Generating Long Root Subgroup Geometries of Classical Groups Over Finite Prime Fields

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Abstract

The generating rank is determined for the (long) root subgroup geometries of $SL(n,\mathbb{F}), \Omega^+(2n,\mathbb{F})$, $\Omega(2n+1,\mathbb{F})$, and $\Omega^-_{2n}(\mathbb{F})$ where $\mathbb{F}$ is a finite prime field. In each instance the generating rank is equal to the universal embedding dimension. We also include a survey of other Lie incidence geometries for which the generating rank is known.

1 Introduction

We assume the reader is familiar with the basic definitions related to a *linear incidence system* or *point-line geometry*, $\Gamma = (P, L)$. As a standard reference see [4]. In particular, the concepts of *a subspace* in Γ, and the subspace $\langle X\rangle_\Gamma$ *generated* by a subset X of P. We define the *generating rank*, $gr(\Gamma)$, of a point-line geometry Γ to be $\min\{|X| \subset P | \langle X\rangle_\Gamma = P\}$, that is, the minimal cardinality of a generating set of Γ.

We further assume familarity with the concept of a *projective embedding* $e : P \to \mathbb{PG}(V)$ of a point-line geometry $\Gamma = (P, L)$ as well as the notion of a *relatively universal embedding*. We say that Γ is *embeddable* if some projective embedding of Γ exists. When this is the case we shall define the *embedding rank*, $er(\Gamma)$, of Γ to

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the maximal dimension of a vector space V for which there exists an embedding into $\mathbb{PG}(V)$. An immediate consequence of these definions is the following:

Lemma 1.1. *Let $\Gamma = (P, L)$ be an embeddable point-line geometry and let $e : P \to \mathbb{PG}(V)$ be an embedding.*

(i) $dim(V) \leq gr(\Gamma)$. Consequently, $er(\Gamma) \leq gr(\Gamma)$.

(ii) If $dim(V) = gr(\Gamma)$ then e is relatively universal.

When the generating and embedding ranks are equal we say that the geometry Γ has a *basis* (see [11]). In this paper we shall investigate several classes of embeddable point-line geometries, specifically the (long) root subgroup geometries of the classical groups - $SL(n\mathbb{F}), \Omega^{\pm}(2n, \mathbb{F})$, and $\Omega(2n+1, \mathbb{F})$ - when the field $\mathbb{F}$ is finite of prime order, and determine their generating ranks. Specifically, we prove

Theorem 1.2. *Let $\mathbb{F}$ be a finite field of prime order.*

(i) For $n \geq 3$ the generating rank of the root subgroup geometry of $SL(n, \mathbb{F})$ is $n^2 - 1$.

(ii) Let $G = \Omega(k, \mathbb{F})$ be a orthogonal group of isometries for a non-singular orthogonal space of dimension k and Witt index $n \geq 3$ (so $k \in \{2n, 2n+1, 2n+2\}$. Then the generating rank of the long root subgroup geometry of G is $\binom{n}{2}$.

In all cases the generating and embedding ranks are equal. As we shall see it is quite clear that these geometries have embeddings with the respective dimensions n^2-1, $\binom{k}{n}$ and consequently, our result provides a simple proof that these embeddings are relatively universal and that these are the universal embedding dimensions.

It does not appear to be the case, however, that the embedding dimension and the generating rank are always equal for embeddable geometries: at least for odd characteristic as it is likely that the usual $G_2(p)$ generalized hexagon, p an odd prime, is a counterexample. However, on the basis of the evidence collected thus far for the prime $p = 2$ we do make the following

Conjecture 1.3. *If $\Gamma = (P, L)$ is a finite embeddable $\mathbb{F}_2$ geometry then the embedding rank of Γ is equal to the generating rank of Γ: $gr(\Gamma) = er(\Gamma)$.*

The outline of this paper is as follows: In section two we record some lemmas on the generation of projective spaces and orthogonal polar spaces which will be used in the subsequent sections. In section three we very briefly review the definition of the long root subgroup geometry of a Chevalley group and identify the long root subgroups in the special linear and orthogonal groups. In section four we investigate the generation of the root subgroup geometry of the group $SL(n, \mathbb{F}), \mathbb{F}$ a finite prime field. In section five we determine the generating rank of the long root subgroup geometry of $\Omega^+(2n, \mathbb{F})$. The following section deals with the generation of the root subgroup geometry of $\Omega(2n+1,\mathbb{F})$. Section seven deals with the case of $\Omega^-(2n,\mathbb{F})$. Finally, in section eight we survey all further instances known to us in which the generating rank of a Lie incidence geometry has been determined.

