Homotopy pull backs, homotopy push outs and joins

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Abstract

We show here how the techniques based on homotopy pull backs and push outs lead to simple proofs for apparently difficult (known or unknown) results. They can be used not only in the category of topological spaces, but also in any Quillen's model category. Many of them rely on the two 'join theorems' we prove here. Further applications are the study of holonomy, or of the Lusternik-Schnirelmann category.

Many of the usual constructions in topology are nothing else but homotopy pull backs, homotopy push outs, or joins (which are a combination of the two formers). Loop spaces, suspensions, mapping cones, Ganea spaces, Whitehead's fat wedges, holonomy, for instance, involve such constructions.

After having defined in section 1 'homotopy pull backs' and 'homotopy push outs' in the general context of a Quillen's model category, we introduce in section 2 the 'join' and 'smash product' constructions and give their properties. Section 3 is quite central as it is devoted to state and prove the two 'join theorems' which are key theorems in the sequel. Section 4 gives applications of the join theorems. Some known difficult results as those of Ganea (4.3) or Marcum (4.1) appear here are easy consequences of the join theorems. In section 5 we go further with Ganea's 'fibre-cofibre' and Whitehead's 'fat wedge' constructions. At last, we study the holonomy of a join of fibration sequences in section 6. All this appears unified by the same kind of techniques that rely on the same small amount of axioms and basic properties.

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1 Basic definitions and properties

In this section, we recall the definitions of homotopy pull back and homotopy push out in topology, and extend them to Quillen's model categories. We also give their few basic properties ; everything in the sequel rely on the 'prism lemma', the 'four (co)fibrations lemma' and the 'cube axiom' we state at the end of the section. We keep the text self-contained and, as often as we know, we give references to similar notions existing in the literature.

We denote by $*$ the base point of any pointed topological space X ; we denote by I_*X the reduced cylinder on X. A pointed topological space is *well-pointed* if the inclusion of the base point is a closed cofibration. Let us consider the category $\mathbf{Top}^w$ of well-pointed topological spaces and pointed continuous maps between them. A map $H : I_*X \to Y$ is a *pointed homotopy* between pointed maps f and g if $H(x,0) = f(x)$ and $H(x,1) = g(x)$. In this case we write $H : f \sim g$. If H and G are two such homotopies, a map $K : I_*I_*X \to Y$ is a *pointed homotopy relative to* (f,g) between H and G if $K(x,s,0) = H(x,s)$, $K(x,s,1) = G(x,s)$, $K(x,0,t) = f(x)$, and $K(x,1,t) = g(x)$. In this case we write $K : H \sim G$.

A *homotopy commutative diagram* in $\mathbf{Top}^w$ is a diagram of pointed continuous maps where each two composites of maps, with same source and target spaces, is equipped with a pointed homotopy between them –the homotopy between f and itself being the obvious homotopy $H(x,s) = f(x)$, that we also denote by f–, and if two composites and/or sum of such homotopies are pointed homotopies between the same two pointed maps f and g, then there is a pointed homotopy relative to (f,g) between them.

Definition 1.1 ([21]) *A homotopy commutative diagram*

$$\begin{tikzcd} P \arrow[r, "f_1"] \arrow[d, "g_1"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

equipped with $H : gf_1 \sim fg_1$, is called a homotopy pull back *when for any homotopy commutative diagram*

$$\begin{tikzcd} D \arrow[r, "f_2"] \arrow[d, "g_2"'] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

equipped with $G : gf_2 \sim fg_2$, the following properties hold :

(i) there exists a map $w : D \to P$ (called whisker map*) and homotopies $K : f_2 \sim f_1 w$ and $L : g_1 w \sim g_2$ such that the whole diagram*

$$\begin{tikzcd}
D \arrow[rr] \arrow[dd] \arrow[dr, dotted] & & C \arrow[dd] \\
& P \arrow[ur] \arrow[dl] & \\
A \arrow[rr] & & B
\end{tikzcd}$$

with all maps and homotopies above is homotopy commutative (which means that $g \circ K + H \circ w + f \circ L \sim G$) ;

(ii) if there exists another map $w' : D \to P$ and homotopies $K' : f_2 \sim f_1 w'$ and $L' : g_1 w' \sim g_2$ such that $g \circ K' + H \circ w' + f \circ L' \sim G$, then there exists a homotopy $M : w \sim w'$ such that the whole diagram with all maps and homotopies above is homotopy commutative (wich means that $K + f_1 \circ M \sim K'$ and $g_1 \circ M + L' \sim L$).

The notion of homotopy pull back dualizes to the notion of *homotopy push out.* 'Dualize' means here 'reverse the direction of arrows'.

There is a 'standard' construction of the homotopy pull back of any two maps $f : A \to B$ and $g : C \to B$ in $\mathbf{Top}^w$ as the *mapping track* :

$$E_{f,g} \cong \{(a, \omega, c) \in A \,\Pi\, B^I \,\Pi\, C \mid f(a) = \omega(0) \text{ and } g(c) = \omega(1)\}$$

(B^I is the free paths space of B) with the obvious maps $E_{f,g} \to A$ and $E_{f,g} \to C$ and homotopy $H : I_* E_{f,g} \to B : ((a, \omega, c), t) \mapsto \omega(t)$.

Dually there is a 'standard' construction of the homotopy push out of any two maps $f : B \to A$ and $g : B \to C$ in $\mathbf{Top}^w$ as the *mapping torus* :

$$Z_{f,g} \cong A \amalg I_* B \amalg C \;/\; f(b) \sim (b, 0) \text{ and } g(b) \sim (b, 1)$$

with the obvious maps $A \to Z_{f,g}$ and $C \to Z_{f,g}$ and homotopy $K : I_* B \to Z_{f,g} : (b, t) \mapsto (b, t)$. (See also [21].)

Remark : The homotopy H is an essential data of the homotopy pull back ; for instance, let ΩX be the loop space of X and consider the diagram

$$\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & X
\end{tikzcd}$$

Equipped with the homotopy $H : I_*(\Omega X) \to X : (\omega, t) \mapsto \omega(t)$, it is a homotopy pull back. It is *not* a homotopy pull back equipped with the 'static' homotopy $G : I_*(\Omega X) \to X : (\omega, t) \mapsto *$.

The notions of homotopy pull back and homotopy push out can be extended to *2-categories* (see [9]), Quillen's *model categories* (see [23]), Baues' *fibration and cofibration categories* (see [1]) or Majewski's *homotopical categories* (see [19]).

From now on, we will assume that $\mathbf{C}$ is a pointed Quillen's model category. 'Pointed' means here that there is a *zero* object (*i.e.* both final and initial) in the category, that we denote by $*$. The first axiom 'M0' of Quillen can be replaced by the slightly weaker one : 'the pull back of any fibration and any map (with same target) exists and the push out of any cofibration and any map (with same source) exists' ; we will keep the terminology 'model category' in this case.

