When are induction and conduction functors isomorphic ?

Claudia Menini* Constantin Nastasescu

Introduction

Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring. It is well known (see e.g. [D], [M_1], [N], [NRV], [NV]) that in the study of the connections that may be established between the categories R-gr of graded R-modules and R_1-mod (1 is the unit element of G), an important role is played by the following system of functors :
$(-)_1 : R\text{-gr} \to R_1\text{-mod}$ given by $M \mapsto M_1$, where $M = \bigoplus_{g\in G} M_g$ is a graded left R-module,
the induced functor, $\mathrm{Ind} : R_1\text{-mod} \to R\text{-gr}$, which is defined as follows : if $N \in R_1$-mod, then $\mathrm{Ind}(N) = R\otimes_{R_1} N$ which has the G-grading given by $(R \otimes_{R_1} N)_g = R_g \otimes_{R_1} N, \forall g \in G$,
and the coinduced functor, $\mathrm{Coind} : R_1\text{-mod} \to R\text{-gr}$, which is defined in the following way : if $N \in R_1$-mod, then $\mathrm{Coind}(N) = \bigoplus_{g\in G}\mathrm{Coind}(N)_g$, where

$$\mathrm{Coind}(N)_g = \{f \in \mathrm{Hom}_{R_1}({}_{R_1}R_R, N) \mid f(R_h) = 0,\ \forall\ h \neq g^{-1}\} .$$

(Note that if G is finite, then $\mathrm{Coind}(N) = \mathrm{Hom}_{R_1}({}_{R_1}R_R, N)$).

It was shown in [N] that the functor Ind is a left adjoint of the functor $(-)_1$ and the unity of the adjunction $\sigma : \mathbf{1}_{R_1\text{-mod}} \to (-)_1 \circ \mathrm{Ind}$ is a functorial isomorphism, and that Coind is a right adjoint of the functor $(-)_1$ and the counity of this adjunction $\tau : (-)_1 \circ \mathrm{Coind} \to \mathbf{1}_{R_1\text{-mod}}$ is a functorial isomorphism.
If the ring R is a G-strongly graded ring (i.e. $R_gR_h = R_{gh} \quad \forall g, h \in G$) then the functors Ind and Coind are isomorphic. Thus the following question naturally

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arises : "if the functors Ind and Coind are isomorphic, does it follow that the ring R is strongly graded ?" A simple example (see Remark 3.3) shows that the answer to this question is negative. So we may ask this other question : "if R is a graded ring and the functors Ind and Coind are isomorphic, then how close is R to being strongly graded ?" The study of this problem is done in §3. The main results of this section are contained in Theorems 3.4, 3.5, 3.6, 3.9, 3.11, 3.12, etc. In particular Theorems 3.9, 3.11 and 3.12 provide the following answer to the above question : if $R = \bigoplus_{g\in G} R_g$ is a G-graded ring and if Ind$\simeq$Coind then $H = \mathrm{Supp}(R) = \{g \in G \mid R_g \neq 0\}$ is a subgroup of G and $R = \bigoplus_{h\in H} R_h$ is an H-strongly graded ring whenever one of the following conditions is satisfied : 1) the category R_1-mod has only one type of simple modules (in particular this holds if R_1 is a local ring), or more generally if 2) every finitely generated and projective module in R_1-mod is faithful, 3) R_1 has only two idempotents 0 and 1 (in particular if R_1 is a domain).

It is obvious that the problem of when the functors Ind and Coind are isomorphic may be considered in the non graded case too. More exactly, if $\psi : R \to S$ is a ring morphism, we can define the Induced functor $S \otimes_R - : R$-mod $\to S$-mod and the Coinduced functor $\mathrm{Hom}_R({}_R S_S, -) : R$-mod$\to S$-mod which are respectively the left and the right adjoint of the restriction of scalar functors $\psi_* : S$-mod$\to R$-mod.
These two functors are isomorphic if and only if (see Theorem 3.15) $\psi : R \to S$ is a (left) Frobenius morphism in the sense of KASCH [K] (see also [NT]). In particular we get that these "left" functors are isomorphic if and only if the corresponding "right" functors are isomorphic. Using results of graded ring theory, we get in 3.18 new examples of Frobenius morphisms.
Although there is a great similarity between the definitions of the functors Ind and Coind in the graded and non graded cases, the problems discussed in Section 3 seem to require different treatments for the two cases. The unification of the two cases may be done using a category which is more general then the categories R-gr and R-mod, namely the category (G, A, R)-gr, where A is a G-set. This is what we do in Section 2. More exactly, if $f : G \to G'$ is a group morphism, A is a G-set, A' is a G'-set and $\varphi : A \to A'$ is a map such that $\varphi(ga) = f(g)\varphi(a)$ for every $g \in G, a \in A$, and R is a G-graded ring, R' is a G'-graded ring and $\psi : R \to R'$ is a ring morphism such that $\psi(R_g) \subseteq R'_{f(g)}$, for every $g \in G$, the system $T = (f, \varphi, \psi)$ allows us to define the functors

$$T_* : (G', A', R')\text{-gr} \to (G, A, R)\text{-gr} ,$$

$$T^* : (G, A, R)\text{-gr} \to (G', A', R')\text{-gr} ,$$

and

$$\tilde{T} : (G, A, R)\text{-gr} \to (G', A', R')\text{-gr} .$$

The functor T_* is exact, T^* is a left adjoint of T_* and $\tilde{T}$ is a right adjoint of T_*. Since T^* (resp. $\tilde{T}$) is a left adjoint (resp. right adjoint) of the functor T_*, we can consider the unity $\sigma : \mathbf{1}_{(G,A,R)\text{-gr}} \to T_* \circ T^*$ (resp. the counity $\tau : T_* \circ \tilde{T} \to \mathbf{1}_{(G,A,R)\text{-gr}}$) of this adjunction.
In §2 we investigate when the morphism σ is an isomorphism. In this case τ is an isomorphism too (Theorem 2.6). The main results are contained in Theorems 2.6,

2.9, 2.18, 2.21, 2.24, 2.27, 2.38. We remark that Theorem 2.24 is a generalization of a well-known result of Dade ([D], Theorem 2.8).
The situations studied in §2 and §3 may be considered in a more general context, namely for functors between Grothendieck categories. This is what we do in the first section. The results of §1 are then used in the other two sections.

Let us add some final remarks concerning section 3. If we have a graded ring $R = \bigoplus_{g\in G} R_g$ such that the group G is finite, then the induced and coinduced functors in the graded and non graded cases are $R \otimes_{R_1} - : R_1\text{-mod} \to R\text{-gr}$ and $\mathrm{Hom}_{R_1}({}_{R_1}R_R, -) : R_1\text{-mod} \to R\text{-gr}$, and $R \otimes_{R_1} - : R_1\text{-mod} \to R\text{-mod}$ and $\mathrm{Hom}_{R_1}({}_{R_1}R_R, -) : R_1\text{-mod} \to R\text{-mod}$, respectively.
Clearly if the functors Ind and Coind, in the graded case, are isomorphic, then the corresponding non graded functors are also isomorphic. Example 3.21 shows that the converse does not hold.

0 Notations and Preliminaries

All rings are associative, with identity $1 \neq 0$ and all modules are unital. Let R be a ring. R-mod (resp. mod-R) will denote the category of left (resp. right) R-modules.
Let G be a multiplicative group with identity element "1". A *G-graded ring* is a ring together with a direct sum decomposition $R = \bigoplus_{g\in G} R_g$ (as additive subgroups) such that $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. The set $\mathrm{Supp}(R) = \{g \in G \mid R_g \neq 0\}$ is called the *support* of R.
R is called a *strongly graded ring* if $R_gR_h = R_{gh}$ for all $g, h \in G$. It is well known (see [NV]) that R is a strongly graded ring $\iff R_gR_{g^{-1}} = R_1$ for every $g \in G$.
A (left) *G-set* is a non-empty set A together with a left action of G on A given by $G\times A \to A$, $(g, a) \mapsto ga$, such that $1a = a$ for all $a \in A$ and $(gg')a = g(g'a)$ for all $g, g' \in G$, $a \in A$.
If H is a subgroup of G then the set of left cosets $G/H = \{gH \ ; \ g \in G\}$ with G-action given by $g(g'H) = gg'H$ for $g, g' \in G$, is a G-set. Given a G-graded ring R, we set $R^{(H)} = \bigoplus_{g\in H} R_g$. Then $R^{(H)}$ is an H-graded ring.
Given a left G-set A, a *left graded R-module of type A* is a left R-module M such that $M = \bigoplus_{a\in A} M_a$ where each M_a is an additive subgroup of M and for all $g \in G$, $a \in A$ it is $R_gM_a \subseteq M_{ga}$.
If $M = \bigoplus_{a\in A} M_a$ and $N = \bigoplus_{a\in A} N_a$ are left graded R-modules of type A, then a *graded morphism* $f : M \to N$ is an R-linear map such that $f(M_a) \subseteq N_a$ for all $a \in A$. If $f : M \to N$ is a graded morphism we will denote by $f_a : M_a \to N_a$ the corestriction to N_a of the restriction of f to M_a, $a \in A$, and we will call it the a-component of f.
(G, A, R)-gr will denote the category of left graded R-modules of type A and graded morphisms. (G, A, R)-gr is a Grothendieck category (see [NRV]).
If $G = A$ with the natural left action of G on itself, then (G, G, R)-gr is just the category R-gr of left graded R-modules.

Let A be a G-set. For each $a \in A$ the *a-suspension* $R(a)$ of R is the object of (G, A, R)-gr which coincides with R as an R-module, but with the graduation defined by

$$R(a)_b = \bigoplus\{R_g \mid g \in G,\ ga = b\} \qquad \text{for } b \in A\ .$$

The family $(R(a))_{a\in A}$ is a system of projective generators of (G, A, R)-gr (see [NRV]).

1 General results on adjoint functors

1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be Grothendieck categories. Throughout this section we will assume that $U : \mathcal{B} \to \mathcal{A}$ is a covariant functor having a left adjoint $T : \mathcal{A} \to \mathcal{B}$ and a right adjoint $H : \mathcal{A} \to \mathcal{B}$. Note that U is exact and that T is right exact, while H is left exact.
Let

$$\alpha_{-,-} : \mathrm{Hom}_{\mathcal{B}}(T, \mathbf{1}_{\mathcal{B}}) \to \mathrm{Hom}_{\mathcal{A}}(\mathbf{1}_{\mathcal{A}}, U)$$

and

$$\gamma_{-,-} : \mathrm{Hom}_{\mathcal{A}}(U, \mathbf{1}_{\mathcal{A}}) \to \mathrm{Hom}_{\mathcal{B}}(\mathbf{1}_{\mathcal{B}}, H)$$

be the adjunction isomorphisms. Let

$$\sigma : \mathbf{1}_{\mathcal{A}} \to U \circ T \quad , \quad \zeta : \mathbf{1}_{\mathcal{B}} \to H \circ U$$

be the unities of these adjunctions and let

$$\rho : T \circ U \to \mathbf{1}_{\mathcal{B}} \quad , \quad \tau : U \circ H \to \mathbf{1}_{\mathcal{A}}$$

be the counities of these adjunctions. For every $L \in \mathcal{A}$ and $M \in \mathcal{B}$ we have :

$$\sigma_L = \alpha_{L,T(L)}(\mathbf{1}_{T(L)}) : L \to U(T(L))\ ,$$

$$\rho_M = \alpha^{-1}_{U(M),M}(\mathbf{1}_{U(M)}) : T(U(M)) \to M\ ,$$
$$\zeta_M = \gamma_{M,U(M)}(\mathbf{1}_{U(M)}) : M \to H(U(M))\ ,$$
$$\tau_L = \gamma^{-1}_{H(L),L}(\mathbf{1}_{H(L)}) : U(H(L)) \to L\ .$$

It is well known that :
1) $\alpha_{L,M}(f) = U(f) \circ \sigma_L$ for every $f \in \mathrm{Hom}_{\mathcal{B}}(T(L), M)$
2) $\alpha^{-1}_{L,M}(h) = \rho_M \circ T(h)$ for every $h \in \mathrm{Hom}_{\mathcal{A}}(L, U(M))$
3) $\gamma_{M,L}(g) = H(g) \circ \zeta_M$ for every $g \in \mathrm{Hom}_{\mathcal{A}}(U(M), L)$
4) $\gamma^{-1}_{M,L}(\ell) = \tau_L \circ U(\ell)$ for every $\ell \in \mathrm{Hom}_{\mathcal{B}}(M, H(L))$
It follows that :
a) $U(\rho_M) \circ \sigma_{U(M)} = \mathbf{1}_{U(M)}$
b) $\rho_{T(L)} \circ T(\sigma_L) = \mathbf{1}_{T(L)}$
c) $H(\tau_L) \circ \zeta_{H(L)} = \mathbf{1}_{H(L)}$
d) $\tau_{U(M)} \circ U(\zeta_M) = \mathbf{1}_{U(M)}$.

1.2. Assume that σ is a functorial isomorphism. For every $L \in \mathcal{A}$ let $\eta_L = \gamma_{T(L),L}(\sigma_L^{-1})$. Then $\eta_L = H(\sigma_L^{-1}) \circ \zeta_{T(L)} : T(L) \to H(L)$ and the η_L's, $L \in \mathcal{A}$, define a functorial morphism $\eta : T \to H$.

Similarly, whenever τ is a functorial isomorphism, we set $\lambda_L = \alpha^{-1}_{L,H(L)}(\tau_L^{-1})$, for every $L \in \mathcal{A}$. Then $\lambda_L = \rho_{H(L)} \circ T(\tau_L^{-1})$ and the λ_L's, $L \in \mathcal{A}$, define a functorial morphism $\lambda : T \to H$.

Theorem 1.3 *Assume that σ and τ are functorial isomorphisms and that $T(L) \simeq H(L)$ in $\mathcal{B}$, for every $L \in \mathcal{A}$. Then η and λ are functorial isomorphisms.*

Proof. For every $L \in \mathcal{A}$, let $\theta_L : T(L) \to H(L)$ be an isomorphism in $\mathcal{B}$. In view of 3) and 4) of 1.1 we get :

$$H(\tau_{U(T(L))} \circ U(H(\sigma_L) \circ \theta_L)) \circ \zeta_{T(L)} =$$

$$= \gamma_{T(L),U(T(L))}(\gamma^{-1}_{T(L),U(T(L))}(H(\sigma_L) \circ \theta_L)) = H(\sigma_L) \circ \theta_L \ .$$

As $\tau_{U(T(L))}$, σ_L and θ_L are isomorphisms it follows that $\zeta_{T(L)}$ and hence also $\eta_L = H(\sigma_L^{-1}) \circ \ \zeta_{T(L)}$ are isomorphisms.
Similarly in view of 1) and 2) of 1.1 we get

$$\rho_{H(L)} \circ T(U(T(\tau_L) \circ \theta_L) \circ \sigma_{U(H(L))}) =$$

$$= \alpha^{-1}_{U(H(L)),H(L)}(\alpha_{U(H(L)),H(L)}(T(\tau_L) \circ \theta_L)) = T(\tau_L) \circ \theta_L$$

so that $\rho_{H(L)}$ and hence also $\lambda_L = \rho_{H(L)} \circ T(\tau_L^{-1})$ are isomorphisms.

1.4. Let

$$\mathcal{C} = \{M \in \mathcal{B} \mid U(M) = 0\}$$

It is well known that $\mathcal{C}$ is a localizing subcategory of $\mathcal{B}$ i.e. $\mathcal{C}$ is closed under subobjects, quotient objects, extensions and under arbitrary direct sums.
Let t be the radical associated to $\mathcal{C}$. For every $M \in \mathcal{B}$, $t(M)$ is the largest subobject of M belonging to $\mathcal{C}$. We will say that M is *$\mathcal{C}$-torsion* if $t(M) = M$ and that M is *$\mathcal{C}$-torsion free* if $t(M) = 0$.

Lemma 1.5 *For every $M \in \mathcal{B}$, we have:*
1) $t(M) \leq Ker(\zeta_M)$;
2) $Im(\rho_M) \leq N$ for every $N \leq M$ such that $M/N \in \mathcal{C}$.