2 Generating Projective Space and Classical Polar Spaces

Here we record some basic lemmas on the generation of an $n-1$ dimensional projective space and the polar space of singular points and totally singular lines in orthogonal space. The first lemma is obvious:

Lemma 2.1. *Let V be an $n-$dimensional vector space over an arbitrary field. Let P consist of the one dimensional spaces and L the two dimensional spaces, the latter identified with the elements of P which it contains. Then $er(P,L)=n$.*

We next take up the generation of non-degenerate orthogonal spaces and for completeness include a general definition and recall some basic concepts. Thus, an *orthogonal space* consists of a pair (V,Q) where V is a vector space over a field $\mathbb{F}$ and $Q:V\to\mathbb{F}$ is a *quadratic form*, that is, it satisfies

(1) $Q(\alpha v)=\alpha^2 Q(v)$ for all $\alpha\in\mathbb{F}, v\in V$; and

(2) The map $(,):V\times V\to\mathbb{F}$ defined by $(v,w)=Q(v+w)-Q(v)-Q(w)$ is a symmetric bilinear form.

We say two vectors are *perpendicular* or *orthogonal* and write $v\perp w$ if $(v,w)=0$. We say that Q is *non-degenerate* if $(,)$ is non-degenerate, that is, if for every $v\in V, v\neq 0$ there is a $w\in V$ such that $(v,w)\neq 0$. We say that Q is *non-singular* if either it is non-degenerate or if $\dim V=2k+1$ and there is a unique one dimensional space $\langle v\rangle$ with $v\perp V$ and $Q(v)\neq 0$. This case can only occur if the characteristic of $\mathbb{F}$ is two.

For a subspace U we set $U^\perp=\{v\in V|v\perp u,\forall u\in U\}$. A subspace U of V is *(totally) singular* if $Q(u)=0$ for all $u\in U$. Assume that $n\geq 3$. It is well known that the dimension of a totally singular subspace cannot exceed $[\frac{dimV}{2}]$ and all maximal singular subspaces have the same dimension, called the *Witt index* of the form. We will be concerned here with three types of non-singular orthogonal spaces and since our ground field is finite we may assume it takes one of the following forms:

(1) (V,Q) is non-degenerate, dim $V=2n$ is even, the Witt index is n. In this case we can find a basis $x_i,y_i,1\leq i\leq n$ of singular vectors such that $x_i\perp x_j, x_i\perp y_j, y_i\perp y_j, i\neq j, (x_i,y_i)=1$. This is called a *hyperbolic space* and such a basis is referred to as a *hyperbolic basis.*

(2) (V,Q) is non-singular, dim $V=2n+1$ is odd, the Witt index is n. In this case we can find a basis of singular vectors x_i,y_i, $1\leq i\leq n-1$, z_j, $1\leq j\leq 3$ such that $x_i\perp x_j,y_j,z_k; y_i\perp y_j,z_k$ for $i\neq j, k=1,2,3$, $(x_i,y_i)=1$, $1\leq i\leq n-1$, and $(z_1,z_2)=(z_1,z_3)=(z_2,z_3)=1$.

(3) (V, Q) is non-degenerate, dim $V = 2n + 2$ even, the Witt index is n. There is then a basis of singular vectors x_i, y_i, $1 \leq i \leq n-1$, z_j, $1 \leq j \leq 4$ such that $x_i \perp x_j, y_j, z_k; y_i \perp y_j, z_k$ for $i \neq j, k = 1,2,3,4$; $(x_i, y_i) = 1$, $1 \leq i \leq n-1$, $(z_i, z_j) = 1$ for $1 \leq i < j \leq 4$ and $(i,j) \neq (3,4)$, and $(z_3, z_4) = d \neq 0$.

The orthogonal polar space has as its points the collection P of singular one dimensional subspaces and as lines the set L of totally singular two dimensional subspaces. In the next three lemmas we show the bases given above, in the respective cases, span a set of singular points which generate the respective geometries. The first result is well known and can be found in [11] but we include a proof for completeness.