We say that an object X of $\mathbf{C}$ is *fibrant* (respectively *cofibrant*) if $X \to *$ is a fibration (respectively $* \to X$ is a cofibration). Let $f : A \to B$ and $g : C \to B$ be two maps in $\mathbf{C}$; let us build the following commutative diagram (†) :

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\simeq"] & B \arrow[d, "\simeq"] & C \arrow[l, "g"'] \arrow[d, "\simeq"] \\
A' \arrow[r, "p"'] & B' & C' \arrow[l, "q"]
\end{tikzcd}$$

where A', B', C' are *fibrant*, either p or q is a fibration, and the maps marked with $\simeq$ are weak equivalences. Then the pull back P of p and q is called the *homotopy pull back* of f and g. (See also [2], [1], [18], [6], [7], [16], [8].)

Remark : The condition 'A', B', C' fibrant' is not necessary if $\mathbf{C}$ is *proper* (see [2] 1.2), so in this case one can choose $p = f$ or $q = g$.

The construction of the *homotopy push out* is dual. 'Dualize' in a model category means 'reverse the direction of arrows, keep weak equivalences, change fibrations to cofibrations, pull backs to push outs, fibrant objects to cofibrant ones'.

The category $\mathbf{Top}^w$ is a model category where the weak equivalences are the homotopy equivalences, and all spaces are both fibrant and cofibrant objects (see [25]). The standard homotopy pull back is a particular case of the above construction (†) where $p = f$, $C' = E_g \cong \{(\omega, c) \in B^I \amalg C \mid g(c) = \omega(1)\}$, $q : E_g \to B : (\omega, c) \mapsto \omega(0)$ and $P = E_{f,g}$.

Remark : The construction (†) of the homotopy pull back in the model category $\mathbf{C}$ does *not* necessarily give a homotopy commutative diagram

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & C \arrow[d] \\
A \arrow[r] & B
\end{tikzcd}$$

in $\mathbf{C}$ (even if it *does* in $\mathbf{Top}^w$). Indeed the map $P \to A'$ (respectively $P \to C'$) can not be 'lifted' to a map $P \to A$ (respectively $P \to C$) because weak equivalences are not necessarily homotopy equivalences. This makes the theory of homotopy pull backs very unconfortable to build in $\mathbf{C}$. However there exists such a commutative diagram in the homotopy category $Ho(\mathbf{C})$, and a homotopy commutative representative in $\mathbf{C}_{cf}$ –we describe $Ho(\mathbf{C})$ and $\mathbf{C}_{cf}$ just below.

The *homotopy category* $Ho(\mathbf{C})$ of $\mathbf{C}$ is the category whose objects are the same as those of $\mathbf{C}$ and whose maps are obtained from those of $\mathbf{C}$ by formally inverting the weak equivalences ([23] I.1.12). Let $\mathbf{C}_{cf}$ be the full subcategory of $\mathbf{C}$ whose objects are those of $\mathbf{C}$ which are both cofibrant and fibrant ; the category $Ho(\mathbf{C})$ is equivalent to the quotient category $\mathbf{C}_{cf}/\sim$ whose objects are those of $\mathbf{C}_{cf}$ and whose maps are the homotopy classes of maps of $\mathbf{C}_{cf}$. The obvious functor $\gamma : \mathbf{C} \to Ho(\mathbf{C})$ carries weak equivalences to isomorphisms, and homotopy commutative diagrams to commutative ones.

Remark : The functor γ does *not* carry a homotopy pull back in $\mathbf{C}$ to a pull back in $Ho(\mathbf{C})$. For instance, the zero object $*$ of $\mathbf{C}$ is also the zero object of $Ho(\mathbf{C})$, so for any X there is a pull back in $Ho(\mathbf{C})$

$$\begin{tikzcd} * \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & * \arrow[d] \\ * \arrow[r] & X \end{tikzcd}$$

which is *not* the homotopy pull back ΩX of $* \to X$ and $* \to X$.

As there is a homotopy equivalence relation in $\mathbf{C}_{cf}$, the notion of homotopy commutative diagram also exists in $\mathbf{C}_{cf}$. The diagrams we shall draw from now on are homotopy commutative diagrams in $\mathbf{C}_{cf}$. The definition 1.1 of homotopy pull back now gets sense for homotopy commutative squares in the setting of a model category. The construction (†) in $\mathbf{C}$ above –or more precisely any representative in $\mathbf{C}_{cf}$ of its image by γ in $Ho(\mathbf{C})$, equipped with the appropriate homotopy– satisfies this definition.

Definitions and notations 1.2 *The sign $\simeq$ denotes an isomorphism in $Ho(\mathbf{C})$.*

We write $P \simeq A \times_B C$ if there is a homotopy pull back

$$\begin{tikzcd} P \arrow[r, "f_1"] \arrow[d, "g_1"'] \arrow[dr, phantom, "h.p.b."] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

The map f_1 will be called homotopy base extension of f (by g). *If $B \simeq *$, we write $P \simeq A \times C$. If $C \simeq *$, we call P (or the map g_1) the* homotopy fibre of f, *and we call the sequence of maps $P \to A \to B$ a* fibration sequence. *In particular $\Omega X = * \times_X *$ is called the* loops of X.

Dually, we write $S \simeq A \vee_B C$ if there is a homotopy push out

$$\begin{tikzcd} B \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "h.p.o."] & C \arrow[d] \\ A \arrow[r] & S \end{tikzcd}$$

If $C \simeq *$, we call S the homotopy cofibre *of f and we write $S \simeq A/B$. In particular $\Sigma X = * \vee_X * \simeq */X$ is called the* suspension of X.

Although a homotopy pull back is not a pull back in $Ho(\mathbf{C})$, it has a similar behaviour : The homotopy base extension of an isomorphism in $Ho(\mathbf{C})$ is an isomorphism in $Ho(\mathbf{C})$; isomorphic maps in $Ho(\mathbf{C})$ have isomorphic homotopy base extensions in $Ho(\mathbf{C})$; the homotopy pull back of maps with same target is symmetric and associative up to isomorphism in $Ho(\mathbf{C})$. This of course dualizes to homotopy push outs.

The three properties we describe now are basic ones for the techniques using homotopy pull back and push outs. We call them the *prism lemma*, the *four (co)fibrations lemma*, and the *cube axiom*. We don't give the proof here but we give references for them.

Lemma 1.3 (Prism Lemma.) (Compare [21] lemmas 12 and 14, [7] 2.5.) *Let be a homotopy commutative diagram*

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, dotted] \arrow[dd] & & C \arrow[dd] \\
& B \arrow[ur] \arrow[dd] & \\
A' \arrow[rr] \arrow[dr] & & C' \\
& B' \arrow[ur] &
\end{tikzcd}$$

where $B - C - C' - B'$ is a homotopy pull back and the dotted map is the whisker map. Then $A - B - B' - A'$ is a homotopy pull back if and only if $A - C - C' - A'$ is a homotopy pull back.

Note the particular case where $A' \simeq *$. In this case, the lemma asserts that if $B - C - C' - B'$ is a homotopy pull back, then the maps $B \to B'$ and $C \to C'$ have common homotopy fibre A.

