 of Manasevich-Mawhin [8], we know that the equation $G_0(\xi) = 0$ has a unique solution $\hat{\xi} \in \mathbb{R}$. Let $P : C(T) \to \mathbb{R}$ and $H : L^1(T) \to C(T)$ be the continuous linear maps defined by

$$P(x) = x(0) \quad \text{for all} \quad x \in C(T)$$

and

$$H(\sigma)(t) = \int_0^t \sigma(s)ds \quad \text{for all} \quad \sigma \in L^1(T).$$

Then, problem (10) has solutions $u \in W^{1,p}_{per}(T)$ given by

$$u(t) = Pu + H(a_0^{-1}(\hat{\xi}(H(h)) - H(h)))(t).$$

Let $\varphi(t) = u(t) + \gamma$ with $\gamma = \|u\|_\infty + M_0 + \eta$, where M_0 and η are from $H(f)_2(iv)$ and (v), respectively. Evidently, $\varphi(t) > M_0$ for all $t \in T$. So, we have, since $\bar{g} \leq 0$ and because of $H(f)_2(iv)$,

$$\begin{aligned} -\Big(|\varphi'(t)|^{p-2}\varphi'(t)\Big)' &= -\Big(|u'(t)|^{p-2}u'(t)\Big)' \\ &= h(t) \\ &\geq h(t) + \bar{g} \\ &= g(t) \\ &\geq f(t, \varphi(t)) \quad \text{a.e. on} \quad T. \end{aligned}$$

Hence $\varphi \in C^1(T)$ is an upper solution of problem (1).

Also, let $\psi(t) = \eta$, where η is from $H(f)_2(v)$. We have $f(t, \psi(t)) = f(t, \eta)$ on T and so, $\psi \in C^1(T)$ is a lower solution of problem (1). Moreover, $\psi(t) = \eta < \varphi(t)$ on T.

Next, let $w : T \times \mathbb{R} \to \mathbb{R}_+$ be the truncation function defined by

$$w(t,x) = \begin{cases} \psi(t) & \text{if} \ \ x < \psi(t), \\ x & \text{if} \ \ \psi(t) \leq x \leq \varphi(t), \\ \varphi(t) & \text{if} \ \ \varphi(t) < x. \end{cases}$$

Evidently, w is a Caratheodory function, i.e., measurable in t and continuous in x, thus jointly measurable, see Hu-Papageorgiou [7, p.142]. So, $|w(t,x)| = w(t,x) \leq \|\varphi\|_\infty$ for a.e. $t \in T$ and all $x \in \mathbb{R}$. Also, if $r = \max\{p, s\}$, we introduce the penalty function $\beta : T \times \mathbb{R} \to \mathbb{R}$, defined by

$$\beta(t,x) = \begin{cases} |\psi(t)|^{r-2}\psi(t) - |x|^{r-2}x & \text{if} \ \ x < \psi(t), \\ 0 & \text{if} \ \ \psi(t) \leq x \leq \varphi(t), \\ |\varphi(t)|^{r-2}\varphi(t) - |x|^{r-2}x & \text{if} \ \ \varphi(t) < x. \end{cases}$$

Set $f_1(t,x) = f(t, w(t,x))$ and let $G : W^{1,p}_{per}(T) \to L^{r'}(T)$, with $r' = \min\{q, s'\}$, be defined by

$$G(x) = N_{f_1}(x) + N_\beta(x).$$

Here N_{f_1} and N_β are the Nemitskii operators corresponding to f_1 and β respectively, i.e., $N_{f_1}(x)(\cdot) = f_1(\cdot, x(\cdot))$ while $N_\beta(x)(\cdot) = \beta(\cdot, x(\cdot))$ for all $x \in W^{1,p}_{per}(T)$. From Krasnoselskii's theorem we know that G is continuous. Also, let

$$\mathcal{D} = \left\{x \in C^1(T) : |x'|^{p-2}x' \in W^{1,r'}(T), x(0) = x(b), x'(0) = x'(b)\right\},$$