Lemma 2.2. *Assume (V, Q) is as in (1) with $n \geq 2$. Then $\{\langle x_i \rangle, \langle y_i \rangle | 1 \leq i \leq n\}$ generates (P, L).*

Proof: Let Z be the set of points spanned by the basis in (1) and let S be the subspace of the polar geometry (P, L) generated by these points. Set $X = \langle x_i | 1 \leq i \leq n \rangle, Y = \langle y_i | 1 \leq i \leq n \rangle$. Then $V = X \oplus Y$. X, Y are maximal totally singular subspaces. By (2.1) every point of X and every point of Y is in S. Now let $\langle u \rangle$ be an arbitrary singular point. Then there are unique $x \in X, y \in Y$ such that $u = x + y$. Now $0 = Q(u) = (x, y)$ so that $x \perp y$. Then $\langle u \rangle$ lies on the line $\langle x, y \rangle \subset S$. ∎

Lemma 2.3. *Assume that (V, Q) is as in (2) with $n \geq 2$. Then $\{\langle x_i \rangle, \langle y_i \rangle | 1 \leq i \leq n-1\} \cup \{\langle z_j \rangle | j = 1,2,3\}$ generates (P, L).*

Proof: Let $Z\{\langle x_i \rangle, \langle y_i \rangle | 1 \leq i \leq n-1\} \cup \{\langle z_j \rangle | j = 1,2,3\}$ and S be the subspace of P spanned by Z. Let $X = \langle x_i | 1 \leq i \leq n-1 \rangle, Y = \langle y_i | 1 \leq i \leq n-1 \rangle$. For $\{i, j, k\} = \{1, 2, 3\}$ set $U_i = X \oplus Y \oplus \langle z_j, z_k \rangle$. The orthogonal spaces $(U_i, Q|U_i)$ are of type (1). By 2.3 every singular point in $U_i, i = 1,2,3$ is in S. Now suppose $\langle u \rangle$ is an arbitrary singular point. Without loss of generality we may assume u does not belong to $U_i, i = 1,2,3$. Now dim $u^\perp$ is four and, consequently, $u^\perp$ is not contained in $X \oplus Y + \langle u \rangle$. Since $u^\perp / \langle u \rangle$ is non-singular there must be a singular point in $\langle w \rangle$ in $u^\perp$ such that $\langle u, w \rangle \cap$ $[(X \oplus Y) + \langle u \rangle] = \langle u \rangle$. Now the line $\langle u, w \rangle$ intersects each of $U_i, i = 1,2,3$ in a point. These points cannot all be identical since $U_1 \cap U_2 \cap U_3 = X \oplus Y$. Suppose $U_i \cap \langle u, w \rangle = \langle v_i \rangle$, $U_j \cap \langle u, w \rangle = \langle v_j \rangle$ are distinct for a some pair $i \neq j \in \{1,2,3\}$. Then $\langle v_i \rangle, \langle v_j \rangle \in S$ and consequently $\langle u \rangle \in \langle u, w \rangle = \langle v_i, v_j \rangle \subset S$. ∎

Remark: Over a finite field the point-line geometry of any two non-singular orthogonal spaces of odd dimension are isomorphic.

Lemma 2.4. *Assume (V, Q) is as in (3). Let $Z = \{\langle x_i \rangle, \langle y_i \rangle | 1 \leq i \leq n-1\} \cup \{z_1, z_2, z_3, z_4\}$ and let S be the subspace of P generated by Z. Then $S = P$.*

Proof: Set $X = \langle x_i | 1 \leq i \leq n-1 \rangle, Y = \langle y_i | 1 \leq i \leq n-1 \rangle$. For $j \in \{1,2,3,4\}$ set $U_j = X \oplus Y \oplus \langle z_k | 1 \leq k \leq 4, k \neq j \rangle$. Then each U_j is non-degenerate. By (2.3) and the above remark every singular point contained of $U_j, j = 1, 2, 3, 4$ is contained in S. Also, $\cap_{j=1}^{4} U_j = X \oplus Y$. Now let $\langle u\rangle$ be an arbitrary singular point. Without loss of generality $\langle u\rangle$ does not belong to any of the U_j. As argued in (2.3) there must be a singular vector w such that $\langle u, w\rangle \cap [X \oplus Y + \langle u\rangle] = \langle u\rangle$. Then $\langle u, w\rangle$ cannot meet all U_j in the same point and by an argument similar to that in (2.3) we get $\langle u\rangle \in S$. ∎

3 The Long Root Subgroup Geometry of Classical Groups

The following is essentially (12.1) in [1] and is also proved explicitly for the exceptional groups in [9]. As an additional reference see [14].