As another example, if $A \to B \to C$ is a cofibration sequence, we see that the homotopy cofibre of $B \to C$ is ΣA applying the dual lemma (called 'prism lemma', too) to the diagram :

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & C \arrow[r] & \Sigma A
\end{tikzcd}$$

Lemma 1.4 (Four fibrations lemma.) ([3]) *Let be a homotopy commutative diagram*

$$\begin{tikzcd}
 & D \arrow[rr] \arrow[dl] \arrow[dd] & & C \arrow[dl] \arrow[dd] \\
A \arrow[rr] \arrow[dd] & & B \arrow[dd] & \\
 & D' \arrow[rr] \arrow[dl] \arrow[dd] & & C' \arrow[dl] \arrow[dd] \\
A' \arrow[rr] \arrow[dd] & & B' \arrow[dd] & \\
 & D'' \arrow[rr] \arrow[dl] & & C'' \arrow[dl] \\
A'' \arrow[rr] & & B'' &
\end{tikzcd}$$

where $D' - C' - B' - A'$ and $D'' - C'' - B'' - A''$ are homotopy pull backs, the sequences $A \to A' \to A'', B \to B' \to B''$, and $C \to C' \to C''$ are fibration sequences, and the maps $D \to D'$ and $D' \to D''$ are the whisker maps. Then $D - C - B - A$ is a homotopy pull back if and only if $D \to D' \to D''$ is a fibration sequence.

Note the particular case where $A \simeq A' \simeq A'' \simeq *$. In this case the three horizontal squares are fibration sequences, symmetrically to the three remaining vertical ones.

These two lemmas are plain transposition of properties of the true pull back in any category, and are consequences of the axioms of model category. As these axioms are autodual, the dual lemmas are also true. 'Dualize' here means 'reverse the directions of the arrows, replace homotopy pull backs by homotopy push outs, fibration sequences by cofibration sequences'.

Definition 1.5 (Compare [8].) *A* cube-map *is a map $f : A \to B$ such that for any homotopy base extension $f' : A' \to B'$ of f and any homotopy commutative diagram*

$$\begin{tikzcd}
 & \cdot \arrow[rr] \arrow[dl] \arrow[dd] & & \cdot \arrow[dl] \arrow[dd] \\
\cdot \arrow[rr] \arrow[dd] & & A' \arrow[dd, "f'"] & \\
 & \cdot \arrow[rr] \arrow[dl] & & \cdot \arrow[dl] \\
\cdot \arrow[rr] & & B' &
\end{tikzcd}$$

where f' is the front map, if the bottom face is a homotopy push out, the four vertical faces are homotopy pull backs, then the top face is a homotopy push out.

Note that the homotopy base extension of a cube-map is a cube-map, and that the composite of cube-maps is a cube-map.

Now let us state :

Axiom 1.6 (Cube axiom.) *All maps are cube-maps.*

The category **Top*** of pointed topological spaces, and in fact also the category **Top**^{w} of well-pointed topological spaces, satisfy the cube axiom ([21] theorem 25, compare [6] chapter 6). Also the category **S*** of pointed simplicial sets satisfies the cube axiom ([7] A.8). The category **Chain** of graded differential modules over a ring R with unit (bounded below, differential of degree -1) satisfies both the cube axiom and its dual ([6] chapter 6).

Note the cube axiom is *not* a consequence of the axioms of model category : if it was the case, the dual of the cube axiom would also be true in any model category, but it is not. For instance, in the cube :

$$\begin{tikzcd}
& S^2\times S^3 \arrow[rr] \arrow[dl] \arrow[dd] & & S^3 \arrow[dl] \arrow[dd] \\
S^2 \arrow[rr] \arrow[dd] & & * \arrow[dd] & \\
& * \arrow[rr] \arrow[dl] & & \cdot \arrow[dl] \\
\cdot \arrow[rr] & & \cdot &
\end{tikzcd}$$

where the top face is a homotopy pull back and the vertical faces are homotopy push outs, the bottom face is *not* a homotopy pull back ; thus the dual of the cube axiom is not true in **Top**^{w}. So the cube axiom breaks the duality between homotopy pull backs and homotopy push outs.

In order to obtain results valid also in categories where the cube axiom is *not* satisfied, we will *not* assume that **C** satisfies it, and will always specify when we need a map to be a cube-map. It is clear that isomorphisms in $Ho(\mathbf{C})$, so weak equivalences in **C**, are cube-maps. However, applications require that there are 'as many cube-maps as possible'. Fortunately this is the case for all the model categories we work with ; more precisely, algebraic categories where topological spaces are modelized via a covariant functor have many cube-maps, while categories where topological spaces are modelized via a contravariant functor have many *dual* cube-maps (dual notion of 1.5).

Let **S**(r) be the category of r-reduced simplicial sets ($r \geq 1$), and let be $\mathcal{S}$ a multiplicative system in $\mathbf{Z}$ ($\mathcal{S} = \{1\}$ if $r = 1$). The category **S**(r) is a model category where weak equivalences are maps f such that $\mathcal{S}^{-1}\pi_*(f)$ is an isomorphism (see [24] II.2). The maps f such that $\mathcal{S}^{-1}\pi_r(f)$ is surjective are cube-maps ([8] proposition 11). Take $\mathcal{S} = \mathbf{Z} - \{0\}$; the diagonal $\Delta : S_0^r \to S_0^r \times S_0^r$, where S_0^r is the rational sphere of dimension r, is *not* a cube map.

Let R be a (commutative) principal ideal domain. Let $\mathbf{DA}_*$(flat) be the category of augmented differential algebras with unit over R (bounded below in degree 0, differential of degree -1), which are flat as R-modules. The category $\mathbf{DA}_*$(flat) is a model category where weak equivalences are maps f such that $H_*(f)$ is an isomorphism (see [22], compare [1] I.7.10 and [7] A.15). The maps f such that $H_0(f)$ is surjective are cube-maps ([7] A.15).

Let $\mathbf{CDA}_*$(c0) be the category of augmented differential commutative algebras over a field k of caracteristic 0 (bounded below in degree 0, differential of degree +1), whose augmentation ϵ induces an isomorphism $H^0(\epsilon)$. The category $\mathbf{CDA}_*$(c0) is a model category where weak equivalences are maps f such that $H^*(f)$ is an isomorphism (see [13]). The maps f such that $H^0(f)$ is an isomorphism and $H^1(f)$ is injective are dual cube-maps ([7] A.19).

2 Joins and smash products

This section is devoted to present the definitions and basic facts about joins and smash products. It may serve as a reference for the next sections. Almost everything here rely on the 'prism lemma' (1.3) and the 'four (co)fibrations lemma' (1.4).