Proof. 1) Let $i : t(M) \to M$ be the canonical injection. Then, from the commutative diagram

$$\begin{array}{ccc} t(M) & \xrightarrow{\quad i \quad} & M \\ \zeta_{t(M)}\big\downarrow & & \big\downarrow\zeta_M \\ H(U(t(M))) & \xrightarrow{(H\circ U)(i)} & H(U(M)) \end{array}$$

as $U(t(M)) = 0$, we get $\zeta_M \circ i = 0$ i.e. $t(M) \leq \mathrm{Ker}(\zeta_M)$.

2) Consider now an exact sequence

$$0 \to N \xrightarrow{j} M \xrightarrow{\pi} M/N \to 0$$

and assume that $M/N \in \mathcal{C}$. Then $U(j) : U(N) \to U(M)$ is an isomorphism so that, from the commutative diagram

$$\begin{array}{ccc} (T \circ U)(N) & \xrightarrow[\sim]{(T\circ U)(j)} & (T \circ U)(M) \\ \rho_N \downarrow & & \downarrow \rho_M \\ N & \xrightarrow{j} & M \end{array}$$

we get $\mathrm{Im}(\rho_M) = \mathrm{Im}(j \circ \rho_N)$ so that $N = \mathrm{Im}(j) \supseteq \mathrm{Im}(\rho_M)$.

Proposition 1.6 *Let $M \in \mathcal{B}$.*

1) If σ is a functorial isomorphism, then Ker(ρ_M) and Coker(ρ_M) belong to $\mathcal{C}$ and Im(ρ_M) is the smallest subobject N of M such that M/N belongs to $\mathcal{C}$.

2) If τ is a functorial isomorphism, then Ker(ζ_M) and Coker(ζ_M) belong to $\mathcal{C}$ and Ker(ζ_M) $= t(M)$.

Proof.

1) In view of a) of 1.1 we have

$$U(\rho_M) \circ \sigma_{U(M)} = \mathbf{1}_{U(M)} \ .$$

Therefore, as $\sigma_{U(M)}$ is an isomorphism, $U(\rho_M)$ is an isomorphism too. Since U is an exact functor we get that Ker(ρ_M) and Coker(ρ_M) belong to $\mathcal{C}$. Apply now Lemma 1.5.

2) In view of d) of 1.1 we have :

$$\tau_{U(M)} \circ U(\zeta_M) = \mathbf{1}_{U(M)} \ .$$

The conclusion follows from this fact in a way analogous as in 1).

Proposition 1.7 *1) For every $L \in \mathcal{A}$, $H(L)$ is $\mathcal{C}$-torsion free. Moreover it has the following property : for any diagram in $\mathcal{B}$ of the form*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \xrightarrow{u} & X & \longrightarrow & Coker(u) & \longrightarrow & 0 \\ & & f\downarrow & & & & & & \\ & & H(L) & & & & & & \end{array}$$

with Coker(u) $\in \mathcal{C}$, there exists a unique morphism in $\mathcal{B}$, $g : X \to H(L)$, such that $g \circ u = f$ (i.e. $H(L)$ is $\mathcal{C}$-closed in the sense of Gabriel).

2) If τ is a functorial isomorphism, for every $M \in \mathcal{B}$, Im(ζ_M) is an essential subobject of $H(U(M))$.

Proof.
1) In view of [G], Lemma 1 page 370, it is enough to show that for every map $u : X' \to X$ such that $\mathrm{Ker}(u)$ and $\mathrm{Coker}(u)$ belong to $\mathcal{C}$, the map $\mathrm{Hom}(u, H(L))$ is an isomorphism. We have the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(U(X), L) & \xrightarrow{\gamma_{X,L}} & \mathrm{Hom}_{\mathcal{B}}(X, H(L)) \\ {\scriptstyle \mathrm{Hom}_{\mathcal{A}(U(u),L)}}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{B}(u,H(L))}} \\ \mathrm{Hom}_{\mathcal{A}}(U(X'), L) & \xrightarrow{\gamma_{X',L}} & \mathrm{Hom}_{\mathcal{B}}(X', H(L)) \end{array}$$

As $U(u)$ is an isomorphism, $\mathrm{Hom}(u, H(L))$ is an isomorphism too.

2) Let $H(U(M)) \xrightarrow{f} Y$ be a morphism such that $M \xrightarrow{\zeta_M} H(U(M)) \xrightarrow{f} Y$ is a monomorphism. We have to prove that f is a monomorphism. As U is a left exact functor, we have that $U(f \circ \zeta_M) = U(f) \circ U(\zeta_M)$ is a monomorphism. Since $\tau_{U(M)}$ is an isomorphism, by d) of 1.1 we get that $U(\zeta_M)$ is an isomorphism. Thus $U(f)$ turns out to be a monomorphism and hence $\mathrm{Ker}(f) \in \mathcal{C}$. By 1), $H(U(M))$ is $\mathcal{C}$-torsion free and therefore $\mathrm{Ker}(f) = 0$.

Proposition 1.8 *Assume that σ and τ are functorial isomorphisms. Let $L \in \mathcal{A}$. Then :*
1) Ker(η_L) and Coker(η_L) belong to $\mathcal{C}$;
2) Ker(λ_L) and Coker(λ_L) belong to $\mathcal{C}$.
Moreover Ker(η_L) $= t(T(L)) =$Ker(λ_L) and Im(η_L) is essential in $H(L)$.

Proof. As $\eta_L = H(\sigma_L^{-1}) \circ \zeta_{T(L)}$ and $\lambda_L = \rho_{H(L)} \circ T(\tau_L^{-1})$, by Proposition 1.6 we get 1) and 2). By 1) of Proposition 1.6, $H(L)$ is $\mathcal{C}$-torsion free and hence $T(L)/\mathrm{Ker}(\eta_L)$ and $T(L)/\mathrm{Ker}(\lambda_L)$ are $\mathcal{C}$-torsion free so that $\mathrm{Ker}(\eta_L) = t(T(L)) = \mathrm{Ker}(\lambda_L)$.
Finally, by 2) in Proposition 1.7, $\mathrm{Im}(\eta_L)$ is essential in $H(L)$.

Proposition 1.9 *Assume that σ is a functorial isomorphism and that, for every $M \in \mathcal{B}$, $T(U(M)) \simeq H(U(M))$. Then, for every $M \in \mathcal{B}$, we have*

$$M \simeq t(M) \oplus M/t(M) \ .$$

Moreover $M/t(M)$ is $\mathcal{C}$-closed.

Proof. Let $M \in \mathcal{B}$. By 1) of Proposition 1.6, the kernel of $\rho_M : T(U(M)) \to M$ belongs to $\mathcal{C}$. As $T(U(M)) \simeq H(U(M))$ and as, by Proposition 1.7, $H(U(M))$ is $\mathcal{C}$-torsion free, we get that $\mathrm{Ker}(\rho_M) = 0$.
Consider now the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & T(U(M)) & \xrightarrow{\rho_M} & M \\ & & {\scriptstyle \mathbf{1}_{T(U(M))}}\downarrow & & \\ & & T(U(M)) & & \end{array}$$

Again by 1) of Proposition 1.6, $\mathrm{Coker}(\rho_M) \in \mathcal{C}$ so that, being $T(U(M)) \simeq H(U(M))$ $\mathcal{C}$-closed we have a morphism $\delta_M : M \to T(U(M))$ such that $\delta_M \circ \rho_M = \mathbf{1}_{T(U(M))}$. Hence $M = \mathrm{Im}(\rho_M) \oplus X$ for some object $X \in \mathcal{B}$ where $X \simeq \mathrm{Coker}(\rho_M) \in \mathcal{C}$. Hence $X \subseteq t(M)$. Since $M/X \simeq \mathrm{Im}(\rho_M)$, then M/X is $\mathcal{C}$-torsion free and therefore we get $X = t(M)$. Hence $M \simeq t(M) \oplus M/t(M)$.

Lemma 1.10 *Let $F : \mathcal{A} \to \mathcal{B}$ be a left adjoint of a functor $G : \mathcal{B} \to \mathcal{A}$. Then F is a category equivalence iff G is a category equivalence. In this case $F \circ G \simeq \mathbf{1}_\mathcal{B}$ and $G \circ F \simeq \mathbf{1}_\mathcal{A}$.*

Proof. Assume that F is a category equivalence and let $G' : \mathcal{B} \to \mathcal{A}$ be a functor such that $G' \circ F \simeq \mathbf{1}_\mathcal{A}$ and $F \circ G' \simeq \mathbf{1}_\mathcal{B}$. Then G' is a right adjoint of F so that $G \simeq G'$. It follows that $F \circ G \simeq \mathbf{1}_\mathcal{B}$ and $G \circ F \simeq \mathbf{1}_\mathcal{A}$. In particular, we get that G is a category equivalence. The converse follows by duality.

Proposition 1.11 *The following assertions are equivalent :*
(a) T is a category equivalence ;
(b) U is a category equivalence ;
(c) H is a category equivalence.
Moreover, if one of these conditions is satisfied, $T \simeq H$.

Proof. The equivalence among (a),(b) and (c) follows from Lemma 1.10. The last statement follows by the uniqueness of the left (or right) adjoint.

Proposition 1.12 *If $\sigma : \mathbf{1}_\mathcal{A} \to U \circ T$ (resp. $\tau : U \circ H \to \mathbf{1}_\mathcal{B}$) is a functorial isomorphism, then U is a category equivalence $\iff$ $\rho : T \circ U \to \mathbf{1}_\mathcal{B}$ (resp. $\zeta : \mathbf{1}_\mathcal{B} \to H \circ U$) is a functorial isomorphism.*

Proof. Assume that U is a category equivalence. Then, by Lemma 1.10, there is a functorial isomorphism $\rho' : T \circ U \to \mathbf{1}_\mathcal{B}$.
Let M be an object in $\mathcal{B}$. Then we have, by 1) and 2) of 1.1,

$$\rho'_M = \alpha^{-1}_{U(M),M}(\alpha_{U(M),M}(\rho'_M)) = \rho_M \circ T(U(\rho'_M) \circ \sigma_{U(M)}) \; .$$

As ρ'_M and $\sigma_{U(M)}$ are isomorphisms we get that ρ_M is an isomorphism.
The converse is trivial.
The proof for τ instead of σ is analogous.

Corollary 1.13 *Assume that $\sigma : \mathbf{1}_\mathcal{A} \to U \circ T$ and $\tau : U \circ H \to \mathbf{1}_\mathcal{B}$ are natural equivalences.*
Then the following statements are equivalent :
(a) $\rho : T \circ U \to \mathbf{1}_\mathcal{B}$ is a functorial isomorphism;
(b) $\zeta : \mathbf{1}_\mathcal{B} \to H \circ U$ is a functorial isomorphism;
(c) T is a category equivalence;
(d) U is a category equivalence;
(e) H is a category equivalence.
Moreover, if one of these conditions is satisfied, $\eta : T \to H$ and $\lambda : T \to H$ are functorial isomorphisms.

Proof. It follows from Propositions 1.11, 1.12 and Theorem 1.3.

We end up this section with some general facts that will be useful in the sequel.

Lemma 1.14 *Let $\mathcal{C}$ be a Grothendieck category, $\xi : C_1 \to C_2$ be a morphism in $\mathcal{C}$. Then ξ is an isomorphism iff $Hom_{\mathcal{C}}(\xi, \cdot) : Hom_{\mathcal{C}}(C_2, \cdot) \to Hom_{\mathcal{C}}(C_1, \cdot)$ is a functorial isomorphism.*

Proof. Assume that $\mathrm{Hom}_{\mathcal{C}}(\xi, \cdot)$ is a functorial isomorphism and let K be a cogenerator of $\mathcal{C}$. Then $\mathrm{Hom}_{\mathcal{C}}(\xi, K) : \mathrm{Hom}_{\mathcal{C}}(C_2, K) \to \mathrm{Hom}_{\mathcal{C}}(C_1, K)$ is an isomorphism. As the functor $\mathrm{Hom}_{\mathcal{C}}(-, K)$ is faithful, it is easy to get that ξ is both an epimorphism and a monomorphism in $\mathcal{C}$.

Proposition 1.15 *Let $\mathcal{C}$ and $\mathcal{D}$ be Grothendieck categories and let $F, G : \mathcal{C} \to \mathcal{D}$ be rightexact covariant functors that commute with arbitrary direct sums. Assume that $(L_i)_{i\in I}$ is a family of generators of $\mathcal{C}$ and that, for every $i \in I$, an isomorphism $\theta_{L_i} : F(L_i) \to G(L_i)$ is given such that, for every morphism $\alpha : L_i \to L_j$ in $\mathcal{C}$, $i, j \in I$, it is $\theta_{L_j} \circ F(\alpha) = G(\alpha) \circ \theta_{L_i}$.*

Then, for every $L \in \mathcal{C}$, there is an isomorphism $\theta_L : F(L) \to G(L)$. Moreover if the L_i's are projective, the θ_L's, $L \in \mathcal{C}$, define a functorial isomorphism $\theta : F \to G$.

Proof. Assume that $L = \bigoplus_{t\in T} \Lambda_t$ where $\Lambda_t \in \{L_i \mid i \in I\}$ for every t. Let $\epsilon_t : \Lambda_t \to L$ and $\pi_t : L \to \Lambda_t,\ t \in T$, be resp. the t-th canonical injection and projection. As F and G commute with direct sums there is a (unique) morphism $\theta_L : F(L) \to G(L)$ such that $\theta_L \circ F(\epsilon_t) = G(\epsilon_t) \circ \theta_{\Lambda_t}$ for every $t \in T$. It follows that $G(\pi_t) \circ \theta_L = \theta_{\Lambda_t} \circ F(\pi_t)$ for every $t \in T$.
Let now $L' = \bigoplus_{t'\in T'} \Lambda_{t'}$ where $\Lambda_{t'} \in \{L_i \mid i \in I\}$ for every t' and let $\alpha : L \to L'$ be a morphism in $\mathcal{C}$. Then $\theta_{L'} \circ F(\alpha) = G(\alpha) \circ \theta_L$. In fact, in view of our assumptions, for every $t \in T$ and $t' \in T'$ we have :

$$G(\pi_{t'}) \circ (\theta_{L'} \circ F(\alpha)) \circ F(\epsilon_t) = \theta_{\Lambda_{t'}} \circ F(\pi_{t'} \circ \alpha \circ \epsilon_t) = G(\pi_{t'} \circ \alpha \circ \epsilon_t) \circ \theta_{\Lambda_t} =$$

$$= G(\pi_{t'}) \circ G(\alpha) \circ G(\epsilon_t) \circ \theta_{\Lambda_t} = G(\pi_{t'}) \circ (G(\alpha) \circ \theta_L) \circ F(\epsilon_t) \ .$$

Let now L be an arbitrary object of $\mathcal{C}$. Then, in $\mathcal{C}$, we have an exact sequence of the form :

$$L_1 \xrightarrow{\alpha} L_2 \xrightarrow{\beta} L \to 0$$

where L_1 and L_2 are direct sums of objects belonging to $\{L_i \mid i \in I\}$. By the foregoing, we have the commutative diagram with exact rows :

$$\begin{array}{ccccccc}
F(L_1) & \xrightarrow{F(\alpha)} & F(L_2) & \xrightarrow{F(\beta)} & F(L) & \longrightarrow & 0 \\
\theta_{L_1}\downarrow & & \theta_{L_2}\downarrow & & & & \\
G(L_1) & \xrightarrow{G(\alpha)} & G(L_2) & \xrightarrow{G(\beta)} & G(L) & \longrightarrow & 0
\end{array}$$

where θ_{L_1} and θ_{L_2} are isomorphisms. It follows that there is a unique morphism

$\theta_L : F(L) \to G(L)$ such that $\theta_L \circ F(\beta) = G(\beta) \circ \theta_{L_2}$. Moreover θ_L is an isomorphism. Assume now that the L_i's, $i \in I$, are projective and let $f : L \to L'$ be a morphism in $\mathcal{C}$. We have to prove that

$$\theta_{L'} \circ F(f) = G(f) \circ \theta_L \qquad (1)$$

As before, for L' we have an exact sequence in $\mathcal{C}$:

$$L'_1 \overset{\alpha'}{\to} L'_2 \overset{\beta'}{\to} L' \to 0$$

where L'_1 and L'_2 are direct sums of objects belonging to $\{L_i \mid i \in I\}$ and $\theta_{L'} \circ F(\beta') = G(\beta') \circ \theta_{L'_2}$.
Since L_2 is projective, there exists a morphism $g : L_2 \to L'_2$ such that $f \circ \beta = \beta' \circ g$:

$$\begin{array}{ccccc} L_2 & \xrightarrow{\beta} & L & & \\ g\downarrow & & \downarrow f & & \\ L'_2 & \xrightarrow{\beta'} & L' & \to & 0 \end{array}$$

As $F(\beta)$ is an epimorphism, to prove (1) is equivalent to prove :

$$\theta_{L'} \circ F(f) \circ F(\beta) = G(f) \circ \theta_L \circ F(\beta) \qquad (2)$$

We have :

$$\theta_{L'} \circ F(f) \circ F(\beta) = \theta_{L'} \circ F(\beta') \circ F(g) = G(\beta') \circ \theta_{L'_2} \circ F(g) =$$
$$= G(\beta') \circ G(g) \circ \theta_{L_2} = G(f) \circ G(\beta) \circ \theta_{L_2} = G(f) \circ \theta_L \circ F(\beta)$$

where $\theta_{L'_2} \circ F(g) = G(g) \circ \theta_{L_2}$, in view of the foregoing results concerning direct sums of L_i's.