Theorem 3.1. *Let G be any finite Chevalley group of rank at least two, other than ${}^2F_4(q)$. Let X and Y be centers of distinct long roots subgroups, of order q. Then one of the following holds:*

(i) $\langle X, Y\rangle$ is elementary Abelian and is the union of $q + 1$ long roots subgroups which pairwise intersect trivially;

(ii) $\langle X, Y\rangle$ is elementary Abelian and $X \cup Y$ are the only long root elements contained in $\langle X, Y\rangle$;

(iii) $\langle X, Y\rangle$ is isomorphic to a Sylow subgroup of order q^3 in $SL(3, q)$, $Z = Z(\langle X, Y\rangle)$ is a conjugate long root subgroup (hence conjugate to X and Y), and each of XZ, YZ are a union of $q + 1$ long roots subgroups as in (i);

(iv) $\langle X, Y\rangle \cong SL(2, q)$ (or $PSL(2, q)$ in $P\Omega^+(4, q)$).

Before proceeding we introduce some notation. If V is a vector space, $v \in V$ and $\tau : V \to V$ an endomorphism, then $[\tau, v] = \tau(v) - v = (\tau - I_V)(v)$. Also, by $[\tau, V]$ we shall mean the subspace of V spanned by all $[\tau, v], v \in V$.

The linear groups, $SL(n, \mathbb{F})$ and the orthogonal groups, $\Omega^{\pm}(2n, \mathbb{F}), \Omega(2n + 1, \mathbb{F})$ are Chevalley groups. We now describe the long root subgroups in $SL(n, \mathbb{F}) \cong SL(V), V$ an $n-$ dimensional vector space over $\mathbb{F}$. Thus, let $\langle v\rangle$ be a one-dimensional subspace of V and H a hyperplane containing $\langle v\rangle$. The group $\chi(\langle v\rangle, H) = \{\tau : V \to V | [\tau, V] \subset \langle v\rangle, [\tau, H] = 0\}$ is a full root subgroup. The elements of $\chi(\langle v\rangle, H)$ are called *transvections* with *axis* H and *center* $\langle v\rangle$. Two such subgroups, $\chi(\langle v_i\rangle, H_i), i = 1, 2$ bear the relation (i) of (3.1) if either $\langle v_1\rangle = \langle v_2\rangle$ or $H_1 = H_2$. In the first instance the root subgroups partitioning the subgroup they generate, $\langle \chi(\langle v_1\rangle, H_1), \chi(\langle v_1\rangle, H_2)\rangle$, are $\{\chi(\langle v_1\rangle, H) | H \supset H_1 \cap H_2\}$. In the second instance the partition is by the subgroups $\{\chi(\langle v\rangle, H_1) | \langle v\rangle \subset \langle v_1, v_2\rangle\}$.

We next describe the long root subgroup geometry of an orthogonal group $\Omega(V)$ where (V, Q) is a non-singular orthogonal space of Witt index at least three. $\Omega(V)$ will be the subgroup of $O(V) = \{\tau : V \to V | Q(\tau(v)) = Q(V), \forall v \in V\}$ generated by the root elements. Let $U = \langle v, w\rangle$ be a totally singular projective line of V. The group $\chi(U) = \{\tau : V \to V | [\tau, V] \subset U, [\tau, U^{\perp}] = 0\}$ is a long root subgroup. These are the root subgroups. Two distinct such subgroups, $\chi(U_i), i = 1, 2$, are related as in (i) of 3.1 if $U_1 \cap U_2 \neq 0$ and $U_2 \subset U_1^{\perp}$. In this case if $X = U_1 \cap U_2, Y = \langle U_1, U_2\rangle$ then the subgroup $\langle \chi(U_1), \chi(U_2)\rangle$ is partitioned by $\{\chi(U) | X \subset U \subset Y\}$ and these are the lines of the root subgroup geometry.

4 Generating the Root Subgroup Geometry of $SL(n, \mathbb{F})$

In light of the description given in section two of the root subgroup geometry of $SL(V), V$ an $n-$dimensional vector space over a field $\mathbb{F}$, we can make the following identification:

Let Π_k denote the subspaces of V of dimension k and set $\mathcal{P} = \{(\langle v\rangle, H) \in \Pi_1 \times \Pi_{n-1} | v \in H\}$. The lines are then in one-to-one correspondence with

$$\{(U, H) \in \Pi_2 \times \Pi_{n-1} | U \subset H\} \cup \{(\langle v\rangle, M) \in \Pi_1 \times \Pi_{n-2} | v \in M\}.$$

The line corresponding to the first type, $(U, H) \in \Pi_2 \times \Pi_{n-1}$ is the set $\{(\langle v\rangle, H) | v \in U\}$ and to the second type, $(\langle v\rangle, M)$, the set $\{(\langle v\rangle, H) | H \subset M\}$. Thus two points $(\langle v_i\rangle, H_i) \in \mathcal{P}, i = 1, 2$ are collinear if and only if either $\langle v_1\rangle = \langle v_2\rangle$ in which case the line on this is the set corresponding to $(\langle v_1\rangle, H_1 \cap H_2)$ or if $H_1 = H_2$ and then the line corresponds to $(\langle p_1, p_2\rangle, H_1)$. We let $\mathcal{L}$ be the set of all such lines and denote by Γ the pair $(\mathcal{P}, \mathcal{L})$. For convenience of notation we set $\Pi = \Pi_1$ and $\mathcal{H} = \Pi_{n-1}$.