and let $L : \mathcal{D} \subset L^r(T) \to L^{r'}(T)$ be defined by

$$L(x) = -(|x'|^{p-2}x')', \quad \text{for} \quad x \in \mathcal{D}.$$

We claim that L is maximal monotone. An easy application of Green's identity shows that L is monotone. Now let $J : L^r(T) \to L^{r'}(T)$ be defined by $J(x) = |x|^{r-2}x$. Clearly, this is continuous and strictly monotone. To show the maximality of L, it suffices to show that $L + J$ is surjective, i.e., $R(L + J) = L^{r'}(T)$. Indeed, suppose that $L + J$ is surjective. Let $(\cdot, \cdot)_{rr'}$ denote the duality brackets for the pair $\left(L^r(T), L^{r'}(T)\right)$. Let $y \in L^r(T)$ and $v \in L^{r'}(T)$ be such that

$$0 \leq (L(x) - v, x - y)_{rr'}. \tag{11}$$

Since we assumed that $L + J$ to be surjective, we can find $x_1 \in \mathcal{D}$ such that $L(x_1) + J(x_1) = v + J(y)$. So, in (11) let $x = x_1 \in \mathcal{D}$, to obtain

$$\begin{aligned} 0 &\leq (L(x_1) - L(x_1) - J(x_1) + J(y), x_1 - y)_{rr'} \\ &= (J(y) - J(x_1), x_1 - y)_{rr'}. \end{aligned}$$

But recall that J is strictly monotone. So from the last inequality it follows that $y = x_1 \in \mathcal{D}$ and $v = L(x_1)$, which proves the maximality of L.

Thus it remains to prove that $R(L + J) = L^{r'}(T)$. This is equivalent to saying that for every $v \in L^{r'}(T)$ the following periodic problem has a solution:

$$\begin{cases} -\left(|x'(t)|^{p-2}x'(t)\right)' + |x(t)|^{r-2}x(t) = v(t) \text{ a.e. on } T \\ x(0) = x(b), \ x'(0) = x'(b). \end{cases} \tag{12}$$

But the solvability of (12) follows from Corollary 3.1 of Manasevich-Mawhin [8]. This proves the maximality of the strictly monotone operator $L + J$. Since $0 \in \mathcal{D}$ and $L(0) = 0$, for each $x \in \mathcal{D}$ we have

$$(L(x), x)_{rr'} + (J(x), x)_{rr'} \geq \|x\|_r^r,$$

so $L + J$ is coercive.

Because $L + J$ is maximal monotone and coercive, it is surjective, see Hu-Papageorgiou [7, p.322]. This, together with the strict monotonicity of $L+J$, implies that $K = (L + J)^{-1} : L^{r'}(T) \to \mathcal{D} \subset W^{1,p}(T)$ is well-defined and single-valued. We claim that K is completely continuous. To this end, we need to show that if $v_n \xrightarrow{w} v$ in $L^{r'}(T)$, then $K(v_n) \to K(v)$ in $W^{1,p}(T)$. Set $x_n = K(v_n)$ and $x = K(v)$. Then, $x_n \in \mathcal{D}$ for all $n \geq 1$ and, we have

$$L(x_n) + J(x_n) = v_n \text{ for all } \ n \geq 1,$$

which implies that $(L(x_n), x_n)_{rr'} + (J(x_n), x_n)_{rr'} = (v_n, x_n)_{rr'}$. Thus by Green's identity and Hölder's inequality we have

$$\|x_n'\|_p^p + \|x_n\|_r^r \leq \|v_n\|_{r'} \|x_n\|_r,$$

hence for some $c_1 > 0$, due to $p \leq r$ and $W^{1,p}(T) \subset L^r(T)$,

$$c_1 \|x_n\|_{1,p}^p \leq \|v_n\|_{r'} \|x_n\|_{1,p}.$$

Therefore, $\{x_n\}_{n\geq 1} \subset W^{1,p}(T)$ is bounded since $\sup_{n\geq 1} \|v_n\|_{r'} < \infty$.