Lemma 1.16 *Let $(L_i)_{i \in I}$ be a family of generators of a Grothendieck category $\mathcal{C}$. Let $F, G : \mathcal{C} \to \mathcal{D}$ be right exact covariant functors that commute with arbitrary direct sums.*
Let $\epsilon : F \to G$ be a functorial morphism. If $\epsilon_{L_i} : F(L_i) \to G(L_i)$ is an isomorphism for every $i \in I$, then ϵ is a functorial isomorphism.

Proof. For every $L \in \mathcal{C}$ we have an exact sequence of the form

$$L_1 \overset{\alpha}{\to} L_2 \overset{\beta}{\to} L \to 0$$

where L_1 and L_2 are direct sums of objects belonging to $\{L_i \mid i \in I\}$. By our assumptions, we have a commutative diagram with exact rows :

$$\begin{array}{ccccccc} F(L_1) & \xrightarrow{F(\alpha)} & F(L_2) & \xrightarrow{F(\beta)} & F(L) & \longrightarrow & 0 \\ \epsilon_{L_1}\downarrow & & \epsilon_{L_2}\downarrow & & \epsilon_L\downarrow & & \\ G(L_1) & \xrightarrow{G(\alpha)} & G(L_2) & \xrightarrow{G(\beta)} & G(L) & \longrightarrow & 0 \end{array}$$

Since ϵ_{L_1} and ϵ_{L_2} are isomorphisms, ϵ_L is an isomorphism too.

2 Applications

2.1. Let $f : G \to G'$ be a morphism of groups, let A be a left G-set, A' a left G'-set and assume that a map $\varphi : A \to A'$ is given such that

$$\varphi(ga) = f(g)\varphi(a) \qquad \text{for every } g \in G, a \in A\ .$$

Let R and R' be graded rings over the groups G and G' respectively and let $\psi : R \to R'$ be a ring morphism such that

$$\psi(R_g) \subseteq R'_{f(g)} \qquad \text{for every } g \in G\ .$$

Set $T = (f, \varphi, \psi)$.
Let $T_* : (G', A', R')\text{-gr} \to (G, A, R)\text{-gr}$ be the functor defined by setting, for every $M \in (G', A', R')\text{-gr}$

$$T_*(M) = \bigoplus_{a\in A} {}^{a}M_{\varphi(a)}$$

where ${}^{a}M_{\varphi(a)} = M_{\varphi(a)}$ for every $a \in A$.
For every $m \in M_{\varphi(a)}$ let ${}^{a}m$ denote the element of ${}^{a}M_{\varphi(a)}$ which coincides with m.
The multiplication by the elements of R on $T_*(M)$ is defined by setting

$$r_g^a m = {}^{ga}(\psi(r_g)m)$$

for every $g \in G,\ a \in A,\ r_g \in R_g,\ m \in M_{\varphi(a)}$.
For every $a \in A$, we have

$$(T_*(M))_a \simeq \mathrm{Hom}_{(G',A',R')\text{-gr}}(R'(\varphi(a)), M)$$

so that

$$T_*(M) \simeq \bigoplus_{a\in A} \mathrm{Hom}_{(G',A',R')\text{-gr}}(R'(\varphi(a)), M)\ .$$

The functor T_* is *a covariant exact functor*.
Let $T^* : (G, A, R)\text{-gr} \to (G', A', R')\text{-gr}$ be the functor defined by setting, for every $L \in (G, A, R)\text{-gr}$,

$$T^*(L) = R' \otimes_R L$$

endowed with the A'-gradation defined in the following way :
$(T^*(L))_{a'} =$ subgroup of $R' \otimes_R L$ spanned by the elements of the form $r'_\lambda \otimes \ell_a$, where $\lambda \in G',\ a \in A,\ \lambda\varphi(a) = a',\ r'_\lambda \in R'_\lambda,\ \ell_a \in L_a$, for every $a' \in A'$.
The functor T^* *is a covariant right exact functor and it is a left adjoint of the functor* T_* (see [M$_2$]).
The adjunction isomorphism

$$\alpha : \mathrm{Hom}_{(G',A',R')\text{-gr}}(T^*, \mathbf{1}_{(G',A',R')\text{-gr}}) \to \mathrm{Hom}_{(G,A,R)\text{-gr}}(\mathbf{1}_{(G,A,R)\text{-gr}}, T_*)$$

is defined as follows : for every $L \in (G, A, R)\text{-gr}$, $M \in (G', A', R')\text{-gr}$

$$\alpha_{L,M} : \mathrm{Hom}_{(G',A',R')\text{-gr}}(T^*(L), M) \to \mathrm{Hom}_{(G,A,R)\text{-gr}}(L, T_*(M))$$

is defined by

$$(\alpha_{L,M}(u))(\ell_a) = u(1 \otimes \ell_a) \in M_{\varphi(a)} = (T_*(M))_a$$

for every $u : T^*(L) \to M$ morphism in (G', A', R')-gr, $a \in A,\ \ell_a \in L_a$, and extending it by linearity.
Moreover, we have :

$$\alpha^{-1}_{L,M} : \mathrm{Hom}_{(G,A,R)\text{-gr}}(L, T_*(M)) \to \mathrm{Hom}_{(G',A',R')\text{-gr}}(T^*(L), M)$$

is defined by

$$(\alpha^{-1}_{L,M}(v))(r' \otimes \ell_a) = r'v(\ell_a)$$

for every $v : L \to T_*(M)$ morphism in (G, A, R)-gr, $a \in A,\ r' \in R',\ \ell_a \in L_a$, and extending it by linearity.
Consider now the functor $\tilde{T} : (G, A, R)\text{-gr} \to (G', A', R')\text{-gr}$ defined by

$$\tilde{T}(L) = \bigoplus_{a' \in A'} (\tilde{T}(L))_{a'} \qquad L \in (G, A, R)\text{-gr}$$

where

$$(\tilde{T}(L))_{a'} = \mathrm{Hom}_{(G,A,R)\text{-gr}}(T_*(R'(a')), L)$$

for every $a' \in A'$, and given $g' \in G',\ r'_{g'} \in R'_{g'},\ a' \in A',\ \xi \in (\tilde{T}(L))_{a'}$

$$r'_{g'}\xi : T_*(R'(g'a')) \to L$$

is defined by setting

$$r'_{g'}\xi = \xi \circ T_*(\mu_{r'_{g'}})$$

where $\mu_{r'_{g'}} : R'(g'a') \to R'(a')$ is the right multiplication by $r'_{g'}$ on R'.

The functor $\tilde{T}$ is left exact and it is a right adjoint of the functor T_* (see [M$_2$]).
The adjunction isomorphism

$$\gamma : \mathrm{Hom}_{(G,A,R)\text{-gr}}(T_*, \mathbf{1}_{(G,A,R)\text{-gr}}) \to \mathrm{Hom}_{(G',A',R')\text{-gr}}(\mathbf{1}_{(G',A',R')\text{-gr}}, \tilde{T})$$

is defined as follows : for every $M \in (G', A', R')$-gr, $L \in (G, A, R)$-gr

$$\gamma_{M,L} : \mathrm{Hom}_{(G,A,R)\text{-gr}}(T_*(M), L) \to \mathrm{Hom}_{(G',A',R')\text{-gr}}(M, \tilde{T}(L))$$

is defined by

$$(\gamma_{M,L}(u))(m_{a'}) = u \circ T_*(\mu_{m_{a'}})\ ,$$

where $\mu_{m_{a'}} : R'(a') \to M$ is the right multiplication by $m_{a'}$ on M , for every $u : T_*(M) \to L$ morphism in (G, A, R)-gr, $a' \in A',\ m_{a'} \in M_{a'}$, and extending it by linearity.
Moreover we have :

$$\gamma^{-1}_{M,L} : \mathrm{Hom}_{(G',A',R')\text{-gr}}(M, \tilde{T}(L)) \to \mathrm{Hom}_{(G,A,R)\text{-gr}}(T_*(M), L)$$

is defined by

$$(\gamma^{-1}_{M,L}(v))(m_{\varphi(a)}) = v(m_{\varphi(a)})(1)$$

for every $v : M \to \tilde{T}(L)$ morphism in (G', A', R')-gr, $a \in A$, $m_{\varphi(a)} \in (T_*(M))_a = M_{\varphi(a)}$ and extending it by linearity.
Let $L \in (G, A, R)$-gr. We have :

$$\sigma_L = \alpha_{L,T^*(L)}(\mathbf{1}_{T^*(L)}) : L \to T_*(T^*(L))$$

$$\sigma_L(\ell_a) = 1 \otimes \ell_a \in T_*(T^*(L))_a = (R' \otimes_R L)_{\varphi(a)}$$

for every $a \in A$, $\ell_a \in L_a$,

$$\tau_L = \gamma^{-1}_{\tilde{T}(L),L}(\mathbf{1}_{\tilde{T}(L)}) : T_*(\tilde{T}(L)) \to L$$

$\tau_L(\xi) = \xi(1)$, for every $\xi : T_*(R'(\varphi(a)) \to L$ morphism in (G, A, R)-gr, $a \in A$.
Let $M \in (G', A', R')$-gr. We have :

$$\rho_M = \alpha^{-1}_{T_*(M),M}(\mathbf{1}_{T_*(M)}) : T^*(T_*(M)) \to M$$

$$\rho_M(r' \otimes x_{\varphi(a)}) = r'x_{\varphi(a)} \ ,$$

for every $a \in A$, $r' \in R'$, $x_{\varphi(a)} \in M_{\varphi(a)} = (T_*(M))_a$

$$\zeta_M = \gamma_{M,T_*(M)}(\mathbf{1}_{T_*(M)}) : M \to \tilde{T}(T_*(M))$$

$$\zeta_M(x_{a'}) = T_*(\mu_{x_{a'}})$$

where $a' \in A'$, $x_{a'} \in M_{a'}$ and $\mu_{x_{a'}} : R'(a') \to M$ is the right multiplication by $x_{a'}$ on M.

2.2 Examples

1. Let A be a left G-set, H a subgroup of G, B a subset of A such that $hB \subseteq B$ for every $h \in H$. Set $T = (f, \varphi, \psi)$ where $f : H \to G$, $\varphi : B \hookrightarrow A$, $\psi : R^{(H)} \to R$ are the canonical injections. Then the left and right adjoints of the functor

$$\begin{array}{rcccl} T_* = T^B & : & (G,A,R)\text{-gr} & \to & (H,B,R^{(H)})\text{-gr} \\ & & M & \mapsto & M^{(B)} = \bigoplus_{x\in B} M_x \end{array}$$

are the functors $T^* = S^B$ and $\tilde{T} = S_B$ as introduced in [NRV].
Given $L \in (H, B, R^{(H)})$-gr, one has :

$$S^B(L) = R \otimes_{R^{(H)}} L$$

equipped with the A-grading : $(S^B(L))_a$ = subgroup of $R \otimes_{R^{(H)}} L$ spanned by the elements of the form $r_g \otimes \ell_b$, $g \in G$, $b \in B$, $gb = a$, $r_g \in R_g$, $\ell_b \in L_b$.
$S_B(L) = \bigoplus_{a\in A}(S_B(L))_a$, where, for each $a \in A$,

$$(S_B(L))_a = \mathrm{Hom}_{(H,B,R^{(H)})\text{-gr}}(T^B(R(a)), L) = \{f \in \mathrm{Hom}_{R^{(H)}}(R, L) \mid f(R_g) = 0$$

$$\text{if } ga \notin B \text{ and } f(R_g) \subseteq L_{ga} \text{ if } ga \in B\}$$

In particular if $A = G,\ B = H = \{1\}$, then the left and right adjoints of the functor $T^{\{1\}} = (-)_1 : R\text{-gr} \to R_1\text{-mod},\ M \mapsto M_1$, are those defined in [N]. Following the terminology in [N], the functor S^B will be denoted in this case by Ind and called the (left) *induced* functor, while the functor S_B will be denoted by Coind and called the (left) *coinduced* functor.
Note that, given $L \in R_1$-mod and $g \in G$, one has :

$$(\mathrm{Coind}(L))_g = \{f \in \mathrm{Hom}_{R_1}(R, L) \mid f(R_h) = 0 \ \ \forall h \neq g^{-1}\} \ .$$

2. In the situation of 2.1 assume that $G = G',\ R = R'$ and $T = (\mathbf{1}_G, \varphi, \mathbf{1}_R)$. Then the left adjoint of the functor

$$\begin{array}{cccc} T_* = S_\varphi & : \ (G, A', R)\text{-gr} & \to & (G, A, R)\text{-gr} \\ & M & \mapsto & \bigoplus_{a\in A} {}^aM_{\varphi(a)} \end{array}$$

where ${}^aM_{\varphi(a)} = M_{\varphi(a)}$, is the functor $T^* = T_\varphi$ as introduced in [NRV]. Given $L \in (G, A, R)$-gr $T_\varphi(L)$ is the R-module L with the A'-gradation defined by

$$(T_\varphi(L))_{a'} = \bigoplus\{L_a \mid a \in A,\ \varphi(a) = a'\} \qquad \text{for } a' \in A' \ .$$

Moreover one has, in this case,

$$(\tilde{T}(L))_{a'} = \prod\{L_a \mid a \in A,\ \varphi(a) = a'\} \qquad \text{for } a' \in A' \ .$$

Given $g \in G,\ r_g \in R_g,\ \bar{x}^{a'} = (x_a)_{\substack{a\in A \\ \varphi(a)=a'}}$ it is

$$r_g \bar{x}^{a'} = \bar{y}^{ga'}$$

where

$$\bar{y}^{ga'} = (y_b)_{\substack{b\in A \\ \varphi(b)=ga'}} \qquad \text{and} \qquad y_b = r_g x_{g^{-1}b} \ .$$

In particular if A' is a singleton with G acting trivially on it, then (G, A', R)-gr $=$ R-mod, and $T^* = U$, the " *forgetful functor*"

$$U : (G, A, R)\text{-gr} \to R\text{-mod} \ .$$

T_* is its right adjoint, usually denoted by F,

$$F : R\text{-mod} \to (G, A, R)\text{-gr} \ .$$

Given $M \in R$-mod, $F(M) = \bigoplus_{a\in A} {}^aM$ where ${}^aM = M$ for every $a \in A$.
For $g \in G,\ a \in A,\ r_g \in R_g,\ m \in M$ we have :

$$r_g^a m = {}^{ga}(r_g m) \ .$$

3. In the situation of 2.1, assume that $G = G' = A = A' = \{1\}$ and $T = (\mathbf{1}_G, \mathbf{1}_A, \psi)$. Then (G, A, R)-gr=R-mod, $(G', A', R') = R'$-mod, and $\psi : R \to R'$ is just a ring morphism. In this case T_* is the *restriction of scalars functor* $\psi_* : R'\text{-mod} \to R\text{-mod}$, where if $M \in R'$-mod, $\psi_*(M) = M$ and the structure of left R-module is given by $r * m = \psi(r)m$ for any $r \in R$. Moreover $T^* = R' \otimes_R - : R\text{-mod} \to R'\text{-mod}$, where here ${}_{R'}R'_R$ is considered as an R'-R-bimodule and also $\tilde{T} = \mathrm{Hom}_R({}_RR'_{R'}, -) : R\text{-mod} \to R'\text{-mod}$, where here ${}_RR'_{R'}$ is considered as an R-R'-bimodule.
The functor $R' \otimes_R -$ (resp. $\mathrm{Hom}_R({}_RR'_{R'}, -)$) is called the (left) *Induction* (resp. (left) *Coinduction*) functor.