Definition 2.1 *Let $A \to B$ and $C \to B$ be two maps, and let $P \simeq A \times_B C$, $J \simeq A \vee_P C$:*

$$\begin{tikzcd}
P \arrow[rr] \arrow[dd] & & C \arrow[dl] \arrow[dd] \\
& J \arrow[dr, dotted] & \\
A \arrow[ur] \arrow[rr] & & B
\end{tikzcd}$$

Then we call J (or the whisker map $J \to B$) the join of A and C over B, *and we write $J \simeq A \bowtie_B C$. If $B \simeq *$, we just write $J \simeq A \bowtie C$.*

Here is the dual notion :

Definition 2.2 *Let $B \to A$ and $B \to C$ be two maps, and let $S \simeq A \vee_B C$, $K \simeq A \times_S C$:*

$$\begin{tikzcd}
B \arrow[rr] \arrow[dd] \arrow[dr, dotted] & & C \arrow[dd] \\
& K \arrow[ur] \arrow[dl] & \\
A \arrow[rr] & & S
\end{tikzcd}$$

Then we call K (or the whisker map $B \to K$) the cojoin of A and C under B, *and we write $K \simeq A \diamond_B C$. If $B \simeq *$, we just write $K \simeq A \diamond C$.*

Definition 2.3 *An object X is B-*sectioned *if there is a homotopy commutative diagram*

$$\begin{tikzcd}
B \arrow[rr, "1_B"] \arrow[dr, "s"'] & & B \\
& X \arrow[ur, "r"'] &
\end{tikzcd}$$

*A map $f : X \to Y$ is said B-*sectioned *if X and Y are B-sectioned and the following diagram is homotopy commutative*

$$\begin{tikzcd}
& X \arrow[dr, "r"] \arrow[dd, "f"] & \\
B \arrow[ur, "s"] \arrow[dr, "s'"'] & & B \\
& Y \arrow[ur, "r'"'] &
\end{tikzcd}$$

In particular, all objects and all maps are $*$-sectioned.

The notion we introduce now is a generalization of the *smash product* to B-sectioned objects. In particular, this leads to the generalization of the decomposition of $\Sigma(A \times C)$ to a decomposition of $\Sigma(A \times_B C)$ for B-sectionned objects A and C (2.12).

Let A and C be B-sectioned. Using the prism lemma, we see that all squares in the following diagram are homotopy push outs :

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & B \arrow[d] \\
A \arrow[r] \arrow[d] & A \vee_B C \arrow[r] \arrow[d] & A \arrow[d] \\
B \arrow[r] & C \arrow[r] & B
\end{tikzcd}$$

Let us consider the following diagram :

$$\begin{tikzcd}
A \vee_B C \arrow[rr] \arrow[dd] \arrow[dr, dotted] & & C \arrow[dd] \\
& A \times_B C \arrow[ur] \arrow[dl] & \\
A \arrow[rr] & & B
\end{tikzcd}$$

where the exterior square is the homotopy push out above and the dotted map is the whisker map of the pull back. Let $P \simeq A \times_B C$ and $S \simeq A \vee_B C$. We have $P \simeq A \Diamond_S C$ and $S \simeq A \bowtie_P C$, and the whisker map $S \to P$ is both the join map of A and C over P and the cojoin map of A and C under S.

Definition 2.4 *Let A and C be B-sectioned. The homotopy cofibre of the map $A \vee_B C \to A \times_B C$ above is called the* smash pull back of A and C over B *and denoted by $A \wedge_B C$. If $B \simeq *$, we call it the* smash product *and denote it by $A \wedge C$.*

Dually, the homotopy fibre of the map $A \vee_B C \to A \times_B C$ is denoted by $A \flat_B C$. If $B \simeq *$, we just note it $A \flat C$.

Proposition 2.5 *Let A and C be B-sectioned. Then there is a cofibration sequence :*

$$B \to A \bowtie_B C \to \Sigma(A \wedge_B C).$$

Proof. Apply the four cofibrations lemma to

$$\begin{tikzcd}
& A \vee_B C \arrow[rr] \arrow[dl] \arrow[dd] & & C \arrow[dl] \arrow[dd] \\
A \arrow[rr, crossing over] \arrow[dd] & & B & \\
& A \times_B C \arrow[rr] \arrow[dl] \arrow[dd] & & C \arrow[dl] \arrow[dd] \\
A \arrow[rr] \arrow[dd] & & A \bowtie_C B \arrow[from=uu, crossing over] & \\
& A \wedge_B C \arrow[rr] \arrow[dl] & & * \arrow[dl] \\
* \arrow[rr] & & \Sigma(A \wedge_B C) \arrow[from=uu, crossing over] &
\end{tikzcd}$$

■

Corollary 2.6 *For any objects A and C, $A \bowtie C \simeq \Sigma(A \wedge C)$.*

The dual of the previous proposition 2.5 also holds :

Proposition 2.7 *Let A and C be B-sectioned. Then there is a fibration sequence :*

$$\Omega(A \flat_B C) \to A \diamond_B C \to B.$$

Corollary 2.8 *For any objects A and C, $A \diamond C \simeq \Omega(A \flat C)$.*

Proposition 2.9 *Let A be B-sectioned and let $X \to Y$ be a B-sectioned map. Let $S \simeq Y \vee_X B$. If $A \to B$ is a cube-map, then there is a cofibration sequence*

$$A \wedge_B X \to A \wedge_B Y \to A \wedge_B S.$$

Proof. By the prism lemma, we have a homotopy push out (†) :

$$\begin{tikzcd}
A \vee_B X \arrow[r] \arrow[d] & A \arrow[d] \\
A \vee_B Y \arrow[r] & A \vee_B S
\end{tikzcd}$$

On the other hand, as $A \times_B S \to S$ is the base extension of the cube-map $A \to B$, the top square of the following cube is a homotopy push out (‡) :

$$\begin{tikzcd}
& A \times_B X \arrow[rr] \arrow[dl] \arrow[dd] & & A \arrow[dl] \arrow[dd] \\
A \times_B Y \arrow[rr, crossing over] \arrow[dd] & & A \times_B S & \\
& X \arrow[rr] \arrow[dl] & & B \arrow[dl] \\
Y \arrow[rr] & & S \arrow[from=uu, crossing over] &
\end{tikzcd}$$

Finally apply the four cofibrations lemma where (†) is the first horizontal face, (‡) is the second horizontal face and the third horizontal face is the expected homotopy push out. ∎

If X is B-sectioned, we note $\Sigma_B X = B \vee_X B$, and $\Omega_B X = B \times_X B$.