It follows from [16] and [17] that for finite fields $\mathbb{F}$ the embedding rank of the root subgroup geometry of $SL(n, \mathbb{F})$ is $n^2 - 1$ and consequently the generating rank of this geometry is at least $n^2 - 1$ by (1.1).

The principal result of this section is part (i) of our main theorem:

Theorem 4.1. *Let $\mathbb{F}$ be a prime field. Then the root subgroup geometry of $SL(n, \mathbb{F})$ $= SL(V)$ has generating rank $n^2 - 1$.*

For a point $x \in \Pi$ we let $(x) = \{(x, H) | H \in \mathcal{H}, x \subset H\}$ and similarly for $H \in \mathcal{H}$ we set $(H) = \{(x, H) | x \in \Pi, x \subset H\}$. Then the geometry induced on any (x) or (H) is a projective space of rank $n - 2$ and therefore by (2.1) can be generated by $n - 1$ points. Suppose now that $v_1, v_2, \ldots, v_n$ is a basis for V. Then each of the subspaces $(\langle v_i\rangle), 1 \leq i \leq n$ is a projective space of rank $n - 2$ and can be generated by $n - 1$ points. Likewise the subspace $(\langle v_1 + v_2 + \cdots + v_n\rangle)$ can be generated by $n - 1$ points. Taking generating set of each of these we obtain $(n+1)(n-1) = n^2 - 1$ points. This will be the set which we shall show generates our geometry. That this set generates $\Gamma = (\mathcal{P}, \mathcal{L})$ when our underlying field is a prime field $\mathbb{F}$ will be a consequence of the next several lemmas. For convenience set $p_i = \langle v_i\rangle, 1 \leq i \leq n, p_{n+1} = \langle v_1 + \cdots + v_n\rangle$.

Lemma 4.2. *The subspace S of $\mathcal{P}$ generated by $(p_i), 1 \leq i \leq n+1$ contains $(\langle v_i + v_j\rangle)$ for every pair $i \neq j$.*

Proof: Without loss of generality we may assume that $i = 1, j = 2$. Set $\Omega = \{1, 2, \ldots, n\}$. First note that if $\Phi \in \Omega^{\{n-1\}}$ and $H_\Phi = \langle v_i | i \in \Phi\rangle$ then $(H_\Phi) \subset S$ since $(p_i, H_\Phi) \in S$ for each $i \in \Phi$ and these points generate (H_Φ) by (2.1). It then follows that S contains $(\langle v_1 + v_2\rangle, \langle v_1, v_2\rangle \oplus M_\Delta)$ where $\Delta \in [\Omega \setminus \{1, 2\}]^{\{n-3\}}$ and $M_\Delta = \langle v_k | k \in \Delta\rangle$. These $n - 2$ points span a hyperplane T in $(\langle v_1 + v_2\rangle)$. Now let $H = \langle v_1 + v_2, v_3, \ldots, v_n\rangle$. X contains the points $(p_i, H), 3 \leq i \leq n$, which generate a hyperplane in (H). However, (p_{n+1}) contains $(\langle v_1 + v_2 + \cdots + v_n\rangle, H)$. Therefore, $S \supset (H)$. But $(\langle v_1 + v_2\rangle, H) \in (H)$. Now this point together with T generate $(\langle v_1 + v_2\rangle)$ completing the lemma. ∎

Now, set $X_0 = \{\langle v_1\rangle, \langle v_2\rangle, \ldots, \langle v_n\rangle, \langle v_1 + v_2 + \cdots + v_n\rangle\}$. Assume that X_k has been defined for $k \in \mathbb{N}$. Then X_{k+1} will consist of all those points $\langle u\rangle \in \mathbb{PG}(V)$ for which there exists a basis $u_1, u_2, \ldots, u_n$ such that $u = u_1 + u_2$ and such that $\langle u_1\rangle, \langle u_2\rangle, \ldots, \langle u_n\rangle, \langle u_1 + u_2 + \cdots + u_n\rangle \in X_k$. Let $X = \cup_{k\in\mathbb{N}} X_k$. 4.1 will be a direct consequence of

Lemma 4.3. $X = \{\langle \sum_{i=1}^n z_i v_i\rangle | (z_1, z_2, \ldots, z_n) \in \mathbb{Z}^n\}$.