2.3. Our present aim is to apply the foregoing results in §1 to the functors $T_*, T^*, \tilde{T}$. For this purpose we investigate whenever σ and τ are isomorphisms. In the following we will use the notations and hypotheses of 2.1.

Lemma 2.4 *Let $\omega : R' \otimes_R R \to R'$ be the map defined by*

$$\omega(r' \otimes r) = r'\psi(r) \qquad r' \in R,\ r \in R\ .$$

Then, for every $a \in A$, ω can be regarded as an isomorphism in (G', A', R')-gr,

$$\omega_a : T^*(R(a)) \xrightarrow{\sim} R'(\varphi(a))\ .$$

Proof. It is well known that ω is an isomorphism of left R'-modules. Let $a' \in A'$, $\lambda \in G'$, $\alpha \in A$ such that $\lambda\varphi(\alpha) = a'$ and let $r'_\lambda \in R'_\lambda$, $g \in G$ such that $ga = \alpha$ and $r_g \in R_g$. Then $r'_\lambda\psi(r_g) \in R'_{\lambda f(g)}$ and

$$\lambda f(g)\varphi(a) = \lambda\varphi(ga) = \lambda\varphi(\alpha) = a'\ .$$

Therefore $\omega_a\Big((T^*(R(a)))_{a'}\Big) \subseteq (R'(\varphi(a)))_{a'}$ and ω_a is a morphism in (G', A', R')-gr.

Lemma 2.5 *For every $a \in A$ let $\nu_a : Hom_{(G,A,R)\text{-}gr}(R(a), \cdot) \to (\ \ )_a$ be the functorial isomorphism which evaluates the morphisms in $1_R \in R(a)_a$. Then*

$$\nu_a \circ Hom_{(G,A,R)\text{-}gr}(T_*(\omega_a) \circ \sigma_{R(a)},\ \cdot\ )$$

is the restriction of τ to $Hom_{(G,A,R)\text{-}gr}(T_*(R'(\varphi(a)),\ \cdot\ ) = (T_* \circ \tilde{T})_a$.

Proof. Let $L \in (G, A, R)$-gr and $\xi \in \mathrm{Hom}_{(G,A,R)\text{-gr}}(T_*(R'(\varphi(a))), L)$. Then

$$\Big(\nu_a \circ \mathrm{Hom}_{(G,A,R)\text{-gr}}(T_*(\omega_a) \circ \sigma_{R(a)}, L)\Big)(\xi) = (\xi \circ T_*(\omega_a) \circ \sigma_{R(a)})(1_R) =$$

$$= \xi(T_*(\omega_a)(1_{R'} \otimes 1_R)) = \xi(1_{R'}\psi(1_R)) = \xi(1_{R'}) = \tau_L(\xi)\ .$$

Theorem 2.6 *$\sigma : \mathbf{1}_{(G,A,R)\text{-}gr} \to T_* \circ T^*$ is a functorial isomorphism if and only if $\tau : T_* \circ \tilde{T} \to \mathbf{1}_{(G,A,R)\text{-}gr}$ is a functorial isomorphism.*

Proof. In view of Lemma 2.5, τ is a functorial isomorphism whenever σ is. Conversely, assume that τ is a functorial isomorphism. Then, always by Lemma 2.5, for each $a \in A$, $\mathrm{Hom}_{(G,A,R)\text{-gr}}(T_*(\omega_a) \circ \sigma_{R(a)}, \cdot\,)$ is a functorial isomorphism, so that, in view of Lemma 1.14, $T_*(\omega_a) \circ \sigma_{R(a)}$ is an isomorphism in (G, A, R)-gr. As, by Lemma 2.4, $T_*(\omega_a)$ is an isomorphism, we get that $\sigma_{R(a)}$ is an isomorphism in (G, A, R)-gr. Therefore, by Lemma 1.16, σ is a functorial isomorphism.

2.7. Given a left G-set A, for every $a, \alpha \in A$ we set

$$C^a_\alpha = \{g \in G \mid ga = \alpha\} \ .$$

Clearly, given $g \in C^a_\alpha$, $f(g)\varphi(a) = \varphi(ga) = \varphi(\alpha)$ i.e. $f(g) \in C^{\varphi(a)}_{\varphi(\alpha)}$. Therefore

$$\psi\Big(\bigoplus\{R_g \mid g \in C^a_\alpha\}\Big) \subseteq \bigoplus\{R'_{g'} \mid g' \in C^{\varphi(a)}_{\varphi(\alpha)}\} \ .$$

We denote by ψ^a_α the corestriction to $\bigoplus\{R'_{g'} \mid g' \in C^{\varphi(a)}_{\varphi(\alpha)}\}$ of the restriction of ψ to $\bigoplus\{R_g \mid g \in C^a_\alpha\}$.
Moreover we set

$$\chi^a = T_*(\omega_a) \circ \sigma_{R(a)} : R(a) \to T_*(R'(\varphi(a))) \ .$$

Lemma 2.8 *For every $a, \alpha \in A$ the α-component, χ^a_α of χ^a coincides with ψ^a_α .*

Proof. We have

$$\begin{aligned}\chi^a_\alpha : R(a)_\alpha = \bigoplus\{R_g \mid g \in C^a_\alpha\} \to T_*(R'(\varphi(a)))_\alpha &= R'(\varphi(a))_{\varphi(\alpha)} \\ &= \bigoplus\{R'_{g'} \mid g' \in C^{\varphi(a)}_{\varphi(\alpha)}\}\end{aligned}$$

and, for every $r \in R(a)_\alpha$, it is :

$$\begin{aligned}\chi^a_\alpha(r) = \chi^a(r) &= (T_*(\omega_a) \circ \sigma_{R(a)})(r) = T_*(\omega_a)(1 \otimes r) = \omega_a(1 \otimes r) \\ &= 1 \cdot \psi(r) = \psi(r) = \psi^a_\alpha(r) \ .\end{aligned}$$

Theorem 2.9 *$\sigma : \mathbf{1}_{(G,A,R)\text{-}gr} \to T_* \circ T^*$ is a functorial isomorphism if and only if, for every $a, \alpha \in A$,*

$$\psi^a_\alpha : \bigoplus\{R_g \mid g \in C^a_\alpha\} \to \bigoplus\{R'_{g'} \mid g' \in C^{\varphi(a)}_{\varphi(\alpha)}\}$$

is bijective.

Proof. In view of Lemma 1.16, σ is a functorial isomorphism if and only if, for every $a \in A$, $\sigma_{R(a)} : R(a) \to T_*(T^*(a))$ is an isomorphism in (G, A, R)-gr. By Lemma 2.4 this holds if and only if, for every $a \in A$, $\chi^a = T_*(\omega_a) \circ \sigma_{R(a)} : R(a) \to T_*(R'(\varphi(a)))$ is an isomorphism in (G, A, R)-gr. Given $a \in A$, as χ^a is a morphism in (G, A, R)-gr, it is an isomorphism iff, for every $\alpha \in A$, its α-component χ^a_α is bijective. By Lemma 2.8, for every $a, \alpha \in A$, $\chi^a_\alpha = \psi^a_\alpha$ and we conclude.

Corollary 2.10 *If* $\sigma : \mathbf{1}_{(G,A,R)}\text{-}gr \to T_* \circ T^*$ *is an isomorphism, then*
1) for every $g \in G$, $\psi_{|R_g}$ *is injective ;*
2) for every $a \in A$, *if* $g' \in Supp(R')$ *and* $g'\varphi(a) \in Im(\varphi)$, *then* $g' \in Im(f)$ *;*
3) $\psi : R \to R'^{(Im(f))}$ *is surjective.*

Proof.
1) Let $g \in G$. Given $a \in A$, then $g \in C^a_\alpha$ for $\alpha = ga$ so that, by Theorem 2.9, $\psi_{|R_g}$ is injective.
Note now that, as $\psi(R_g) \subseteq R'_{f(g)}$ for every $g \in G$, given $a, \alpha \in A$, ψ^a_α is surjective iff, for every $g' \in C'^{\varphi(a)}_{\varphi(\alpha)}$, we have

$$R'_{g'} = \sum\{\psi(R_g) \mid g \in C^a_\alpha \cap f^{-1}(g')\} \ .$$

2) Let $g' \in \mathrm{Supp}(R')$ and assume that, for a certain $a \in A$, $g'\varphi(a) \in \mathrm{Im}(\varphi)$. Then $g'\varphi(a) = \varphi(\alpha)$ for a suitable $\alpha \in A$ i.e. $g' \in C'^{\varphi(a)}_{\varphi(\alpha)}$. As $R'_{g'} \neq 0$, by the foregoing we get $f^{-1}(g') \neq \emptyset$.

3) Let $g' \in \mathrm{Im}(f)$, $g' = f(g)$ for a suitable $g \in G$. Given $a \in A$ set $\alpha = ga$. Then $\varphi(\alpha) = f(g)\varphi(a)$ so that $g' \in C'^{\varphi(a)}_{\varphi(\alpha)}$ and hence

$$R'_{g'} = \sum\{\psi(Rg) \mid g \in C^a_\alpha \cap f^{-1}(g')\} \subseteq \mathrm{Im}(\psi) \ .$$

2.11 Example Let $f : G \to G'$ be a group morphism and let K be any ring. Then f induces, in a natural way, a ring homomorphism

$$\psi = \psi_f : K[G] \to K[G'] \ ,$$

where $K[G]$ and $K[G']$ are the usual group rings over G and G' respectively, such that $\psi_{|K} = \mathbf{1}_K$ and $\psi(g) = f(g)$ for every $g \in G$. Let $A = G$, $A' = G'$, $\varphi = \psi$.

Then, given $a, \alpha \in G$ we have $C^a_\alpha = \{\alpha a^{-1}\}$ and $C'^{\varphi(a)}_{\varphi(\alpha)} = \{f(a\alpha^{-1})\}$. Since for every $g \in G$ and $k \in K$ it is

$$\psi(kg) = kf(g) \ ,$$

we conclude that ψ^a_α is bijective. Therefore, by Theorem 2.9 σ is an isomorphism. Note that ψ is not injective if f is not injective.

2.12. Given $a' \in A'$, we set

$$\nabla^{a'} : \bigoplus\{R(b) \mid b \in \varphi^{-1}(a')\} \to T_*(R'(a'))$$

the codiagonal morphism of the family of morphisms $\{\chi^b \mid b \in \varphi^{-1}(a')\}$. Clearly, if $\varphi^{-1}(a') = \emptyset$, $\nabla^{a'} = 0$.

Proposition 2.13 *1) If* $\psi : R \to R'$ *is injective, then, for every* $a' \in A'$, $\nabla^{a'}$ *is injective.*
2) If $\psi : R \to R'^{(Im(f))}$ *is surjective and* $a' \in A'$ *is such that* $g'a' \in Im(\varphi)$, *with* $g' \in Supp(R')$, *implies* $g' \in Im(f)$, *then* $\nabla^{a'}$ *is surjective.*

Proof. Let $a' \in A'$. As $\nabla^{a'}$ is a morphism in (G,A,R)-gr it is injective (resp. surjective) if and only if, for every $\alpha \in A$, its α-component $\nabla^{a'}_\alpha$ is injective (resp. surjective). By definition of $\nabla^{a'}$, given $\alpha \in A$, $\nabla^{a'}_\alpha$ is the codiagonal morphism of the family of morphisms $\{\chi^b_\alpha \mid b \in \varphi^{-1}(a')\}$. By Lemma 2.8, $\chi^b_\alpha = \psi^b_\alpha$, $\alpha, b \in A$. Moreover

$$\left(\bigoplus\{R(b) \mid b \in \varphi^{-1}(a')\}\right)_\alpha = \bigoplus\{R_g \mid g \in C^b_\alpha\ ,\ b \in \varphi^{-1}(a')\} \leq R$$

and

$$\bigoplus\{R'_{g'} \mid g' \in C^{a'}_{\varphi(\alpha)}\} \leq R'\ .$$

It follows that $\nabla^{a'}_\alpha$ coincides with the corestriction $\psi^{a'}$ to $\bigoplus\{R'_{g'} \mid g' \in C^{a'}_{\varphi(\alpha)}\}$ of the restriction of ψ to $\left(\bigoplus\{R_g \mid g \in C^b_\alpha\ ,\ b \in \varphi^{-1}(a')\}\right)$. Therefore $\nabla^{a'}$ is injective whenever ψ is and 1) is proved.

2) Let $\alpha \in A$, $g' \in C^{a'}_{\varphi(\alpha)} \cap \mathrm{Supp}(R')$. Then, in view of our assumptions, $g' \in \mathrm{Im}(f)$ so that $R'_{g'} \subseteq \mathrm{Im}(\psi)$. By the foregoing, we get that $\nabla^{a'}_\alpha = \psi^{a'}$ is surjective. Hence $\nabla^{a'}$ is surjective.

Corollary 2.14 *Assume that $\psi : R \to R'^{(Im(f))}$ is a ring isomorphism and that, for every $g' \in Supp(R')$ and $a \in A$, $g'\varphi(a) \in Im(\varphi)$ implies $g' \in Im(f)$, then, for every $a \in A$,*

$$\nabla^{\varphi(a)} : \bigoplus\{R(b) \mid b \in \varphi^{-1}(\varphi(a))\} \to T_*(R'(\varphi(a))$$

is an isomorphism in (G,A,R)-gr.

Proposition 2.15 *Assume that $\sigma : \mathbf{1}_{(G,A,R)\text{-}gr} \to T_* \circ T^*$ is a functional isomorphism. Then $\psi : R \to R'$ is injective if and only if $\varphi : A \to A'$ is injective. Moreover, in this case, $\nabla^{\varphi(a)} : R(a) \to T_*(R'(\varphi(a)))$ is an isomorphism in (G,A,R)-gr.*

Proof. Assume that $\psi : R \to R'$ is injective. Then, in view of Corollary 2.10, the hypotheses of Corollary 2.14 are fulfilled and hence

$$\bigoplus\{R(b) \mid b \in \varphi^{-1}(\varphi(a))\} \cong T_*(R'(\varphi(a)))\ ,$$

for every $a \in A$. On the other hand, given $a \in A$,

$$\chi^a = T_*(\omega_a) \circ \sigma_{R(a)} : R(a) \to T_*(R'(\varphi(a)))$$

is an isomorphism and hence we get :

$$R(a) \cong \bigoplus\{R(b) \mid b \in \varphi^{-1}(\varphi(a))\}\ .$$

Hence $|\varphi^{-1}(\varphi(a))| = 1$ for every $a \in A$, i.e. φ is injective.
Assume now that $\varphi : A \to A'$ is injective. Then, given $a \in A$, we have

$$T_*(R'(\varphi(a))) = \bigoplus_{\alpha \in A} R'(\varphi(a))_{\varphi(\alpha)} \leq R'(\varphi(a))$$

and hence $\chi^a : R(a) \to T_*(R'(\varphi(a)))$ coincides, in view of Lemma 2.8, with ψ.

Proposition 2.16 *Assume that $\sigma : \mathbf{1}_{(G,A,R)}$-gr $\to T_* \circ T^*$ is a functorial isomorphism. If $f_{|Supp(R)}$ is injective, then also $\psi : R \to R'$ and $\varphi : A \to A'$ are injective.*

Proof. By Corollary 2.10, $\psi_{|R_g}$ is injective, for every $g \in G$.