Corollary 2.10 *Let A and X be B-sectioned. If $A \to B$ is a cube-map, then $\Sigma(A \wedge_B X) \simeq A \wedge_B (\Sigma_B X)$.*

Proof. Apply the previous proposition to the B-sectioned map $r : X \to B$. ∎

Corollary 2.11 *For any objects A and X, if $A \to *$ is a cube-map, then we have $\Sigma(A \wedge X) \simeq A \wedge (\Sigma X)$.*

Proposition 2.12 *Let A and C be B-sectioned. Then*

$$\Sigma(A \times_B C) \simeq (A \bowtie_B C) \vee_B (B/A) \vee_B (B/C).$$

Proof. Immediate by lemmas 2.16 and 2.18 below. ∎

Corollary 2.13 *For any objects A and C,*

$$\Sigma(A \times C) \simeq (A \bowtie C) \vee \Sigma A \vee \Sigma C.$$

The dual of the above proposition 2.12 also holds :

Proposition 2.14 *Let A and C be B-sectioned, and let F and F' be the homotopy fibres of $B \to A$ and $B \to C$ respectively. Then*

$$\Omega(A \vee_B C) \simeq (A \diamond_B C) \times_B F \times_B F'.$$

Corollary 2.15 *For any objects A and C,*

$$\Omega(A \vee C) \simeq (A \diamond C) \times \Omega A \times \Omega C.$$

We say that a map $f : X \to Y$ *factorizes through B* if there is a homotopy commutative diagram :

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \\ & B \arrow[ur] & \end{tikzcd}$$

In particular, f is said to be *null-homotopic* if f factorizes through $*$.

Lemma 2.16 *If A and C are B-sectioned, then the maps $A \to A \bowtie_B C$ and $C \to A \bowtie_B C$ in the homotopy push out*

$$\begin{tikzcd} A \times_B C \arrow[r] \arrow[d] \arrow[dr, phantom, "h.p.o."] & C \arrow[d] \\ A \arrow[r] & A \bowtie_B C \end{tikzcd}$$

factorize through B.

Proof. Look at the top cube of the diagram in the proof of proposition 2.5. ∎

Lemma 2.17 *If $X \to Y$ factorizes through B, then $(Y/X) \simeq (B/X) \vee_B Y$.*

Proof. Use the prism lemma. ∎

Lemma 2.18 *Let $S \simeq X \vee_Y Z$ a homotopy push out, and assume the maps $X \to S$ and $Z \to S$ factorize through B. Then $\Sigma Y \simeq (B/X) \vee_B (B/Z) \vee_B S$.*

Proof. As $S \simeq X \vee_Y Z$, we have $Z/Y \simeq S/X$, and as $X \to S$ factorizes through B, we have $S/X \simeq (B/X) \vee_B S$. Recall from the Puppe sequence that the homotopy cofibre of $Z \to Z/Y$ is ΣY. As $Z \to S$ factorizes through B, so does $Z \to Z/Y$. Thus we have $\Sigma Y \simeq (B/Z) \vee_B (Z/Y) \simeq (B/Z) \vee_B (B/X) \vee_B S$. ∎

3 Join theorems

In this section, we prove two main theorems about joins. They will be key theorems in the sequel. The first one asserts that the 'join of homotopy pull backs' is a homotopy pull back. The second asserts that the 'join of a homotopy pull back and a homotopy push out' is a homotopy push out. The 'cube axiom' –or the notion of 'cube-map' (1.5)– plays an essential role here.

Theorem 3.1 (Join theorem I.) (Compare [7], 2.7.) *Let $B \to B'$ be a cube-map, and let be two homotopy pull backs*

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "h.p.b."] & B \arrow[d] \arrow[dr, phantom, "h.p.b."] & C \arrow[l] \arrow[d] \\ A' \arrow[r] & B' & C' \arrow[l] \end{tikzcd}$$

Then we have a homotopy pull back

$$\begin{tikzcd} A \bowtie_B C \arrow[r] \arrow[d] \arrow[dr, phantom, "h.p.b."] & B \arrow[d] \\ A' \bowtie_{B'} C' \arrow[r] & B' \end{tikzcd}$$

Moreover we have two homotopy pull backs

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "h.p.b."] & A \bowtie_B C \arrow[d] \\ A' \arrow[r] & A' \bowtie_{B'} C' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C \arrow[r] \arrow[d] \arrow[dr, phantom, "h.p.b."] & A \bowtie_B C \arrow[d] \\ C' \arrow[r] & A' \bowtie_{B'} C' \end{tikzcd}$$

Proof. Let us consider the following construction :

$$\begin{tikzcd}
 & P \arrow[rr] \arrow[dl] \arrow[dd] & & C \arrow[dl] \arrow[dr] \arrow[dd] & \\
A \arrow[rr, crossing over] \arrow[dd] & & Q \arrow[rr, crossing over] & & B \arrow[dd] \\
 & P' \arrow[rr] \arrow[dl] & & C' \arrow[dl] \arrow[dr] & \\
A' \arrow[rr] & & J' \arrow[from=uu, crossing over] \arrow[rr] & & B'
\end{tikzcd}$$

where $P' \simeq A' \times_{B'} C'$, $J' \simeq A' \vee_{P'} C' \simeq A' \bowtie_{B'} C'$, $Q \simeq J' \times_{B'} B$ and $P \simeq P' \times_{A'} A$.

By the prism lemma, we have also $A \simeq A' \times_{J'} Q$, $C \simeq C' \times_{J'} Q$ and $P \simeq P' \times_{C'} C$, $P \simeq A \times_B C$. The map $Q \to J'$ is a cube map as it is the homotopy base extension of the cube-map $B \to B'$. So $P - C - Q - A$ is a homotopy push out. Thus $Q \simeq A \bowtie_B C$. ∎

Theorem 3.2 (Join theorem II.) *Let $C' \to B'$ be a cube-map, and let be a homotopy push out and a homotopy pull back*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "h.p.o."] & B \arrow[d] \arrow[dr, phantom, "h.p.b."] & C \arrow[l] \arrow[d] \\
A' \arrow[r] & B' & C' \arrow[l]
\end{tikzcd}$$

(i.e. *$B' \simeq A' \vee_A B$ and $C \simeq C' \times_{B'} B$). Then we have a homotopy push out*

$$\begin{tikzcd}
A \bowtie_B C \arrow[r] \arrow[d] \arrow[dr, phantom, "h.p.o."] & B \arrow[d] \\
A' \bowtie_{B'} C' \arrow[r] & B'
\end{tikzcd}$$

Moreover we have a homotopy push out

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "h.p.o."] & A \bowtie_B C \arrow[d] \\
A' \arrow[r] & A' \bowtie_{B'} C'
\end{tikzcd}$$

Proof. Let us consider the following construction :

$$\begin{tikzcd}
 & P \arrow[rr] \arrow[dl] \arrow[dd] & & C \arrow[dl] \arrow[dr] \arrow[dd] & \\
A \arrow[rr] \arrow[dd] & & J \arrow[rr] \arrow[dd] & & B \arrow[dd] \\
 & P' \arrow[rr] \arrow[dl] & & C' \arrow[dl] \arrow[dr] & \\
A' \arrow[rr] & & S \arrow[rr] & & B'
\end{tikzcd}$$

where $P \simeq A \times_B C$, $J \simeq A \vee_P C \simeq A \bowtie_B C$, $S \simeq A' \vee_A J$ and $P' \simeq A' \times_{B'} C'$.