Proof: For the remainder of this section let $Z = \{\langle \sum_{i=1}^n z_i v_i\rangle | (z_1, z_2, \ldots, z_n) \in \mathbb{Z}^n\}$. We begin by showing that $X \supset Z$. We prove this part of the lemma in a sequence of six steps.

(1) Let $u_1, \ldots, u_n$ be a basis for V such that $\langle u_i\rangle \in X$ for each $i, 1 \leq i \leq n$ and $\langle u_1 + \ldots u_n\rangle \in X$. Then, for any subset $I \subset \{1, 2, \ldots, n\}, \langle \sum_{i\in I} u_i\rangle \in X$.

Let $k(I) = min\{|I|, n - |I|\}$. We do induction on k. In the initial case, $k = 0$, by hypothesis $\langle u_1 + u_2 + \ldots u_n\rangle \in X$. Assume then $k(I) = k$ and that the result is true for all subsets J of $\{1, 2, \ldots, n\}$ with $k(J) < k$. Suppose first that $|I| = k$. Without loss of generality we may assume that $I = \{1, 2, \ldots, k\}$ and we can assume that $k > 2$ since $\langle u_1 + u_2\rangle \in X$. By hypothesis $\langle u_1 + \ldots u_{k-1}\rangle, \langle u_{k-1} + \ldots u_n\rangle \in X$. Set $w_1 = u_1 + \ldots u_{k-1}, w_2 = u_k$. For $3 \leq i \leq n - k + 2$ set $w_i = u_{k+i-2}$. For $n - k + 3 \leq i \leq n$ set $w_i = -u_{i+k-n-2}$. Then $w_1, \ldots, w_n$ is a basis for V and $\langle w_i\rangle \in X, 1 \leq i \leq n$. Moreover, $w_1 + \ldots w_n = u_{k-1} + \ldots u_n$ and consequently $\langle w_1 + \ldots w_n\rangle \in X$. Therefore, $\langle u_1 + \ldots u_k\rangle = \langle w_1 + w_2\rangle \in X$.

On the other hand, assume that $k = n - |I|$. Without loss of generality we may assume that $I = \{1, 2, \ldots, n - k\}$. By the inductive hypothesis $\langle u_1 + \ldots u_{n-k} + u_{n-k+1}\rangle \in X$ and
$lngu_1 + \ldots u_{k-1}\rangle \in X$. Now set $w_1 = u_1 + \ldots u_{n-k} + u_{n-k+1}, w_2 = -u_{n-k+1}$. For $3 \leq i \leq n-k+1$ set $w_i = -u_{n-k+3-i}$ and for $n-k+2 \leq i \leq n$ set $w_i = u_{2n+2-k-i}$. Then $w_1, \ldots, w_n$ is a basis for $V, \langle w_i\rangle \in X$ and $\langle w_1 + w_2 + \ldots w_n\rangle = \langle u_{n-k+2} + \ldots u_n\rangle \in X$. Then $\langle u_1 + \ldots u_{n-k}\rangle = \langle w_1 + w_2\rangle \in X$. ∎

(2) If $u_1, \ldots, u_n$ is a basis for V and $\langle u_1 + \ldots u_n\rangle, \langle u_i\rangle \in X, 1 \leq i \leq n$ then $\langle u_i - u_j\rangle$ for $i \neq j$.

Without loss of generality it suffices to prove that $\langle u_1 - u_2\rangle \in X$. Now $w_1 = u_1 + u_3, w_2 = -(u_2 + u_3), w_3 = u_2, w_4 = u_4, \ldots, w_n = u_n$ is a basis for V. The sum of these vectors is $u_1 + u_3 + u_4 + \cdots + u_n$ and by (1) $\langle u_1 + u_3 + u_4 + \cdots + u_n\rangle \in X$. Therefore $\langle w_1 + w_2\rangle = \langle u_1 + u_3 + (-u_2 - u_3)\rangle = \langle u_1 - u_2\rangle \in X$.

For $z \in \mathbb{Z}^n$ set $m(z) = \max\{|z_i| : 1 \le i \le n\}$; $l(z) = |\{i|z_i \ne 0, 1 \le i \le n\}|$; and $w(z) = \sum_{i=1}^n |z_i|$. We now proceed to prove the lemma by induction on $w(z)$.