Thus, we may assume that $x_n \xrightarrow{w} y$ in $W^{1,p}(T)$ and $x_n \to y$ in $C(T)$. Also, from the equation $L(x_n) + J(x_n) = v_n$ it follows that $\{|x_n'|^{p-2}x_n'\}_{n\geq 1} \subset W^{1,r'}(T)$ is bounded and so we may assume that $|x_n'|^{p-2}x_n' \xrightarrow{w} v$ in $W^{1,r'}(T)$, hence $|x_n'|^{p-2}x_n' \to v$ in $C(T)$. Recall that $a_0 : \mathbb{R} \to \mathbb{R}$ is the homeomorphism $a_0(x) = |x|^{p-2}x$. Let $\hat{a_0}^{-1} : C(T) \to C(T)$ be defined by

$$\hat{a_0}^{-1}(x)(\cdot) = a_0^{-1}(x(\cdot)).$$

Clearly, $\hat{a_0}^{-1}$ is continuous and bounded. So, in $C(T)$ we have

$$\hat{a_0}^{-1}(|x_n'|^{p-2}x_n') = x_n' \to \hat{a_0}^{-1}(v).$$

Hence, $v = |y'|^{p-2}y'$. So

$$|x_n'|^{p-2}x_n' \to |y'|^{p-2}y',$$

from which it follows that $x_n' \to y'$ in $C(T)$ and so, $x_n \to y$ in $W^{1,p}(T)$. Therefore, we conclude that $L(y) + J(y) = v$, so $y = K(v)$ and thus $y = x$. Consequently, $x_n \to x$ in $W^{1,p}(T)$ and this proves the complete continuity of K.

Let $G_1(x) = G(x) + J(x)$. Clearly, from the definition of G we see that there exists $M_1 > 0$ such that for all $x \in W^{1,p}(T)$ we have

$$\|G_1(x)\|_{r'} \leq M_1.$$

Define

$$V = \left\{v \in L^{r'}(T) : \|v\|_{r'} \leq M_1\right\}.$$

Then, $KG_1(W^{1,p}(T)) = K(V)$ and the latter is relatively compact in $W^{1,p}(T)$ since K is completely continuous. So, by Schauder fixed point theorem, we can find $x \in \mathcal{D} \subset W^{1,p}(T)$ such that

$$x = KG_1(x),$$

so $L(x) = G(x)$. Therefore, we have

$$\begin{cases} -\left(|x'(t)|^{p-2}x'(t)\right)' = f(t, w(t, x(t))) + \beta(t, x(t)) \text{ a.e. on } T \\ x(0) = x(b),\ x'(0) = x'(b). \end{cases}$$

We need to show now that $\psi(t) = \eta \leq x(t) \leq \varphi(t)$ on T. Recall that $f(t, \psi(t)) = f(t, \eta) \geq 0$ a.e. on T, see $H(f)_2(v)$. So, we have

$$-\left(|x'(t)|^{p-2}x'(t)\right)' \geq f(t, w(t, x(t))) + \beta(t, x(t)) - f(t, \eta) \text{ a.e. on } T,$$

hence

$$\begin{aligned}\int_0^b -(|x'(t)|^{p-2}x'(t))'(\psi - x)_+(t)dt \\ \geq \int_0^b (f(t, w(t, x(t))) - f(t, \eta) + \beta(t, x(t)))(\psi - x)_+(t)dt.\end{aligned}$$

Employing Green's identity and because of the periodic boundary conditions, we obtain

$$\begin{aligned}\int_0^b -(|x'(t)|^{p-2}x'(t))'(\psi - x)_+(t)dt &= \int_0^b |x'(t)|^{p-2}x'(t)(\psi - x)'_+(t)dt \\ &= \int_{\{\psi > x\}} -|x'(t)|^p dt.\end{aligned}$$

Also, by the definition of w and the fact that $\psi(t) = \eta$, we have

$$\begin{aligned}0 &= \int_{\{\psi > x\}} (f(t, \psi(t)) - f(t, \eta))(\psi - x)(t)dt \\ &= \int_0^b (f(t, w(t, x(t))) - f(t, \eta))(\psi - x)_+(t)dt.\end{aligned}$$