Let $r \in R$, $r \neq 0$. Write $r = \sum_{i=1}^{n} r_{g_i}$ where $g_i \neq g_j$ for $i \neq j$, $g_i, g_j \in G$, and $0 \neq r_{g_i} \in R_{g_i}$, for every i. Then $0 \neq \psi(r_{g_i}) \in R'_{f(g_i)}$. As f is injective, the $f(g_i)$'s are all distincts so that we get $0 \neq \sum_{i=1}^{n} \psi(r_{g_i}) = \psi(r)$. Therefore ψ is injective and hence, by Proposition 2.15, also $\varphi : A \to A'$ is injective.

Remark 2.17 *The converse of Proposition 2.16 does not hold. In fact let $N \neq \{1\}$ be a normal subgroup of a group G and let $A = A' = G' = G/N$. Let $R = R'$ be any graded ring over G such that $G = Supp(R)$. Let $f : G \to G/N$ be the canonical projection and $T = (f, \mathbf{1}_A, \mathbf{1}_R)$. Then (G, A, R)-gr $= (G', A', R')$-gr, σ is obviously a functorial isomorphism, $\psi = \mathbf{1}_R$ and $\varphi = \mathbf{1}_A$ are isomorphisms, but f is not injective.*

Theorem 2.18 *Assume that*
1) $\varphi : A \to A'$ is injective;
2) $\psi : R \to R'^{(Im(f))}$ is a ring isomorphism;
3) for every $a \in A$, if $g' \in Supp(R')$ and $g'\varphi(a) \in Im(\varphi)$, then $g' \in Im(f)$.
Then $\sigma : \mathbf{1}_{(G,A,R)}$-gr $\to T_* \circ T^*$ and $\tau : T_* \circ \tilde{T} \to \mathbf{1}_{(G,A,R)}$-gr are functorial isomorphisms.

Proof. Let $a, \alpha \in A$, $g' \in C^{\varphi(a)}_{\varphi(\alpha)} \cap \mathrm{Supp}(R')$. By assumption 3), $g' \in \mathrm{Im}(f)$. Therefore, by 2), $R'_{g'} \subseteq \mathrm{Im}(\psi)$ and hence $R'_{g'} = \sum\{\psi(R_g) \mid g \in f^{-1}(g')\}$. Let $g \in f^{-1}(g')$. Then $\varphi(ga) = f(g)\varphi(a) = g'\varphi(a) = \varphi(\alpha)$ and hence, as φ is injective, $ga = \alpha$. It follows that $R'_{g'} = \sum\{\psi(R_g) \mid g \in C^a_\alpha \cap f^{-1}(g')\}$ and hence ψ^a_α is surjective. By 1) ψ^a_α is injective. The conclusion now follows by Theorem 2.9 and Theorem 2.6.

Remark 2.19 *Note that the assumptions of Theorem 2.18 are, in practice, those of Theorem 3.7 in [NRV]. For a list of examples fulfilling these assumptions see [NRV] Remarks 3.9.*

2.20. Assume that $\sigma : \mathbf{1}_{(G,A,R)}$-gr $\to T_* \circ T^*$ is a functorial isomorphism. Then, by Theorem 2.6, also $\tau : T_* \circ \tilde{T} \to \mathbf{1}_{(G,A,R)}$-gr is a functorial isomorphism and it is straightforward to check that, in this case, $\eta = \lambda : T^* \to \tilde{T}$. Given $L \in (G, A, R)$-gr,

$$\eta_L : T^*(L) = R' \otimes_R L \to \bigoplus_{a' \in A'} \mathrm{Hom}_{(G,A,R)\text{-gr}}(T_*(R'(a')), L)$$

is defined by

$$(\eta_L(r' \otimes \ell))(s') = \left((\psi^a_\alpha)^{-1}(s'r')\right)\ell$$

for every $g' \in G',\ a \in A,\ a' = g'\varphi(a),\ r' \in R'_{g'},\ \ell \in L_a,\ \alpha \in A$,

$$s' \in (T_*(R'(a')))_\alpha = R'(a')_{\varphi(\alpha)} = \bigoplus\{R'_{t'} \mid t' \in C^{a'}_{\varphi(\alpha)}\} .$$

Note that this makes sense as, given $t' \in C^{a'}_{\varphi(\alpha)}$, $t'g'\varphi(a) = t'a' = \varphi(\alpha)$ so that $t'g' \in C^{\varphi(a)}_{\varphi(\alpha)}$ and $s'r' \in \sum\{R'_{v'} \mid v' \in C^{\varphi(a)}_{\varphi(\alpha)}\}$. Moreover, by Theorem 2.9,

$$\psi^a_\alpha : \bigoplus\{R_g \mid g \in C^a_\alpha\} \to \sum\{R'_{v'} \mid v' \in C^{\varphi(a)}_{\varphi(\alpha)}\}$$

is bijective.

Theorem 2.21 *Assume that* $\sigma : \mathbf{1}_{(G,A,R)\text{-}gr} \to T_* \circ T^*$ *is an isomorphism. Then the following conditions are equivalent :*
(a) T_* *is a category equivalence;*
(b) T^* *is a category equivalence;*
(c) $\tilde{T}$ *is a category equivalence;*
(d) $\zeta : \mathbf{1}_{(G',A',R')\text{-}gr} \to \tilde{T} \circ T_*$ *is a functorial isomorphism;*
(e) $\rho : T^* \circ T_* \to \mathbf{1}_{(G',A',R')\text{-}gr}$ *is a functorial isomorphism;*
(f) for every $a' \in A'$, $\rho_{R(a')} : (T^* \circ T_*)(R(a')) \to R(a')$ *is surjective;*
(g) for every $M \in (G', A', R')$*-gr,* $\rho_M : (T^* \circ T_*)(M) \to M$ *is surjective.*
Moreover, if one of this conditions is satisfied $\eta : T^* \to \tilde{T}$ *is a functorial isomorphism.*

Proof. (a) $\iff$ (b) $\iff$ (c) $\iff$ (d) $\iff$ (e) and the last assertion follow from Corollary 1.13, in view of Theorem 2.6.
(e)$\Rightarrow$ (f) is trivial.
(f)$\Rightarrow$(g) Let $M \in (G', A', R')$-gr. Then we have an exact sequence of the form

$$F_1 = \bigoplus_{i\in I} R'(a'_i) \to F_2 = \bigoplus_{j\in J} R'(a'_j) \to M \to 0 .$$

As $T^* \circ T_*$ is right exact and commutes with direct sums we get the commutative diagram with exact rows :

$$\begin{array}{ccccccc}
(T^* \circ T_*)(F_1) & \longrightarrow & (T^* \circ T_*)(F_2) & \longrightarrow & (T^* \circ T_*)(M) & \longrightarrow & 0 \\
\rho_{F_1}\downarrow & & \rho_{F_2}\downarrow & & \rho_M\downarrow & & \\
F_1 & \longrightarrow & F_2 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

where the first two arrows are surjective. As ρ_{F_2} is surjective, also ρ_M is surjective.
(g)$\Rightarrow$(e) In view of Lemma 1.16, ρ is a functorial isomorphism iff, for every $a' \in A'$, $\rho_{R'(a')}$ is an isomorphism. By our assumption, $\rho_{R'(a')}$ is surjective. Let $K = \mathrm{Ker}(\rho_{R'(a')})$. In view of Proposition 1.6, $T_*(K) = 0$. On the other hand, by our assumption, $\rho_K : (T^* \circ\ T_*)(K) \to\ K$ is surjective. Therefore $K = 0$.

Proposition 2.22 *Let $a' \in A'$. Then $\rho_{R'(a')}$ is surjective iff there exist $n \in \mathbf{N},\ \alpha_{\mathbf{1}}, ...,$ $\alpha_n \in A,\ g'_i \in C_{a'}^{\varphi(\alpha_i)},\ m_i \in \mathbf{N},\ \gamma^{\mathbf{i}}_{\mathbf{1}}, ..., \gamma^{\mathbf{i}}_{\mathbf{m_i}} \in \mathbf{C}^{\mathbf{a}'}_{\varphi(\alpha_{\mathbf{i}})},\ \mathbf{z_i} \in \mathbf{R}'_{\mathbf{g}'},\ \omega^{\mathbf{i}}_{\mathbf{j}} \in \mathbf{R}'_{\gamma^{\mathbf{i}}_{\mathbf{j}}}$ such that*

$$\sum_{i=1}^{n}\Big(z_i \sum_{j=1}^{m_i} \omega^i_j\Big) = \sum_{i=1}^{n}\sum_{j=1}^{m_i} z_i\omega^i_j = 1$$

Proof. As $R'(a')$ is a left R'-module spanned by 1 and $\rho_{R'(a')}$ is a morphism of R'-modules, $\rho_{R'(a')}$ is surjective iff $1 \in \mathrm{Im}(\rho_{R'(a')})$ i.e. iff there exists an element $t \in T^*(T_*(R'(a')))$ such that $\rho_{R'(a')}(t) = 1$. As $1 \in R'(a')_{a'}$, we must have $t \in T^*(T_*(R'(a')))_{a'}$. Therefore there exist $n \in \mathbf{N},\ \alpha_{\mathbf{1}}, ..., \alpha_{\mathbf{n}} \in \mathbf{A}$ and, for every $i = 1, ..., n,\ g'_i \in C_{a'}^{\varphi(\alpha_i)},\ z_i \in R'_{g'_i},\ \omega_i \in R'(a')_{\varphi(\alpha_i)}$, such that

$$t = \sum_{i=1}^{n} z_i \otimes \omega_i.$$

For every $i = 1, ..., n$, there exist $m_i \in \mathbf{N},\ \gamma'_{\mathbf{1}}, ..., \gamma'_{\mathbf{m_i}} \in \mathbf{C}^{\mathbf{a}'}_{\varphi(\alpha_{\mathbf{i}})},\ \omega^{\mathbf{i}}_{\mathbf{j}} \in \mathbf{R}'_{\gamma^{\mathbf{i}}_{\mathbf{j}}}$ such that

$$\omega_i = \sum_{j=1}^{m_i} \omega^i_j\ .$$

Therefore

$$t = \sum_{i=1}^{n}\sum_{j=1}^{m_i} z_i \otimes \omega^i_j \qquad \text{and} \qquad 1 = \rho(t) = \sum_{i=1}^{n}\sum_{j=1}^{m_i} z_i\omega^i_j\ .$$

Corollary 2.23 *Assume that $G'Im(\varphi) = A'$ and R' is strongly graded. Then, for every $a' \in A',\ \rho_{R'(a')}$ is surjective.*

Proof. Let $a' \in A'$. There exist $g' \in G', \alpha \in A$ such that $a' = g'\varphi(\alpha)$. As R' is strongly graded we have $1 = \sum_{i=1}^{n} z_i\omega_i$ where $n \in \mathbf{N},\ \mathbf{z_i} \in \mathbf{R}'_{\mathbf{g}'}$ and $\omega_i \in R'_{(g')^{-1}}$.
Set $\alpha_1 = ... = \alpha_n = \alpha,\ g'_1 = ... = g'_n = g',\ m_1 = ... = m_n = 1\ , \gamma^1_1 = ... = \gamma^n_1 = (g')^{-1},\ \omega^i_1 = \omega^i$. As $g'\varphi(\alpha) = a'$ we have $(g')^{-1}a' = \varphi(\alpha)$ so that $\gamma^1_1, ..., \gamma^n_1 \in C^{a'}_{\varphi(\alpha)}$, $\omega^i_1 \in R'_{\gamma^i_1}$ and

$$\sum_{i=1}^{n}\sum_{j=1}^{m_i} z_i\omega^i_j = \sum_{i=1}^{n} z_i\omega_i = 1\ .$$

Theorem 2.24 *Assume that*
1) φ is injective;
2) $\psi : R \to R'^{(Im(f))}$ is an isomorphism;
3) for every $a \in A$, if $g' \in Supp(R')$ and $g'\varphi(a) \in Im(\varphi)$, then $g' \in Im(f)$;

4) R' is strongly graded;
5) $G' Im(\varphi) = A'$.
Then $\sigma : \mathbf{1}_{(G,A,R)}\text{-}gr \to T_* \circ T^*$, $\rho : T^* \circ T_* \to \mathbf{1}_{(G',A',R')}\text{-}gr$ and $\lambda : T^* \to \tilde{T}$ are functorial isomorphisms.

Proof. It follows by Theorem 2.18, Theorem 2.21 and Corollary 2.23.

Corollary 2.25 *Let H be a subgroup of a group G, let A be a G-set and let B be a subset of A such that $hB \subseteq B$, for all $h \in H$. Assume that :*
1) $gb \in B$, for $g \in G$ and $b \in B$, implies $g \in H$;
2) $GB = A$.
Let R be a strongly G-graded ring and let $T = (f, \varphi, \psi)$ where f = canonical injection $H \hookrightarrow G$, φ = canonical injection $B \hookrightarrow A$, ψ = canonical injection $R^{(H)} \hookrightarrow R$.
Then the categories (G, A, R)-gr and $(H, B, R^{(H)})$-gr are equivalent, and this equivalence is given by the functors T^* and T_*. Moreover $\lambda : T^* \to \tilde{T}$ is a functorial isomorphism.

Remark 2.26 *The hypotheses of Corollary 2.25 are, in particular, fulfilled when $B = \{x\}$, $H = G_x$, the stabilizer subgroup of x. Therefore Proposition 3.10 and Corollary 3.11 in [NRV] can be derived from this result.*

Theorem 2.27 *Let $T = (f, \varphi, \psi)$ be as in 2.1. Then the functors T^* and $\tilde{T}$ are isomorphic if and only if the following conditions are satisfied :*
1) for every $a' \in A'$, $T_*(R'(a'))$ is finitely generated and projective in R-mod;
2) for every $a \in A$, there exists an isomorphism in (G', A', R')-gr

$$\theta_a : R'(\varphi(a)) \xrightarrow{\sim} \tilde{T}(R(a))$$

such that, given $a_1, a_2, \alpha \in A$, $r \in R(a_2)_{a_1}$, $s \in R'(\varphi(a_1))_{\varphi(\alpha)}$ it is

$$\Big((\theta_{a_1}(1))(s)\Big) \cdot r = (\theta_{a_2}(1))(s \cdot \psi(r)) \qquad (*)$$

Proof. First of all note that, as $\tilde{T} = \bigoplus_{a' \in A'} \mathrm{Hom}_{(G,A,R)}\text{-gr}(T_*(R'(a')), -)$, $\tilde{T}$ is right exact and commute with direct limits if and only if, for every $a' \in A'$, the functor $\mathrm{Hom}_{(G,A,R)}\text{-gr}(T_*(R'(a')), -)$ is right exact and commute with direct limits. Let $a' \in A'$. Then the functor $\mathrm{Hom}_{(G,A,R)}\text{-gr}(T_*(R'(a')), -)$ is right exact if and only if $T_*(R'(a'))$ is projective in (G, A, R)-gr, if and only if - by Corollary 2.9 in [NRV] - it is projective in R-mod. On the other hand, by slightly changing the usual proof in R-mod, it is easy to show that the functor $\mathrm{Hom}_{(G,A,R)}\text{-gr}(T_*(R'(a')), -)$ commute with direct limits iff $T_*(R'(a'))$ is finitely generated.