(3) Assume $m(z) = 1$. We know the result is true if $l(z) = 2$ by (2). Therefore we may assume that $l(z) > 2$. Set $p(z) = |\{i|z_i > 0\}|, n(z) = |\{i|z_i < 0\}|$. By (1) we may assume that $p(z)n(z) \ne 0$. Without loss of generality we may assume that $n(z) \ge p(z)$ so $n(z) > 1$.

Suppose $p(z) = 1, n(z) = 2$. Without loss of generality we can take $z = (1, -1, -1, 0, 0, \ldots, 0)$. Now set $u_1 = v_1 - v_2; u_i = -v_{i+1}, 2 \le i \le n-1, u_n = -v_1$. Then $u_1, \ldots, u_n$ are independent, $\langle u_i\rangle \in X$. Additionally, $\langle u_1 + u_2 + \cdots + u_n\rangle = \langle -(v_2 + v_3 + \cdots + v_n)\rangle \in X$ by (1). Then $\langle v_1 - v_2 - v_3\rangle = \langle u_1 + u_2\rangle \in X$.

Suppose now that $p(z) = n(z) = 2$. Without loss of generality we can take $z = (1, 1, -1, -1, 0, \ldots, 0)$. Now set $u_1 = v_1 + v_2 - v_3; u_i = -v_{i+2}, 2 \le n-2; u_{n-1} = -(v_1 + v_2), u_n = -(v_3 + v_4 + \cdots + v_n)$. Then $u_1, \ldots, u_n$ is independent and $\langle u_i\rangle \in X$. Now $\sum_{i=1}^n u_i = -2(v_3 + v_4 + \cdots + v_n)$ and consequently $\langle \sum_{i=1}^n u_i\rangle \in X$. Thus $\langle v_1 + v_2 - v_3 - v_4\rangle = \langle u_1 + u_2\rangle \in X$.

We may assume $n(z) > 2$. Set $p(z) = s, n(z) = t$. Without loss of generality $z = (1, 1, \ldots, 1, -1, -1, \ldots, -1, 0, \ldots, 0)$. Note that $z_{s+t-2} = -1$. Now set $u_1 = v_1 + \ldots v_s - v_{s+1} - \cdots - v_{s+t-1}; u_i = -v_{s+t+i-2}, 2 \le i \le n+2-s-t; u_{n+2-s-t+j} = -v_j, 1 \le j \le s+t-3$, and $u_n = -(v_{s+t-2} + v_{s+t-1} + \ldots v_n)$. Then $u_1, \ldots, u_n$ are independent and $\langle u_i\rangle \in X$. $\sum_{i=1}^n u_i = -2(v_{s+t-2} + v_{s+t-1} + \cdots + v_n)$ and hence by (1) $\langle \sum_{i=1}^n u_i\rangle \in X$. Then $\langle v_1 + v_2 + \cdots + v_s - v_{s+1} - v_{s+2} - \cdots - v_t\rangle = \langle u_1 + u_2\rangle \in X$.

(4) May now assume that $m = m(z) > 1$. By reordering if necessary we may assume that $|z_1| \ge |z_2| \ge \ldots |z_n|$ and also if $|z_i| = |z_{i+1}|$ then $z_i \ge z_{i+1}$. We may also assume that $z_1 > 0$ so that $z_1 = m$.

Assume first that $|z_2| = m$. In this case set $u_1 = \sum_{i=1}^n z_i v_i - (z_1 v_1 + z_2 v_2)$. Note that $w(0, 0, z_3, z_4, \ldots, z_n) < w(z)$ so that by induction $\langle u_1\rangle \in X$. Next set $u_2 = z_1 v_1 + z_2 v_2 = m(v_1 \pm v_2)$ so that, by (2), $\langle u_2\rangle \in X$. Now for $3 \le i \le n$ let $u_i = -z_i v_i$ if $z_i \ne 0$ and $m v_i$ if $z_i = 0$. Then $u_1, \ldots, u_n$ is independent and for each $i, \langle u_i\rangle \in X$. Also $\sum_{i=1}^n u_i = m(v_1 \pm v_2 + \sum_{j>2, z_j=0} v_j)$ and by (3) $\langle \sum_{i=1}^n u_i\rangle \in X$. Then $\langle \sum_{i=1}^n z_i v_i\rangle = \langle u_1 + u_2\rangle \in X$.

(5) We may now assume that $|z_2| < m$. Suppose that $l(z) = 2$. We first treat the case that $m = z_1 = 2$. Take $u_1 = v_1 + z_2 v_2, u_2 = -v_1, u_i = z_2 v_i, i > 2$. Then the u_i are independent, $\langle u_i\rangle \in X$, and $\langle \sum_{i=1}^n u_i\rangle = \langle \pm (v_2 + v_3 + \cdots + v_n)\rangle \in X$ by (1). Then by (3) $\langle z_1 v_1 + z_2 v_2\rangle = \langle 2v_1 + z_2 v_2\rangle = \langle u_1 - u_2\rangle \in X$ by (3).