So, we obtain

$$\begin{aligned}0 &\geq -\int_{\{\psi > x\}} |x'(t)|^p dt \\ &\geq \int_0^b \beta(t, x(t))(\psi - x)_+(t)dt \\ &= \int_{\{\psi > x\}} (|\psi(t)|^{r-2}\psi(t) - |x(t)|^{r-2}x(t))(\psi - x)_+(t)dt \\ &> 0,\end{aligned}$$

a contradiction. This proves that $\psi(t) \leq x(t)$ on T. Similarly, we can show that $x(t) \leq \varphi(t)$ on T. Therefore, $w(t, x(t)) = x(t)$ and $\beta(t, x(t)) = 0$ for all $t \in T$. Thus,

$$\begin{cases} -(|x'(t)|^{p-2}x'(t))' = f(t, x(t)) \text{ a.e. on } T, \\ x(0) = x(b), \ x'(0) = x'(b). \end{cases}$$

Hence, $x \in C^1(T)$ is a solution of problem (1) and, $x(t) \geq \psi(t) = \eta > 0$ for all $t \in T$.

■

4 Pairs of Solutions of Constant Sign

Combining Propositions 1 and 2, we can prove a multiplicity result for problem (1) with an unbounded nonlinearity f. The hypotheses on $f(t,x)$ are the following:

$\mathbf{H}(f)_3$: $f : T \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(t,0) \leq 0$ a.e. on T and

(i) $t \to f(t,x)$ is measurable for all $x \in \mathbb{R}$;

(ii) $x \to f(t,x)$ is continuous for almost all $t \in T$;

(iii) $|f(t,x)| \leq a(t) + c|x|^{s-1}$ for a.e. $t \in T$ and all $x \in \mathbb{R}$, with some $c > 0$ and $a \in L^{s'}(T)$ with $\frac{1}{s} + \frac{1}{s'} = 1$ and $1 \leq s < \infty$;

(iv) $\lim_{x \to -\infty} \frac{pF(t,x)}{|x|^p} = 0$ uniformly for a.e. $t \in T$ with $F(t,x) = \int_0^x f(t,r)dr$, and there exists $M > 0$ such that $f(t,x) \geq 0$ or $f(t,x) \leq 0$ for a.e. $t \in T$ and all $x \leq -M$;

(v) $\lim_{x \to -\infty}(xf(t,x) - pF(t,x)) = \infty$ uniformly for almost all $t \in T$;

(vi) $f(t,x) \leq g(t)$ for a.e. $t \in T$ and all $x \geq M_0$ with some $M_0 > 0$, and some $g \in L^1(T)$ with $\int_0^b g(t)dt \leq 0$;

(vii) $F(t,\eta_1) > 0$ and $f(t,\eta_2) \geq 0$ a.e. on T, for some $\eta_1 < 0 < \eta_2$.

Theorem 3. *If hypothesis $H(f)_3$ holds, then problem (1) has two solutions $x, y \in C^1(T)$ such that*

$$x \neq 0, \ x(t) \leq 0 \ \textit{and} \ y(t) > 0 \ \textit{for all} \ t \in T.$$

Consider the following function (the t-dependence is dropped for simplicity):

$$f(x) = \begin{cases} x^2 & \text{if } x < -1, \\ x & \text{if } -1 \leq x \leq 1, \\ \sin x^2 - \sin 1 - x \ln x & \text{if } x > 1. \end{cases}$$

Then if $p > 3$, it is easy to check that f satisfies $H(f)_3$ with the first option in $H(f)_3(iv)$ valid. Similarly, with $p = 2$, we can have the function

$$f(x) = \begin{cases} \sqrt{|x|} & \text{if } x < -1, \\ x & \text{if } x \in [-1,0], \\ 0 & \text{if } x \in [0,5], \\ -x^4 & \text{if } x > 5. \end{cases}$$

Finally, if $p > 7$ the function

$$f(x) = \begin{cases} -x^6 & \text{if } x < -1, \\ 2x+1 & \text{if } x \in [-1,1], \\ 4-x & \text{if } x > 1. \end{cases}$$

satisfies $H(f)_3$ with the second option in $H(f)_3(iv)$ valid.

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