Now assume that $\tilde{T}$ is right exact and commute with direct limits. Then, by Proposition 1.15, there is a functorial isomorphism $\theta : T^* \to \tilde{T}$ if and only if for every $a \in A$, there is an isomorphism

$$\theta_{R(a)} : T^*(R(a)) \to \tilde{T}(R(a))$$

such that, for every morphism $\mu : R(a_1) \to R(a_2)$ in (G, A, R)-gr, $a_1, a_2 \in A$, it is

$$\theta_{R_{a_2}} \circ T^*(\mu) = \tilde{T}(\mu) \circ \theta_{R_{a_1}} \ . \qquad (**)$$

Given a family of isomorphisms satisfying (**), set

$$\theta_a = \theta_{R_a} \circ \omega_a^{-1} \qquad\qquad a \in A$$

where ω_a is as in 2.4. Then $\theta_a : R'(\varphi(a)) \to \tilde{T}(R(a))$ is an isomorphism.
Let $a_1, a_2 \in A,\ r \in R(a_2)_{a_1}$, and let $\mu = \mu_r : R(a_1) \to R(a_2)$ be the right multiplication by r. Then, from (**), we get

$$\theta_{R_{a_2}} \circ T^*(\mu) \circ \omega_{a_1}^{-1} = \tilde{T} \circ \theta_{R_{a_1}} \circ \omega_{a_1}^{-1}$$

and hence

$$\theta_{a_2} \circ \omega_{a_2} \circ T^*(\mu) \circ \omega_{a_1}^{-1} = \tilde{T} \circ \theta_{a_1} \ .$$

It follows that

$$\mu \circ (\theta_{a_1}(1)) = (\tilde{T}(\mu) \circ \theta_{a_1})(1) = (\theta_{a_2} \circ \omega_{a_2} \circ T^*(\mu) \circ \omega_{a_1}^{-1})(1) =$$

$$= (\theta_{a_2} \circ \omega_{a_2} \circ T^*(\mu))(1 \otimes 1) = (\theta_{a_2} \circ \omega_{a_2})(1 \otimes r) = \theta_{a_2}(\psi(r)) \ .$$

Hence $\mu_r \circ (\theta_{a_1}(1)) = \theta_{a_2}(\psi(r))$.
Given $\alpha \in A,\ s \in R'(\varphi(a_1))_{\varphi(\alpha)}$ we get :

$$\Big((\theta_{a_1}(1))(s)\Big) \cdot r = (\mu_r \circ \theta_{a_1}(1))(s) = (\theta_{a_2}(\psi(r)))(s) =$$

$$= (\psi(r)\theta_{a_2}(1))(s) = \theta_{a_2}(1)(s\psi(r))$$

and hence (*) is satisfied.
Conversely let $\theta_a : R'(\varphi(a)) \xrightarrow{\sim} \tilde{T}(R(a)),\ a \in A$, be a family of isomorphisms satisfying (*). For every $a \in A$, set $\theta_{R(a)} = \theta_a \circ \omega_a$. Then $\theta_{R(a)} : T^*(R(a)) \to \tilde{T}(R(a))$ is an isomorphism.
Let $\mu : R(a_1) \to R(a_2)$ be a morphism in (G, A, R)-gr. Set $r = \mu(1)$. Then $r \in R(a_2)_{a_1}$ and $\mu = \mu_r$, the right multiplication by r. Given $\alpha \in A,\ s \in R'(\varphi(a_1))_{\varphi(\alpha)}$, we have

$$[(\theta_{R(a_2)} \circ T^*(\mu_r))(1 \otimes 1)](s) = [(\theta_{a_2} \circ \omega_{a_2})(1 \otimes r)](s) =$$

$$= (\theta_{a_2}(\psi(r)))(s) = (\theta_{a_2}(1))(s \cdot \psi(r)) = \Big((\theta_{a_1}(1))(s)\Big) \cdot r =$$

$$= \Big((\tilde{T}(\mu_r) \circ \theta_{a_1})(1)\Big)(s) = [(\tilde{T}(\mu_r) \circ \theta_{R(a_1)})(1 \otimes 1)](s) \ .$$

Therefore

$$(\theta_{R(a_2)} \circ T^*(\mu_r))(1 \otimes 1) = (\tilde{T}(\mu_r) \circ \theta_{R(a_1)})(1 \otimes 1) \ .$$

As $T^*(R(a))$ is a cyclic module spanned by $1 \otimes 1$, we get

$$\theta_{R(a_2)} \circ T^*(\mu_r) = \tilde{T}(\mu_r) \circ \theta_{R(a_1)}$$

and hence (**) is satisfied.

2.28. Assume that $A = \{a\}$ is a singleton with G acting trivially on it. Then, for each $a' \in A$,

$$\mathrm{Hom}_{(G,A,R)\text{-gr}}(T_*(R'(a')), R(a))$$

is a right R-module with respect to

$$(\xi r)(x) = \xi(x) \cdot r \ ,$$

$\xi : T_*(R'(a')) \to R(a)$ morphism in (G, A, R)-gr, $r \in R,\ x \in T_*(R'(a'))$. It follows that $\tilde{T}(R(a))$ has a natural structure of R-module.
On the other hand $T_*(R'(a')) = R'(a') \otimes_R R$ is also a right R-module.

Corollary 2.29 *Assume that $A = \{a\}$ is a singleton. Then the functors T^* and $\tilde{T}$ are isomorphic if and only if the following conditions are satisfied :*

1) for every $a' \in A'$, $R'(a')_{\varphi(a)}$ is finitely generated and projective in R-mod;

2) there exists an isomorphism in (G', A', R')-gr

$$\theta : R'(\varphi(a)) \overset{\sim}{\to} \tilde{T}(R(a))$$

that is also an isomorphism of right R-modules.

2.30. Assume that $G = G',\ f = \mathbf{1}_G,\ R = R'$ and $\psi = \mathbf{1}_R$. Then $T = (\mathbf{1}_G, \varphi, \mathbf{1}_R)$ and, in view of Proposition 2.16 and Theorem 2.18, σ is a functorial isomorphism iff $\varphi : A \to A'$ is injective. The following proposition shows that, even if φ is not injective, the functors T^* and $\tilde{T}$ can be isomorphic.

Proposition 2.31 *Let $T = (\mathbf{1}_G, \varphi, \mathbf{1}_R)$ be as in 2.29. Then we have :*

1) for every $a' \in A'$ the morphism

$$\nabla^{a'} : \bigoplus\{R(b) \mid b \in \varphi^{-1}(a')\} \to T_*(R(a'))$$

defined in 2.12, is an isomorphism in (G, A, R)-gr;

2) the functors T^* and $\tilde{T}$ are isomorphic iff, for every $a' \in A'$, the set $\varphi^{-1}(a')$ is finite.

Proof. 1) follows directly by Proposition 2.13.
In the sequel of the proof, for each $a' \in A'$, we identify, through the isomorphism $\nabla^{a'}$, the direct sum $\bigoplus\{R(b) \mid b \in \varphi^{-1}(a')\}$ with $T_*(R(a'))$.
2) From 1) we get that, given $a' \in A',\ T_*(R'(a'))$ is always projective in R-mod, while it is finitely generated iff the set $\varphi^{-1}(a')$ is finite. Assume that this holds for every $a' \in A'$. Given $a \in A$, we define

$$\theta_a : R(\varphi(a)) \to \tilde{T}(R(a)) = \bigoplus_{a' \in A'} \mathrm{Hom}_{(G,A,R)\text{-gr}}(\bigoplus\{R(b) \mid b \in \varphi^{-1}(a')\}, R(a))$$

by setting, for every $a' \in A',\ g \in G$ such that $g\varphi(a) = a'$ and $r_g \in R_g \subseteq R(\varphi(a))_{a'}$

$$\theta_a(r_g) : \bigoplus\{R(b) \mid b \in \varphi^{-1}(a')\} \to R(a)$$

to be the morphism which is the right multiplication by r_g on $R(ga)$ and 0 elsewhere. Note that, since $\varphi(ga) = g\varphi(a) = a',\ ga \in \varphi^{-1}(a')$. Clearly $\theta_a(r_g)$ is a morphism in (G, A, R)-gr.
A routine checking shows that θ_a is an isomorphism in (G, A', R)-gr and that condition (*) of theorem 2.27 is fulfilled so that $T^* \simeq \tilde{T}$.

Corollary 2.32 *Let $U : (G, A, R)$-gr $\to$ R-mod be the forgetful functor. The right adjoint functor F of U, F : R-mod $\to (G, A, R)$-gr is also a left adjoint functor of U iff A is finite.*

Proof. It follows by Proposition 2.31 after the Remarks in 2.2.2.

Remark 2.33 *1) Part of Proposition 2.31 and Corollary 2.32 can be found in [NRV].*
2) The proof of 2) in Proposition 2.31 can be also done directly using the Remarks in 2.2.2.

2.34. For every $M \in (G, A, R)$-gr, let

$$\mathrm{Supp}(M) = \{a \in A \mid M_a \neq 0\} \ .$$

Supp(M) will be called the *support* of M.

Proposition 2.35 *Let $T = (f, \varphi, \psi)$ be as in 2.1. For every $L \in (G, A, R)$-gr, Supp$(T^*(L))$ and Supp$(\tilde{T}(L))$ are contained in G'Im(φ) .*

Proof. Let $L \in (G, A, R)$-gr. Recall that, given $a' \in A'$, $(T^*(L))_{a'}$ = subgroup of $R' \otimes_R L$ spanned by the elements of the form $r'_\lambda \otimes \ell_a$, where $\lambda \in G',\ a \in A,\ \lambda\varphi(a) = a',\ r'_\lambda \in R'_\lambda,\ \ell_a \in L_a$. It follows that $a' \in G'\mathrm{Im}(\varphi)$ whenever $(T^*(L))_{a'} \neq 0$.
Assume now that $0 \neq (\tilde{T}(L))_{a'} = \mathrm{Hom}_{(G,A,R)\text{-gr}}(T_*(R'(a')), L)$. Then $T_*(R'(a')) \neq 0$ so that there is an $\alpha \in A$ such that

$$0 \neq (T_*(R'(a'))_\alpha = R'(a')_{\varphi(\alpha)} = \bigoplus_{g' \in C^{a'}_{\varphi(\alpha)}} R'_{g'} \ .$$

Hence $C^{a'}_{\varphi(\alpha)} \neq \emptyset$ and therefore $a' \in G'\mathrm{Im}(\varphi)$.

Corollary 2.36 *Assume that $\sigma : \mathbf{1}_{(G,A,R)\text{-}gr} \to T_* \circ T^*$ is a functorial isomorphism. Then, for every $L \in (G, A, R)$-gr, Supp(Ker(η_L)) and Supp(Coker(η_L)) are contained in G'Im(φ).*

Proof. Given $L \in (G, A, R)$-gr, we have $\eta_L : T^*(L) \to \tilde{T}(L)$.
Hence $\mathrm{Supp}(\mathrm{Ker}(\eta_L)) \subseteq \mathrm{Supp}(T^*(L))$ and $\mathrm{Supp}(\mathrm{Coker}(\eta_L)) \subseteq \mathrm{Supp}(\tilde{T}(L))$.

Proposition 2.37 *Let $T = (f, \varphi, \psi)$ be as in 2.1. Assume that R' is a strongly graded ring and let $M \in (G', A', R')$-gr be such that Supp$(M) \subseteq G'$Im(φ) . Then, if $M \neq 0$, $T_*(M) \neq 0$.*

Proof. Assume that $M \neq 0$. Let $a' \in \mathrm{Supp}(M)$, $0 \neq m_{a'} \in M_{a'}$. Then we have $a' = g'\varphi(a)$ for suitable $g' \in G'$, $a \in A$. As R' is strongly graded there are $n \in \mathbf{N}$, $\mathbf{r_1}, ..., \mathbf{r_n} \in \mathbf{R_{g'}}$, $\mathbf{s_1}, ..., \mathbf{s_n} \in \mathbf{R_{(g')^{-1}}}$ such that $1 = \sum_{i=1}^{n} r_i s_i$. Then

$$0 \neq m_{a'} = 1 \cdot m_{a'} = \sum_{i=1}^{n} r_i s_i m_{a'}$$

and hence we get $0 \neq s_i m_{a'}$ for some i, $1 \leq i \leq n$. Then

$$0 \neq s_i m_{a'} \in M_{(g')^{-1}a'} = M_{\varphi(a)} = (T_*(M))_a$$

and hence $T_*(M) \neq 0$.

Theorem 2.38 *Let $T = (f, \varphi, \psi)$ be as in 2.1. Assume that $\sigma : \mathbf{1}_{(G,A,R)}\text{-}gr \to T_* \circ T^*$ is a functorial isomorphism and that R' is a strongly graded ring. Then $\eta : T^* \to \tilde{T}$ is a functorial isomorphism.*

Proof. By Theorem 2.6 and Proposition 1.8, for every $L \in (G, A, R)$-gr, $\mathrm{Ker}(\eta_L)$ and $\mathrm{Coker}(\eta_L)$ belong to $\mathcal{C} = \{M \in (G', A', R')\text{-gr} \mid T_*(M) = 0\}$.
By Corollary 2.36, $\mathrm{Supp}(\mathrm{Ker}(\eta_L))$ and $\mathrm{Supp}(\mathrm{Coker}(\eta_L))$ are contained in $G'\mathrm{Im}(\varphi)$, so that, by Proposition 2.37 we get $\mathrm{Ker}(\eta_L) = 0$ and $\mathrm{Coker}(\eta_L) = 0$.

Corollary 2.39 *([NRV] Proposition 3.10)*
Assume that
1) $\varphi : A \to A'$ is injective;
2) $\psi : R \to R'^{(Im(f))}$ is a ring isomorphism;
3) for every $a \in A$, if $g' \in Supp(R')$ and $g'\varphi(a) \in Im(\varphi)$, then $g' \in Im(f)$.
Then, if R' is a strongly graded ring, $\eta : T^* \to \tilde{T}$ is a functorial isomorphism.

Proof. Follows by Theorems 2.18 and 2.38.

2.40 Example Let $T = (f, \varphi, \psi)$ be as in Example 2.11. Then, as we remarked in 2.11, σ is a functorial isomorphism. Since $R' = K[G']$ is a strongly graded ring, by Theorem 2.38 $\eta : T^* \to \tilde{T}$ is a functorial isomorphism.

3 Two particular cases

3.1. Let R be a G-graded ring. Set $\varphi = f =$ canonical injection : $\{1\} \hookrightarrow G$, $\psi =$ canonical injection $R_1 \hookrightarrow R$. Let $T = (f, \varphi, \psi)$. Then, in this case, $T_* = (-)_1$: R-gr $\to R_1$-mod, $M \mapsto M_1$, while $T^* = \mathrm{Ind} : R_1\text{-mod} \to R$-gr, the (left) *induced functor*, and $\tilde{T} = \mathrm{Coind} : R_1\text{-mod} \to R$-gr, the (left) *coinduced functor*.
Recall that, given $N \in R_1$-mod, $\mathrm{Ind}(N)$ is the graded left R-module $M = R \otimes_{R_1} N$, where M has the grading $M_g = R_g \otimes_{R_1} N$, $g \in G$, and $\mathrm{Coind}(N) = \{f \in \mathrm{Hom}_{R_1}(R, N) \mid f(R_g) = 0$ for almost every $g \in G\}$ with the grading :

$$(\mathrm{Coind}(N))_g = \{f \in \mathrm{Hom}_{R_1}(R, N) \mid f(R_h) = 0 \ \ \forall h \neq g^{-1}\} .$$

The right induced functor and the right coinduced functor from mod-R_1 into gr-R, are defined in an analogous way.
From the foregoimg results we know that Ind is a left adjoint of $(-)_1$ and that Coind is a right adjoint of $(-)_1$. Hence $(-)_1$ is an exact functor, Ind is right exact and Coind is left exact. These facts were firstly proved in [N$_1$].
The adjunction and coadjunction morphisms have, in this case, the following form (we use the notations of 2.1). Given $N \in R_1$-mod, we have : $\sigma_N : N \to (\mathrm{Ind}(N))_1 = (R \otimes_{R_1} N)_1$, $x \mapsto 1 \otimes x$, for every $x \in N$, $\tau_N : (\mathrm{Coind}(N_1))_1 \simeq \mathrm{Hom}_{R_1}(R_1, N) \to N$, $\xi \mapsto \xi(1)$.
Given $M \in R$-gr, we have $\zeta_M : M \to \mathrm{Coind}(M_1)$, $\zeta_M(x_g) = (\mu_{x_g})_1 : (R(g))_1 \to M_1$, where $g \in G$, $x_g \in M_g$ and $\mu_{x_g} : R(g) \to M$ is the right multiplication by x_g on M. Therefore $(\zeta_M(x_g))(a) = a_{g^{-1}}x_g$, for every $a = \sum_{g\in G} a_g \in R$. It follows that, given $x \in M$, $x = \sum_{g\in G} x_g$, we have

$$(\zeta_M(x))(a) = \sum_{g\in G} a_{g^{-1}}x_g \qquad \text{for every } a = \sum_{g\in G} a_g \in R .$$

Moreover $\rho_M : \mathrm{Ind}(M_1) = R \otimes_{R_1} M_1 \to M$ is defined by setting

$$\rho_M(r \otimes x_1) = rx_1 \qquad \text{for every } r \in R,\ x_1 \in M_1 .$$

By Theorem 2.18, σ and τ are, in this case, functorial isomorphisms. Hence, from 2.20, we learn that $\eta = \lambda : T^* = \mathrm{Ind} \to \tilde{T} = \mathrm{Coind}$ has the following form. For every $N \in R_1$-mod,

$$\eta_N : \mathrm{Ind}(N) \to \mathrm{Coind}(N)$$

is defined by

$$(\eta_N(r \otimes x))(s) = \sum_{g\in G} (s_{g^{-1}}r_g)x$$

for every $r \in R$, $s \in R$, $x \in N$.
Let

$$\mathcal{C} = \{M \in R\text{-gr} \mid M_1 = 0\}$$

and let t be the radical associated to $\mathcal{C}$. Then, by Proposition 1.8, we have that $\mathrm{Ker}(\eta_N)$ and $\mathrm{Coker}(\eta_N)$ belong to $\mathcal{C}$. Moreover $\mathrm{Ker}(\eta_N) = t(\mathrm{Ind}(N))$ and $\mathrm{Im}(\eta_N)$ is essential in $\mathrm{Coind}(N)$. Still, by Proposition 1.6, we have that, for every $M \in \mathcal{B}$, $\mathrm{Ker}(\rho_M)$, $\mathrm{Coker}(\rho_M)$, $\mathrm{Ker}(\zeta_M)$, $\mathrm{Coker}(\zeta_M)$ belong to $\mathcal{C}$, $\mathrm{Ker}(\zeta_M) = t(M)$ and $\mathrm{Im}(\rho_M)$ is the smallest subobject L of M sucht that M/L belongs to $\mathcal{C}$.