By the prism lemma, we have $P \simeq A \times_{A'} P'$ and $B' \simeq S \vee_J B$. As $C' \to B'$ is a cube map, $P - C - C' - P'$ is a homotopy push out. So by the prism lemma, $S \simeq A' \vee_{P'} C'$. Thus $S \simeq A' \bowtie_{B'} C'$. ∎

4 Applications of the join theorems

In this section, we give a collection of applications of the join theorems in the model category $\mathbf{C}$. In the category $\mathbf{Top}^w$, most are known results, but with our approach, they appear as direct consequences of the join theorems. Note that none of these results are dualizable, because they all rely on the cube axiom (or cube maps).

Theorem 4.1 (Compare [20] theorem 1.3.) *Let $P \to A$ and $P \to B$ be any two maps, $M \simeq A \vee_P B$, and let $N \to M$ be a cube-map and $N \to C$ be any map. Let us build the following diagram*

$$\begin{tikzcd}
 & Q \arrow[rr] \arrow[dl] \arrow[dd] & & F \arrow[dl] \arrow[dd] & \\
E \arrow[rr] \arrow[dd] & & N \arrow[rr] \arrow[dd] & & C \arrow[dd] \\
 & P \arrow[rr] \arrow[dl] & & A \arrow[dl] & \\
B \arrow[rr] & & M \arrow[rr] & & X
\end{tikzcd}$$

where $E \simeq B \times_M N$, $F \simeq A \times_M N$, $Q \simeq P \times_B E \simeq P \times_A F$ and $X \simeq M \vee_N C$. Finally, let be $Z \simeq P \vee_Q F \simeq P \bowtie_A F$ and $Y \simeq B \vee_E C$. Then there is a homotopy push out

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] \arrow[dr, phantom, "h.p.o."] & Y \arrow[d] \\
A \arrow[r] & X
\end{tikzcd}$$

Proof. Let $J \simeq B \vee_E N \simeq B \bowtie_M N$. By the join theorem II, we have $M \simeq A \vee_Z J$ (†). Now by the prism lemma, $Y \simeq B \vee_E C \simeq J \vee_N C$. Applying the prism lemma again in the diagram

$$\begin{tikzcd} N \arrow[r] \arrow[d] & J \arrow[r] \arrow[d] & M \arrow[d] \\ C \arrow[r] & Y \arrow[r] & X \end{tikzcd}$$

we get $X \simeq M \vee_J Y$ (‡). Finally apply the prism lemma to (†) and (‡) to get $X \simeq A \vee_Z Y$. ∎

Proposition 4.2 *Let $F \to E \to B$ and $F' \to E' \to B$ be two fibration sequences. If $* \to B$ is a cube-map, then there is a fibration sequence*

$$F \bowtie F' \to E \bowtie_B E' \to B.$$

Proof. Immediate by the join theorem I. ∎

In particular :

Corollary 4.3 (Compare [10] theorem 1.1.) *Let $F \to E \to B$ be a fibration sequence. If $* \to B$ is a cube-map, then there is a fibration sequence*

$$F \bowtie \Omega B \to E \bowtie_B * \to B.$$

Proposition 4.4 *Let $A \to B \to C$ be a cofibration sequence and let $F \to E \to C$ be a fibration sequence. If $E \to C$ is a cube-map, then the join of B and E over C is*

$$E \bowtie_C B \simeq C \vee (F \bowtie A)$$

and the join map is the obvious map $C \vee (F \bowtie A) \to C$.

Proof. By the join theorem II, we have the following homotopy push outs

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{h.p.o.}"] & F \bowtie A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{h.p.o.}"] & * \arrow[d] \\ B \arrow[r] & E \bowtie_C B \arrow[r] & C \end{tikzcd}$$

But by lemma 2.16 we know that $A \to F \bowtie A$ is null-homotopic, so the left homotopy push out splits into the two ones :

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{h.p.o.}"] & * \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{h.p.o.}"] & F \bowtie A \arrow[d] \\ B \arrow[r] & C \arrow[r] & E \bowtie_C B \end{tikzcd}$$

∎

Corollary 4.5 (Compare [5] proposition 2.1.) *Let $A \to B \to C$ be a cofibration sequence and $F \to E \to C$ be a fibration sequence. Let $P \simeq E \times_C B$. If $E \to C$ is a cube-map, then there is a cofibration sequence*

$$F \bowtie A \to B/P \to C/E.$$

Proof. First note that, as $E \bowtie_C B \simeq E \vee_P B$, the maps $P \to B$ and $E \to E \bowtie_C B$ have common homotopy cofibre B/P. Now apply the four cofibrations lemma to the following diagram :

$$\begin{tikzcd}
 & * \arrow[rr] \arrow[dl] \arrow[dd] & & E \arrow[dl] \arrow[dd] \\
* \arrow[rr] \arrow[dd] & & E \arrow[dd] & \\
 & F \bowtie A \arrow[rr] \arrow[dl] \arrow[dd] & & E \bowtie_C B \arrow[dl] \arrow[dd] \\
* \arrow[rr] \arrow[dd] & & C \arrow[dd] & \\
 & F \bowtie A \arrow[rr] \arrow[dl] & & B/P \arrow[dl] \\
* \arrow[rr] & & C/E &
\end{tikzcd}$$

■

Proposition 4.6 *Let A and C be B-sectioned, and $B \to A$ and $B \to C$ be cube-maps. Then we have a homotopy pull back :*

$$\begin{tikzcd}
\Omega_B A \bowtie_B \Omega_B C \arrow[r] \arrow[d] \arrow[dr, phantom, "h.p.b."] & B \arrow[d] \\
A \vee_B C \arrow[r] & A \times_B C
\end{tikzcd}$$

Proof. Let $P \simeq A \times_B C$. We have $B \simeq A \times_P C$ by the prism lemma, so $A \bowtie_P C \simeq A \vee_B C$, as we did already notice above. The map $C \to P$ is a cube-map, as it is the homotopy base extension of $B \to A$; thus the map $B \to P$ is a cube-map, as it is the composite of $B \to C$ and $C \to P$. Now the join theorem I applied to the diagram

$$\begin{tikzcd}
\Omega_B C \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{h.p.b.}"] & B \arrow[d] \arrow[dr, phantom, "\text{h.p.b.}"] & \Omega_B A \arrow[l] \arrow[d] \\
A \arrow[r] & P & C \arrow[l]
\end{tikzcd}$$

gives the result. ■

Corollary 4.7 *For any objects A and C, if $* \to A$ and $* \to C$ are cube-maps, then*

$$A \flat C \simeq \Omega A \bowtie \Omega C.$$

To end this section, we show that the associativity of the join construction also rely on the join theorems.