From Theorem 2.21 and Theorem 2.24 we deduce the following form of a classical result due to Dade (see [D] Theorem 2.8).

Theorem 3.2 *Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring. Then the following assertions are equivalent :*
(a) R is strongly graded;
(b) $(-)_1$ is a category equivalence;

(c) Ind is a category equivalence;
(d) Coind is category equivalence;
(e) $\zeta : \mathbf{1}_{R\text{-}gr} \to Coind \circ (-)_1$ *is a functorial isomorphism;*
(f) $\rho : Ind \circ (-)_1 \to \mathbf{1}_{R\text{-}gr}$ *is a functorial isomorphism;*
(g) for every $g \in G$, $\rho_{R(g)} : Ind(R_{g^{-1}}) \to R(g)$ *is surjective.*
Moreover, if one of these conditions is satisfied, $\eta : Ind \to Coind$ *is a functorial isomorphism.*

Proof. By Theorems 2.21 and 2.24, it remains to prove that (g)⇒(a). Given $g \in G$, there exists an element $a \in (\mathrm{Ind}(R_{g^{-1}}))_g$ such that $\rho_{R(g)}(a) = 1$. Write $a = \sum_{i=1}^{n} r_i \otimes s_i$ where $n \in \mathbf{N}$, $\mathbf{r_i} \in \mathbf{R_g}$, $\mathbf{s_i} \in \mathbf{R_{g^{-1}}}$. Then we get $\sum_{i=1}^{n} r_i s_i = \rho_{R(g)}(a) = 1$.

Remark 3.3 *Let* G *be a non trivial group i.e.* $G \neq \{1\}$ *and let* R *be an arbitrary ring. Then* R *can be considered as a* G*-graded ring with the trivial grading :* $R_1 = R$ *and* $R_g = 0$ *for every* $g \neq 1$ *. Obviously, in this case we have Ind* $\simeq$ *Coind but* R *is not strongly graded.*
Thus, in this case, we may ask the following question :
"If R *is a graded ring and the functors Ind and Coind are isomorphic, how much does* R *approach a strongly graded ring ?"*

From the foregoing, we deduce the following :

Theorem 3.4 *Let* R *be a* G*-graded ring. The following assertions are equivalent :*

(a) the functors Ind and Coind are isomorphic;

(b) $\eta : Ind \to Coind$ *is a functorial isomorphism;*

(c) $\hat{\eta} = \eta_{R_1} \circ \omega^{-1} : R \to Coind(R_1)$, $(\hat{\eta}(r))(s) = \sum_{g \in G} s_{g^{-1}} r_g$, $r, s \in R$ *is an isomorphism and for every* $g \in G$, R_g *is projective and finitely generated in* R_1*-mod;*

(d) there exists an isomorphism $\theta : R \to Coind(R_1)$ *in* R*-gr that is also a morphism in mod-*R_1 *and for every* $g \in G$, R_g *is finitely generated and projective in* R_1*-mod.*

Proof. As σ and τ are functorial isomorphisms (see 3.1), (a) ⇒ (b) follows by Theorem 1.3.
(b) ⇒ (c) is trivial.
(c) ⇒ (d) It is easy to check that $\hat{\eta}$ is also a morphism in mod-R_1.
(d) ⇒ (a) follows by Corollary 2.29.

Next theorem outlines a nice symmetry we have in this case.

Theorem 3.5 *Let* R *be a* G*-graded ring. Then the left functors Ind and Coind are isomorphic if and only if the right functors Ind and Coind are isomorphic.*

Proof. In the following we will denote the "right version" of whatever we introduced before by using the same letter and $'$. Assume that the left functors Ind and Coind are isomorphic.
Let $g \in G$. As $R_{g^{-1}}$ is finitely generated and projective in R_1-mod, the evaluation morphism, $\nu : R_{g^{-1}} \to \mathrm{Hom}_{\text{mod-}R_1}(\mathrm{Hom}_{R_1\text{-mod}}(R_{g^{-1}}, R_1), R_1)$, is an isomorphism in R_1-mod (see [AF] Prop. 20.17).
On the other hand $\hat{\eta} : R \to \mathrm{Coind}(R_1)$ is an isomorphism in R-gr and, moreover, it is a morphism in mod-R_1. Therefore

$$\mathrm{Hom}(\hat{\eta}_g, R_1) \circ \nu : R_{g^{-1}} \to \mathrm{Hom}_{\text{mod-}R_1}(R_g, R_1)$$

is an isomorphism. Given $s_{g^{-1}} \in R_{g^{-1}}$ and $r_g \in R_g$ we have :

$$[(\mathrm{Hom}(\hat{\eta}_g, R_1) \circ \nu)(s_{g^{-1}})](r_g) = (\nu(s_{g^{-1}}) \circ \hat{\eta}_g)(r_g) =$$

$$= \nu(s_{g^{-1}})(\hat{\eta}(r_g)) = \hat{\eta}(r_g)(s_{g^{-1}}) = s_{g^{-1}} r_g = [\hat{\eta}'_{g^{-1}}(s_{g^{-1}})](r_g) \ .$$

Therefore $\hat{\eta}'_{g^{-1}}$ is an isomorphism. It follows that $\hat{\eta}'$ is an isomorphism.
As $\hat{\eta}_g : R_g \to \mathrm{Hom}_{R_1\text{-mod}}(R_{g^{-1}}, R_1)$ is an isomorphism in mod-R_1, and as $R_{g^{-1}}$ is finitely generated and projective in R_1-mod, we get that R_g is finitely generated and projective in mod-R_1 (see [AF] Prop. 20.17). By Theorem 3.4$'$, the right functors Ind$'$ and Coind$'$ are isomorphic.

Theorem 3.6 *Let R be a G-graded ring. Assume that Ind $\simeq$ Coind and let $g \in$ Supp(R). Then there exist elements $a_i \in R_g$, $b_i \in R_{g^{-1}}$, $1 \leq i \leq n$, such that for every $a \in R_g$, $b \in R_{g^{-1}}$ we have:*

$$a = \Big(\sum_{i=1}^{n} a_i b_i\Big) a \qquad , \qquad b = b\Big(\sum_{i=1}^{n} a_i b_i\Big)$$

Proof. By Theorem 3.5, $\hat{\eta}_g : R_g \to \mathrm{Hom}_{R_1\text{-mod}}(R_{g^{-1}}, R_1)$, $(\hat{\eta}_g(r_g))(s_{g^{-1}}) = s_{g^{-1}} r_g$, $r_g \in R_g$, $s_{g^{-1}} \in R_{g^{-1}}$, is an isomorphism. By the same theorem, $R_{g^{-1}}$ is finitely generated and projective in R_1-mod. Thus, by the Dual Basis Lemma, there exist $b_1, b_2, ..., b_n \in R_{g^{-1}}$ and $f_1, ..., f_n \in \mathrm{Hom}_{R_1}(R_{g^{-1}}, R_1)$ such that for each $b \in R_{g^{-1}}$ we have

$$b = \sum_{i=1}^{n} f_i(b) b_i \ .$$

For every $i = 1, ..., n$, there is an $a_i \in R_g$ such that $f_i = \hat{\eta}_g(a_i)$. Hence

$$b = \sum_{i=1}^{n} (\hat{\eta}_g(a_i))(b) b_i = \sum_{i=1}^{n} b a_i b_i = b\Big(\sum_{i=1}^{n} a_i b_i\Big) \ .$$

Let $c = \sum_{i=1}^{n} a_i b_i$. Then $b = bc$ for every $b \in R_{g^{-1}}$ and thus $R_{g^{-1}}(1 - c) = 0$. It follows that $R_{g^{-1}}(1-c)R_g = 0$ so that $\hat{\eta}_g((1-c)R_g) = 0$. As $\hat{\eta}_g$ is injective, we get $(1-c)R_g = 0$ and hence $a = \Big(\sum_{i=1}^{n} a_i b_i\Big) a$ for every $a \in R_g$.

Lemma 3.7 *Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring. If $R_g R_{g^{-1}} = R_1$ for every $g \in Supp(R)$ then $H = Supp(R)$ is a subgroup of G and $R = \bigoplus_{h\in H} R_h$ is an H-strongly graded ring.*

Proof. Let $g, h \in \mathrm{Supp}(R)$ and assume that $gh \notin \mathrm{Supp}(R)$. Then $0 = R_{gh} R_{h^{-1}} \supseteq R_g R_h R_{h^{-1}} = R_g$. Contradiction.

Proposition 3.8 *Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring. Assume that Ind $\simeq$ Coind. If every R_g, $g \in Supp(R)$ is faithful as a left R_1-module, then $H = Supp(R)$ is a subgroup of G and $R = \bigoplus_{h\in H} R_h$ is an H-strongly graded ring.*

Proof. Let $g \in \mathrm{Supp}(R)$. By Theorem 3.6 there exist elements $a_i \in R_g$, $b_i \in R_{g^{-1}}$ $(1 \le i \le n)$ such that we have $a = \Big(\sum_{i=1}^{n} a_i b_i\Big) a$ for every $a \in R_g$. Set $c = 1 - \sum_{i=1}^{n} a_i b_i$. Then $cR_g = 0$ and hence, in view of our assumption, $c = 0$ i.e. $1 = \sum_{i=1}^{n} a_i b_i$. Therefore $R_g R_{g^{-1}} = R_1$ for every $g \in \mathrm{Supp}(R)$. Apply now Lemma 3.7.

Theorem 3.9 *Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring. Assume that Ind $\simeq$ Coind. If every finitely generated and projective module in R_1-mod is faithful, then $H = Supp(R)$ is a subgroup of G and $R = \bigoplus_{h\in H} R_h$ is an H-strongly graded ring.*

Proof. Let $g \in \mathrm{Supp}(R)$. Then, by Theorem 3.4 R_g is finitely generated and projective in R_1-mod. It follows, by our assumption, that R_g is faithful. The conclusion now follows by Proposition 3.8.

3.10. If A is a ring, we denote by Ω_A the set of all isomorphism classes of simple objects in A-mod, i.e.

$$\Omega_A = \{[S] \mid S \text{ is a simple left } A\text{-module}\}$$

and $[S] = \{S' \in A\text{-mod} \mid S' \simeq S\}$.
The ring A is called *local* if $A/J(A)$ is a simple artinian ring ($J(A)$ is the Jacobson radical).
Clearly if A is local, then $|\Omega_A| = 1$ (in general the converse is not true).

Now we can give one of the main results of this section.

Theorem 3.11 *Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring and assume that Ind $\simeq$ Coind. If- $|\Omega_{R_1}| = 1$ (in particular if R_1 is a local ring) then $H = Supp(R)$ is a subgroup of G and $R = \bigoplus_{h\in H} R_h$ is an H-strongly graded ring.*

Proof. Since $|\Omega_{R_1}| = 1$ every finitely generated and projective module in R_1-mod is a generator (see [AF] Theorem 10.4 and Proposition 17.9) and hence it is faithful. Apply now Theorem 3.9.

Theorem 3.12 *Let $R = \bigoplus_{g \in G} R_g$ be a G-graded ring with the property that R_1 has only two idempotents 0 and 1 (in particular when R_1 is a domain). Assume that Ind $\simeq$ Coind.*
Then $H = Supp(R)$ is a subgroup of G and $R = \bigoplus_{h \in H} R_h$ is an H-strongly graded ring.

Proof. By Theorem 3.6 if $g \in \mathrm{Supp}(R)$, there exist elements $a_i \in R_g,\ b_i \in R_{g^{-1}},\ 1 \leq i \leq n$ such that for every $a \in R_g$ we have

$$a = \Big(\sum_{i=1}^{n} a_i b_i\Big) a \ .$$

In particular we have that, for every $1 \leq k \leq n$, $a_k = \Big(\sum_{i=1}^{n} a_i b_i\Big) a_k$ so that

$$a_k b_k = \Big(\sum_{i=1}^{n} a_i b_i\Big)(a_k b_k) \ .$$

Therefore $e = \sum_{i=1}^{n} a_i b_i$ is an idempotent of R_1 . Since $g \in \mathrm{Supp}(R),\ R_g \neq 0$ and hence $e \neq 0$. It follows that $e = 1$ and so $R_g R_{g^{-1}} = R_1$. Apply now Lemma 3.7.

3.13 Example Let A be a ring and let ${}_A M_A$ be an A-A-bimodule. Assume that $\varphi = [-,-] : M \otimes_A M \to A$ is an A-A-morphism satisfying $[m_1, m_2]m_3 = m_1[m_2, m_3]$ for all $m_1, m_2, m_3 \in M$. We define a multiplication on the abelian group $A \times M$ by setting

$$(a, m)(a', m') = (aa' + [m, m'], am' + ma') \ .$$

In this way $A \times M$ becomes a ring which is called the semi-trivial extension of A by M and φ and will be denoted by $A \times_\varphi M$. The ring $R = A \times_\varphi M$ can be considered as a graded ring of type $\mathbf{Z_2} = \mathbf{Z}/2\mathbf{Z}$ by putting $R_0 = A \times \{0\},\ R_1 = \{0\} \times M$. We have :
$\hat{\eta}_0 : R_0 \to \mathrm{Hom}_{R_0}(R_0, R_0)$ is an isomorphism and $\hat{\eta}_1 : R_1 = M \to \mathrm{Hom}_A(M, A)$, where $\hat{\eta}_1(m)(m') = [m', m],\ m', m \in M$. The map $[-,-] : M \otimes_A M \to A$ is called *left non degenerate* if $m = 0$ if and only if $[m', m] = 0$ for every $m' \in M$. The map $[-,-] : M \otimes_A M \to A$ is called *left onto* if for any $f \in \mathrm{Hom}_A({}_A M, A)$ there exists an $m \in M$ such that $f = [-, m]$ (i.e. $f(m') = [m', m]$ for every $m' \in M$). Therefore we get :

Proposition 3.13.1. *Within the above notations the functors Ind and Coind are isomorphic if and only if the map $[-,-]$ is left non degenerate and left onto and ${}_A M$ is finitely generated and projective.*

3.13.2. A particular case. Let K be a field, $A = K \times K,\ e = (1, 0),\ M = Ae = K \times \{0\}$. We define $\varphi = [-,-] : M \otimes M \to A$ by setting $[m, m'] = mm'$, where $m, m' \in Ae = M$. If $[m, m'] = 0$ for every $m \in M$ then we have $mm' = 0$ for every $m \in Ae$. Thus, for $m = e$ we get $em' = m' = 0$. Therefore the map $[-,-]$ is non degenerate. Let now $f \in \mathrm{Hom}_A(M, A)$. If we put $f(e) = m$, then we have $f(e) = f(e \cdot e) = ef(e) = em = me$ and therefore $m = me \in Ae = M$. On the other hand $f(\lambda e) = \lambda f(e) = \lambda m = (\lambda e)m$ for every $\lambda \in A$ and thus $f = [-, m]$. Hence for the semitrivial extension $A \times_\varphi M$ we have Ind $\simeq$ Coind. We observe that this ring is not strongly graded (as $R_1 R_1 = Ae \neq A$), also $\mathrm{Supp}(R) = \mathbf{Z_2} \neq \{\mathbf{0}\}$. Moreover the ring A is not local.