Theorem 4.8 (Associativity of the join.) (Compare [15] 3.74.) *Let be two cube-maps $A \to D$ and $C \to D$ and any third map $B \to D$. Then*

$$(A \bowtie_D B) \bowtie_D C \simeq A \bowtie_D (B \bowtie_D C).$$

Proof. Let us consider the following construction :

$$\begin{tikzcd}
& Q \arrow[rr] \arrow[dl] \arrow[dd] & & E \arrow[dl] \arrow[dr] \arrow[dd] & \\
F \arrow[rr] \arrow[dd] & & N \arrow[rr] \arrow[dd] & & C \arrow[dd] \\
& P \arrow[rr] \arrow[dl] & & B \arrow[dl] \arrow[dr] & \\
A \arrow[rr] & & M \arrow[rr] & & D
\end{tikzcd}$$

where $P \simeq A \times_D B$, $E \simeq B \times_D C$, $F \simeq A \times_D C$, $Q \simeq P \times_A F \simeq P \times_B E \simeq F \times_C E$, $M \simeq A \vee_P B \simeq A \bowtie_D B$, and $N \simeq M \times_D C$. As $C \to D$ is a cube-map, $N \simeq F \bowtie_C E$ by the join theorem I.

Let $X \simeq M \vee_N C \simeq M \bowtie_D C \simeq (A \bowtie_D B) \bowtie_D C$. Now let $Z \simeq P \bowtie_A F$ and $Y \simeq B \vee_E C \simeq B \bowtie_D C$. As $A \to D$ is a cube-map, $Z \simeq A \times_D Y$ by the join theorem I.

Note that $N \to M$ is a cube-map because it is the homotopy base extension of the cube-map $C \to D$. Applying theorem 4.1 to the interior cube of the above diagram prolonged by the homotopy push out $X \simeq M \vee_N C$, we obtain $X \simeq A \vee_Z Y$. Since $Z \simeq A \times_D Y$, we have $X \simeq A \bowtie_D Y \simeq A \bowtie_D (B \bowtie_D C)$. ∎

5 Ganea and fat wedge constructions

Ganea spaces and Whitehead's fat wedges play a crucial role in the study of the Lusternik-Schnirelmann category. In fact, they are not else but some particular join constructions. We present here these notions in a model category. (See also [7], [14], [3], [4], [5] for further development.)

Definition 5.1 *For any object B, the n^{th}* Ganea object *G_nB and* Ganea map *$g_n : G_nB \to B$ are defined inductively by : $g_0 : G_0B \simeq * \to B$ and*

$$g_n : G_nB \simeq G_{n-1}B \bowtie_B * \to B$$

is the join of g_{n-1} and g_0.

Proposition 5.2 *If $* \to B$ is a cube-map, then $G_{m+n+1}B \simeq G_mB \bowtie_B G_nB$.*

Proof. This is just the associativity of the join. ∎

Proposition 5.3 *If $* \to B$ is a cube-map, then there is a fibration sequence*

$$\Omega B \bowtie \Omega B \bowtie \ldots \bowtie \Omega B \ (n+1 \text{ times}) \to G_n B \to B.$$

Proof. First note that $\Omega B \to *$ is a cube-map as it is the homotopy base extension of $* \to B$, so the join of loops is associative. Now the result follows directly from corollary 4.3 by recursivity. ■

Let us define the *diagonals* $\Delta_n : B \to B^n \simeq B \times B \times \ldots \times B$ (n times) of any object B, inductively by $\Delta_0 : B \to *$ and Δ_n is the whisker map induced by the maps $1_B : B \to B$ and $\Delta_{n-1} : B \to B^{n-1}$ and the following homotopy pull back :

$$\begin{tikzcd} B^n \simeq B \times B^{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{h.p.b.}"] & B^{n-1} \arrow[d] \\ B \arrow[r] & * \end{tikzcd}$$

Definition 5.4 *For any object B, the n^{th}* fat wedge $T_n B$ *and* fat wedge map $t_n : T_n B \to B^{n+1}$ *are defined inductively by : $t_0 : T_0 B \simeq * \to B$ and*

$$t_n : T_n B \simeq B^n \bowtie_{B^{n+1}} (T_{n-1} B \times B) \to B^{n+1}$$

is the join of the obvious (whisker) maps $B^n \simeq B^n \times * \to B^n \times B \simeq B^{n+1}$ and $T_{n-1} B \times B \to B^n \times B \simeq B^{n+1}$.

Theorem 5.5 (Compare [12] 4.3, [7] 3.11.) *Assume $\Delta_n : B \to B^n$ is a cube-map for all $n \geq 1$. Then for all $n \geq 1$ there is a homotopy pull back*

$$\begin{tikzcd} G_n B \arrow[r] \arrow[d, "g_n"'] \arrow[dr, phantom, "h.p.b."] & T_n B \arrow[d, "t_n"] \\ B \arrow[r, "\Delta_{n+1}"'] & B^{n+1} \end{tikzcd}$$

Proof. By induction on n. The case $n = 0$ is immediate as $g_0 = t_0$ and Δ_1 is the identity. Assume the result is known to be true for $n = m - 1$.

Applying the prism lemma in the diagram

$$\begin{tikzcd} G_{m-1} B \arrow[r] \arrow[d] & T_{m-1} B \times B \arrow[r] \arrow[d] & T_{m-1} B \arrow[d] \\ B \arrow[r, "\Delta_{m+1}"'] & B^m \times B \arrow[r] & B^m \end{tikzcd}$$

we obtain that the left square is a homotopy pull back (†).

On the other hand, applying the prism lemma in the diagram

$$\begin{tikzcd}
{*} \arrow[r] \arrow[d] & B^m \arrow[r] \arrow[d] & {*} \arrow[d] \\
B \arrow[r, "\Delta_{m+1}"'] & B^m \times B \arrow[r] & B
\end{tikzcd}$$

we obtain that the left square is a homotopy pull back (‡).

Now apply the join theorem I to (†) and (‡) to get the result for $n = m$. ∎

Corollary 5.6 *Assume $\Delta_n : B \to B^n$ is a cube-map for all $n \geq 1$. Then for all $n \geq 1$, g_n has a homotopy section if and only if Δ_{n+1} admits a homotopy factorization through t_n.*

Proof. The section $s : B \to G_nB$ of g_n is the whisker map induced by $1_B : B \to B$, the map $l : B \to T_nB$ in the factorization of Δ_{n+1} and the homotopy pull back of the theorem 5.5. ∎

6 Holonomy

This section is devoted to study the relation between join and holonomy. The main theorem 6.2 here asserts that the holonomy of the 'join of fibration sequences' is the 'join of the holonomies' of the fibration sequences.

Let $F \to E \to B$ be a fibration sequence. Let us consider the following diagram (†) :

$$\begin{tikzcd}[row sep=small, column sep=small]
& \Omega B \times F \arrow[rr] \arrow[dl, "h"'] \arrow[dd] & & \Omega B \arrow[dl] \arrow[dd] \\
F \arrow[rr, crossing over] \arrow[dd] & & {*} & \\
& F \arrow[rr] \arrow[dl] & & {*} \arrow[dl] \\
E \arrow[rr] & & B \arrow[from=uu, crossing over] &
\end{tikzcd}$$

where the map $h : \Omega B \times F \to F$ is the whisker map of the front homotopy pull back $F \simeq E \times_B *$. The prism lemma shows that the top and left faces of the cube are homotopy pull backs.

Definition 6.1 *The whisker map $h : \Omega B \times F \to F$ in the diagram above is called the* holonomy *of the fibration sequence $F \to E \to B$.*

Let be a homotopy commutative diagram

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & B \arrow[d, "g"] & C \arrow[l] \arrow[d, "h"] \\
A' \arrow[r] & B' & C' \arrow[l]
\end{tikzcd}
$$

The obvious (whisker) map $A \bowtie_B C \to A' \bowtie_{B'} C'$ is called the *join of f and h over g.*

Theorem 6.2 *Let $F \to E \to B$ and $F' \to E' \to B$ be two fibration sequences, and let $h : \Omega B \times F \to F$ and $h' : \Omega B \times F' \to F'$ respectively be their holonomies. If $* \to B$ if a cube-map, then the holonomy of the fibration sequence $F \bowtie F' \to E \bowtie_B E' \to B$ is the join of the holonomies h and h' over $\Omega B \to *$.*

Proof. Let us consider the join

$$J \simeq (\Omega B \times F) \bowtie_{\Omega B} (\Omega B \times F')$$

of the maps $\Omega B \times F \to \Omega B$ and $\Omega B \times F' \to \Omega B$ in the diagram (†) above, and the corresponding one with primes. Let us also consider the join of the maps $F \to *$ and $F' \to *$. Putting all this together we get the following diagram (‡) :

$$
\begin{tikzcd}
 & \Omega B \times (F \times F') \arrow[rr] \arrow[dl] \arrow[dd] & & \Omega B \times F' \arrow[dl] \arrow[dd] & \\
\Omega B \times F \arrow[rr] \arrow[dd] & & J \arrow[rr] \arrow[dd] & & \Omega B \arrow[dd] \\
 & F \times F' \arrow[rr] \arrow[dl] & & F \arrow[dl] & \\
F \arrow[rr] & & F \bowtie F' \arrow[rr] & & *
\end{tikzcd}
$$

where the five vertical squares are homotopy pull backs and the two horizontal squares are homotopy push outs. Indeed this diagram is nothing else but the construction of the join theorem I applied to the two top faces of the diagram (†) and of the corresponding one with primes. By construction (and the prism lemma), the maps $\Omega B \times F \to F$ and $\Omega B \times F' \to F'$ are the holonomies and the map $J \to F \bowtie F'$ is their join.

Furthermore, let us consider the join of the maps $E \to B$ and $E' \to B$. Applying (four times) the join theorem I to the diagram (†) of homotopy pull backs above, and the corresponding one with primes, we get the diagram :

$$
\begin{tikzcd}
 & J \arrow[rr] \arrow[dl, "x"'] \arrow[dd] & & \Omega B \arrow[dl] \arrow[dd] \\
F \bowtie F' \arrow[rr] \arrow[dd] & & * \arrow[dd] & \\
 & F \bowtie F' \arrow[rr] \arrow[dl] & & * \arrow[dl] \\
E \bowtie_B E' \arrow[rr] & & B &
\end{tikzcd}
$$

where the front, top, rear and bottom faces are homotopy pull backs. By the prism lemma, the left face is also a homotopy pull back. Thus by definition, the map $x : J \to F \bowtie F'$ is the holonomy of the fibration sequence $F \bowtie F' \to E \bowtie_B E' \to B$. ∎

Corollary 6.3 *Let $F \to E \to B$ be a fibration sequence. If $* \to B$ if a cube-map, then the holonomy of the fibration sequence $F \bowtie \Omega B \to E \bowtie_B * \to B$ is the join of the holonomies $\Omega B \times F \to F$ and $\Omega B \times \Omega B \to \Omega B$ over $\Omega B \to *$.*

References

[1] Baues H., *Algebraic homotopy*, Cambridge University Press (1989).

[2] Bousfield A.K., Friedlander E., *Homotopy theory of* Γ*-spaces, spectra and bisimplicial sets*, Lecture Notes in Math. **658**, Springer Verlag (1978), 80-130.

[3] Cornea 0., *Cone-length and Lusternik Schnirelmann category*, Topology **33** (1994), 95-111.

[4] Cornea 0., *There is just one rational cone-length*, Trans. A.M.S. **344** (1994), 835-848.

[5] Cornea 0., *Strong L.S. category equals cone-length*, Topology **34** (1995), 377-381.

[6] Doeraene J.-P. *L.S. catégorie dans une catégorie à modèles*, Thesis, Université Catholique de Louvain (1990).

[7] Doeraene J.-P., *L.S. category in a model category*, Journal of Pure and Appl. Algebra **84** (1993), 215-261.

[8] Doeraene J.-P., Tanré D., *Axiome du Cube et Foncteurs de Quillen*, Annales de l'Institut Fourier **45**-4, (1995), 1061-1077.

[9] Gabriel P., Zisman M., *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete 35, Springer-Verlag (1967).

[10] Ganea T., *A generalization of the homology and homotopy suspension*, Comment. Math. Helv. **39** (1965), 295-322.

[11] Ganea T., *Cogroups and Suspensions*, Inventiones Math. **9** (1970), 185-197.

[12] Gilbert W.J., *Some problems in algebraic topology on Lusternik-Schnirelmann categories and cocategories*, Thesis, Oxford (1967).

[13] Halperin S., Watkiss C., *Relative Homotopical Algebra*, Publication I.R.M.A., Lille (1980).

[14] Hess K.P., Lemaire J.-M., *Generalizing a definition of Lusternik and Schnirelmann to model categories*, Journ. of Pure and Applied Algebra **91** (1994), 165-182.

[15] James I.M., *General Topology and Homotopy Theory*, Springer Verlag (1984).

[16] Kahl T., *Lusternik-Schnirelmann-Kategorie und axiomatische Homotopietheorie*, Diplomarbeit, Freie Universität Berlin (1993).

[17] Lusternik L., Schnirelmann L., *Méthodes topologiques dans les problèmes variationnels*, Hermann, Paris (1934).

[18] Majewski M., *Tame homotopy theory via polynomial forms*, Diplomarbeit, Freie Universität Berlin (1988).

[19] Majewski M., *A cellular Lie algebra model for spaces and its equivalence with the models of Quillen and Sullivan*, Thesis, Freie Universität Berlin (1996).

[20] Marcum H.J., *Fibrations over double mapping cylinders*, Illinois Journ. of Math. **24**-2 (1980), 344-358.

[21] Mather M., *Pull-backs in Homotopy Theory*, Can. J. Math. **28**-2 (1976), 225-263.

[22] Munkholm H., *DGA algebras as a Quillen model category*, J. Pure Appl. Algebra **13** (1978), 221-232.

[23] Quillen D., *Homotopical Algebra*, Lecture Notes in Math. **43**, Springer Verlag (1967).

[24] Quillen D., *Rational Homotopy theory*, Ann. of Math. **90** (1969), 205-295.

[25] Strøm A., *The homotopy category is a homotopy category*, Arch. Math. **23** (1972), 435-441.

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