3.14. Let $\psi : R \to S$ be a morphism of rings. Set $G = G' = A = A'$, $R' = S$, $T = (\mathbf{1}_G, \mathbf{1}_A, \psi)$. Then, as we remarked in 2.2.3, (G, A, R)-gr=R-mod, (G', A', R')-gr=S-mod, T_* is the restriction of scalar functors $\psi_* : S$-mod $\to R$-mod, T^* is the (left) Induction functor $S \otimes_R - : R$-mod $\to S$-mod, $\tilde{T}$ is the (left) Coinduction functor $\mathrm{Hom}_R({}_R S_S, -) :$
R-mod $\to S$-mod.

Part of the following theorem can be found in [NT].

Theorem 3.15 *Let $\psi : R \to S$ be a ring homomorphism. The following assertions are equivalent :*

(a) The functors $S \otimes_R -$ and $Hom_R({}_R S_S, -)$ are isomorphic;

(b) 1) ${}_R S$ is finitely generated and projective in R-mod;

2) there exists an isomorphism of S-R-bimodules

$$\theta : {}_S S_R \to Hom_R({}_R S_S, {}_R R_R)$$

(c) 1) ${}_R S$ is finitely generated and projective in R-mod;

2) there exists an R-R-morphism :

$$[-,-] : S \otimes_R S \to R$$

which is left non degenerate and left onto. Also $[-,-]$ is associative in the sense that

$$[ss', s''] = [s, s's''] \quad \text{for all } s, s', s'' \in S \ .$$

Proof. (a) $\iff$ (b) by Corollary 2.29.
(b) $\Rightarrow$ (c) Define $[-,-] : S \otimes_R S \to R$ by setting

$$[s, s'] = \theta(s')(s) \qquad \text{for every } s, s' \in S \ .$$

As θ is an S-R-bimodule morphism, it is easy to prove that $[-,-]$ is an R-R-morphism. As θ is bijective, $[-,-]$ is left non degenerate and left onto. Let us prove that $[-,-]$ is associative. Let $s, s', s'' \in S$. We have :

$$[s, s's''] = (\theta(s's''))(s) = (s'\theta(s''))(s) = \theta(s'')(ss') = [ss', s''] \ .$$

(c)$\Rightarrow$ (b) Define $\theta : S \to \mathrm{Hom}_R({}_R S_S, {}_R R_R)$ by setting $\theta(s')(s) = [s, s']$ for every $s, s' \in S$. It is straightforward to show that θ is an isomorphism of S-R-bimodules.

3.16. Following Kasch [K], (see also [NT]) we say that a ring morphism $\psi : R \to S$ is a *left Frobenius morphism* if it fulfills one of the equivalent conditions of Theorem **3.15**.

Let $\psi : R \to S$ be a *left Frobenius morphism* and $\theta : {}_S S_R \to \mathrm{Hom}_R({}_R S_S, {}_R R_R)$ be an isomorphism of S-R-bimodules. Denote by

$$\nu : {}_R S \to \mathrm{Hom}_R(\mathrm{Hom}_R({}_R S, {}_R R_R), {}_R R_R)$$

the evaluation morphism and let

$$\theta' = \mathrm{Hom}_R(\theta, R) \circ \nu : {}_R S_S \to \mathrm{Hom}_R({}_R S_R, {}_R R_R)\ .$$

Then it is easy to prove (see [NT] Proposition 1) that ν is an isomorphism so that θ' is an isomorphism of R-S-bimodules. Moreover S_R, being isomorphic to $\mathrm{Hom}_R({}_R S, {}_R R_R)$, is projective and finitely generated. Thus $\psi : R \to S$ is also a right Frobenius morphism.

By symmetry, the converse also holds so that one can simply consider Frobenius morphisms without any regard for the side. Moreover it is important to note (see [NT] §2) that if $[[-,-]] : S \otimes_R S \to R$ is the R-R-morphism associated to θ', then for every $a, b \in S$

$$[[a,b]] = \theta'(a)(b) = [\mathrm{Hom}_R(\theta, R)(\nu(a))](b) = \nu(a)(\theta(b)) = \theta(b)(a) = [a,b]\ .$$

Hence $[[-,-]] = [-,-]$.

From these considerations and by Theorem 3.14 we get :

Corollary 3.17 *Let $\psi : R \to S$ be a ring morphism. Then the "left" functors Induction and Coinduction are isomorphic if and only if the "right" functors Induction and Coinduction are isomorphic. Moreover, in this case, every associative R-R-morphism*

$$[-,-] : S \otimes_R S \to R$$

which is left non degenerate and left onto is also right non degenerate and right onto.

3.18 Example

- 1. If $R = \bigoplus_{g\in G} R_g$ is a G-strongly graded ring and $H \leq G$ is a subgroup of finite index then the *canonical injection* $i : R^{(H)} \to R$ is a *Frobenius morphism.* Indeed let $T = (f, \varphi, \psi)$ where $f : H \to G$, $\varphi : \{H\} \hookrightarrow G/H$, $\psi = i : R^{(H)} \to R$ are the canonical injections. Then the categories $R^{(H)}$-mod and $(G/H, R)$-gr are equivalent and this equivalence is given by the functors T^* and T_*. Moreover $\lambda : T^* \to \tilde{T}$ is a functorial isomorphism(see Corollary 2.25).

- Let $F : (G/H, R)$-gr $\to$ R-mod be the forgetful functor. Then $F \circ T^* = R \otimes_{R^{(H)}} -$ and $F \circ \tilde{T} = \mathrm{Hom}_{R^{(H)}}(R, -)$ as H has finite index.

- 2. Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring and let A be a finite G-set. We can define the *smash product* $R\#A$ associated to R and to the G-set A.

- $R\#A$ is defined as follows. It is the free R-module with basis $\{p_x,\ x \in A\}$ where the multiplication is defined by

$$(a_g p_x)(b_h p_y) = \begin{cases} a_g b_h p_y & \text{if } hy = x \\ 0 & \text{if } hy \neq x \end{cases}$$

- for any $g, h \in G,\ a_g \in R_g,\ b_h \in R_h,\ x, y \in A$. This may be extended by $\mathbf{Z}$-bilinearity to a product on all of $R\#A = \bigoplus\{R_g p_x \mid g \in G, x \in A\}$. It turns out that $R\#A$ is a ring with identity $1 = \sum\limits_{x\in A} p_x$ and $\{p_x \mid x \in A\}$ is a set of orthogonal idempotents. The map $\eta : R \to R\#A,\ \eta(a) = a \cdot 1 = \sum\limits_{x\in A} a p_x$ is an injective ring morphism (for details see the Proposition 2.11 in [NRV]). This *morphism is a Frobenius morphism.* In fact let $(-)^\# : (G, A, R)$-gr $\to R\#A$-mod be the functor which assignes to each $M \in (G, A, R)$-gr the abelian group M endowed with the structure of left $R\#A$-module defined by setting

$$(a_g p_x)m = a_g m_x \qquad \text{for } g \in G,\ a_g \in R_g,\ x \in A,\ m = \sum_{x\in A} m_x \in M \ .$$

  Then $(-)^\#$ is a category equivalence (see Theorem 2.13 in [NRV]). Its inverse is the functor $(-)_{\mathrm{gr}} : R\#A$-mod $\to (G, A, R)$-gr which assignes to each $M \in R\#A$-mod the left R-module obtained from M by restriction of scalars via the morphism η and with A-gradation defined by setting $M_x = p_x M$ for every $x \in A$.

- Let $U : (G, A, R)$-gr $\to$ R-mod be the forgetful functor and let $F : R$-mod $\to (G, A, R)$- gr be its right adjoint (see 2.2.2). Since A is finite F is also a left adjoint of U (see Corollary 2.32).

- It follows that the functor

$$(-)^\# \circ F : R\text{-mod} \to R\#A\text{-mod}$$

is a right and left adjoint of the functor

$$U \circ (-)_{\text{gr}} : R\#A\text{-mod} \to R\text{-mod} .$$

Since the functor $U \circ (-)_{\text{gr}}$ is the restriction of scalar functor $\eta_* : R\#A$-mod $\to R$- mod, by the uniqueness of the left adjoint we get that $(-)^{\#} \circ F \simeq R\#A \otimes_R -$ while by the uniqueness of the right adjoint we get that $(-)^{\#} \circ F \simeq \text{Hom}_R(R\#A, -)$. Therefore the Induction and Coinduction functors are isomorphic.

- In particular if R is an arbitrary ring, $|G| = 1$ and A is a set with $|A| = n$, we can consider A as a G set. In this case $R\#A = R^n$ (the cartesian product) and $\eta : R \to R^n$ is the diagonal map $\eta(a) = (a, a, ..., a)$.

- 3. Let $\mathcal{K}$ be a field. Then a ring morphism $\psi : \mathcal{K} \to A$ is a Frobenius morphism iff A is a Frobenius $\mathcal{K}$-algebra as defined in the book by Curtis and Reiner [CR] page 413. Note that every semisimple algebra over a field is a Frobenius algebra.

Proposition 3.19 *If $\psi : R \to S$ and $\varphi : S \to T$ are two Frobenius morphisms, then $\varphi \circ \psi : R \to T$ is a Frobenius morphism.*

Proof. Let $M \in R$-mod. Since

$$T \otimes_S (S \otimes_R M) \simeq T \otimes_R M$$

the Induction functor associated to $\varphi \circ \psi$ is the composition of the induction functors associated to ψ and to φ. On the other hand since

$$\text{Hom}_S({}_ST_T, \text{Hom}_R({}_RS_S, M)) \simeq \text{Hom}_R({}_RS_S \otimes_S {}_ST_T, M) \simeq \text{Hom}_R({}_RT_T, M)$$

we get that the coinduction functor associated to $\varphi \circ \psi$ is the composition of the coinduced functors associated to ψ and to φ. From these facts, the conclusion follows.

3.20. Let now $R = \bigoplus_{g \in G} R_g$ be a G-graded ring. Assume that the group G is finite. Since G is finite, the graded functors induction and coinduction are the functors :

$$R \otimes_{R_1} - : R_1\text{-mod} \to R\text{-gr}$$

$$\text{Hom}_{R_1}({}_{R_1}R_R, -) : R_1\text{-mod} \to R\text{-gr} .$$

We can consider also the non graded functors Induction and Coinduction :

$$R \otimes_{R_1} - : R_1\text{-mod} \to R\text{-mod}$$

$$\text{Hom}_{R_1}({}_{R_1}R_R) : R_1\text{-mod} \to R\text{-mod} .$$

Clearly if the graded functors induction and coinduction are isomorphic, also the non graded functors Induction and Coinduction are isomorphic. Therefore, it is natural to wonder if the converse is true, namely to ask the following question :

"If the non graded functors Induction and Coinduction are isomorphic, is it true that graded functors induction and coinduction are isomorphic ?"

The following example shows that, in general, the answer is no.

3.21 Example Let A be an arbitrary ring and let $R = A[X]$ be the polynomial ring over A. This ring is a $\mathbf{Z}$-graded ring with the natural grading

$$R_n = \begin{cases} AX^n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

If $d > 0$ is a natural number, then R has a natural $\mathbf{Z_d} = \mathbf{Z}/\mathbf{dZ}$-grading. Indeed, if $\mathbf{Z_d} = \{\hat{\mathbf{0}}, \hat{\mathbf{1}}, ..., \widehat{\mathbf{d-1}}\}$, then for any $k \in \{0, 1, ..., d-1\}$ we have $R_{\hat{k}} = A[X^d]X^k$. We have $R_{\hat{0}} = A[X^d]$ and $R = \bigoplus_{\hat{k} \in \mathbf{Z_d}} R_{\hat{k}}$. Note that

$$R_{\hat{1}} R_{\widehat{d-1}} = A[X^d]X \; A[X^d]X^{d-1} = A[X^d]X^d \neq A[X^d]R_{\hat{0}}$$

and therefore R is not a strongly graded ring.
Consider the canonical morphism $\hat{\eta} : R \to \text{Coind}(R_{\hat{0}}) = \text{Hom}_{R_{\hat{0}}}(R, R_{\hat{0}})$. We have

$$(\hat{\eta}(r))(s) = \sum_{k=0}^{d-1} s_{\hat{k}} r_{\widehat{d-k}}.$$

It is easy to see that $\hat{\eta}$ is injective. Nevertheless, $\hat{\eta}$ is not surjective.
In fact, consider $f : R \to R_{\hat{0}}$ defined by setting

$$f(\alpha) = \alpha_{d-1}$$

for every $\alpha = \alpha_0 + \alpha_1 X + ... + \alpha_{d-1} X^{d-1} \in R, \;\; \alpha_i \in R_{\hat{0}}$.
Then $0 \neq f \in (\text{Hom}_{\hat{R}_0}(R, R_{\hat{0}}))_{\hat{1}}$. If $\hat{\eta}$ is surjective, we have $f = \hat{\eta}(r)$ for a suitable $r \in R_{\hat{1}}$. Then we get :

$$1 = f(X^{d-1}) = (\hat{\eta}(r))(X^{d-1}) = rX^{d-1} \quad , \quad \text{contradiction} \; .$$

Therefore, in view of Theorem 3.4, the graded functors Induction and Coinduction are not isomorphic. Note that each $R_{\hat{k}}$ is a free $R_{\hat{0}}$-module with basis X^k.
Now we consider the particular case when $A = K$ is a field. Then $R = K[X]$ and

$$\hat{\eta} : K[X] \to \text{Hom}_{K[X^d]}(K[X], K[X^d]) \qquad d > 0 \; .$$

By Prop. 1.8, $\text{Im}(\hat{\eta})$ is an essential $K[X]$-submodule of $L = \text{Hom}_{K[X^d]}(K[X], K[X^d])$. Therefore L is $K[X]$-*torsion free*. On the other hand, $K[X]$, as $K[X^d]$-module, is free with $\text{rank}_{K[X^d]} K[X] = d$. It follows that also L is free over $K[X^d]$ and $\text{rank}_{K[X^d]} K[X] = d$. Hence L, as $K[X]$-module, is *finitely generated*. Since $K[X]$ is a principal ideal domain, then L, as $K[X]$-module, is free with finite basis. Let $s = \text{rank}_{K[X]} L$. Then $sd = \text{rank}_{K[X^d]} L = d$. Thus $s = 1$ and hence there is an isomorphism

$$\theta : K[X] \to \text{Hom}_{K[X^d]}(K[X], K[X^d])$$

as $K[X]$-modules. Since the ring $K[X]$ is commutative, it follows, by Theorem 3.15, that the non graded functors Induction and Coinduction are isomorphic. In particular, if $d = 2$ it is easy to show that the map

$$\theta : K[X] \to \mathrm{Hom}_{K[X^2]}(K[X], K[X^2])$$

defined by setting

$$(\theta(r))(s) = r_0 s_1 + r_1 s_0$$

where $r = r_0 + r_1 X,\ s = s_0 + s_1 X,\ r_0, r_1, s_0, s_1 \in K[X^2],$ is an isomorphism in $K[X]$-mod. Note that, using these notations, the $K[X]$-$K[X]$-bilinear map associated to θ is

$$\begin{array}{cccc} [-,-] & : & K[X] \otimes_{K[X^2]} K[X] & \to & K[X^2] \\ & & r \otimes s & \mapsto & r_0 s_1 + r_1 s_0 \end{array}$$

It follows that the restriction of θ to $M \otimes_{K[X^2]} M$, where $M = K[X^2]X$, is the 0-map.

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Claudia MENINI
Dipartimento di Matematica
Università di Ferrara
Via Machiavelli 35
I-44100 Ferrara (ITALY)

Constantin NASTASESCU
Universitatea Bucaresti
Facultatea de Matematica
Str. Academici 14
R-70109 Bucarest 1(ROMANIA)