So now assume that $z_1 > 2$. Set $u_1 = (z_1 - 1)v_1, u_2 = -v_1 - z_2 v_2, u_3 = z_2 v_2 + (z_1 - 2)v_3$, and for $i > 3$ set $u_i = (z_1 - 2)v_i$, We remark that $w((-1, -z_2, 0, \ldots, 0))$,

$w((0,z_2,z_1-2,0,\dots,0)) < w((z_1,z_2,0,\dots,0))$ and so by our inductive hypothesis $\langle u_2\rangle,\langle u_3\rangle \in X$. Since $u_1,u_i, i>3$ are multiples of v_j all $\langle u_i\rangle \in X$. Furthermore, since $z_1>2$, $u_1,\dots,u_n$ are independent. Note that $\sum_{i=1}^n u_i = (z_1-2)(v_1+v_3+v_4+\cdots+v_n)$ so that $\langle \sum_{i=1}^n u_i\rangle \in X$ by (1). By (3) $\langle z_1v_2+z_2v_2\rangle = \langle u_1-u_2\rangle \in X$.

(6) We now complete the inclusion $X\supset Z$ by doing induction on $l(z)$. The case $l(z)=1$ is satisfied by hypothesis and $l(z)=2$ in the previous step. Assume that $t\in\mathbb{N}, t>2$ and that for all $z=(z_1,z_2,\dots,z_n)\in\mathbb{Z}^n$ if $l(z)<t$ then $\langle \sum_{i=1}^n z_iv_i\rangle \in X$. Now assume $z=(z_1,z_2,\dots,z_n)\in\mathbb{Z}^n$ satisfies $l(z)=t$. We know that $m=m(z)>1$. For $1\le i\le t$ let $a_i = z_t\frac{z_i}{|z_i|}$. Set $u_1=\sum_{i=1}^t z_iv_i - \sum_{i=1}^t a_iv_i$ and $u_2=\sum_{i=1}^t a_iv_i$. Further, for $3\le i\le t$ set $u_i=-(z_i-a_i)v_{i-1}$ if $z_i-a_i\neq 0$, alternatively, $u_i=(z_1-a_1)v_{i-1}$ if $z_i-a_i=0$. For $t+1\le i\le n$ set $u_i=(z_1-a_1)v_i$. Then $\langle u_1\rangle\in X$ since $l((z_1-a_1,\dots,z_{t-1}-a_{t-1},0,\dots,0))<l(z)$. Also, $\langle u_2\rangle\in X$ since $w((a_1,a_2,\dots,a_t,0,\dots,0))<w(z)$. Clearly for $i\ge 3, \langle u_i\rangle\in X$. Furthermore, $\langle \sum_{i=1}^n u_i\rangle \in X$ by (4) since it is a multiple of a vector $\sum_{i=1}^n e_iv_i$ with $e_i\in\{0,-1,1\}$. Consequently, $\langle \sum_{i=1}^t z_iv_i\rangle = \langle u_1+u_2\rangle \in X$. This completes the proof that $X\supset Z$.

(7) Finally, we prove that $X\subset Z$. Of course, if $\mathbb{F}$ is a prime field (including the rational numbers) then this is obvious and in that case we have already proved (4.1). However since it is of interest to see what subspace of $\mathcal{P}$ is generated by $\cup_{i=1}^{n+1}(p_i)$ we deal with the general case of an arbitrary field. In that case, let $\mathbb{F}_0$ be its prime subfield. Suppose to the contrary that X is not contained in Z. Let $m\in\mathbb{N}$ be minimal such that there exists a point $\langle u\rangle\in X_m, \langle u\rangle\notin Z$. Clearly $m>0$. Then there is a basis $u_1,\dots,u_n$ for V such that $\langle u_i\rangle\in X_{m-1}, 1\le i\le n$, $\langle u_1+u_2+\cdots+u_n\rangle\in X_{m-1}$, and $\langle u\rangle=\langle u_1+u_2\rangle$. Since $m-1<m$ there are vectors $x_i\in Z$ and scalars $a_i\in\mathbb{F}$, $1\le i\le n+1$ such that $u_i=a_ix_i, 1\le i\le n$, $a_{n+1}x_{n+1}=u_1+u_2+\cdots+u_n$. Set $x=x_{n+1}, a=a_{n+1}$.

Since the $x_j\in Z$ they are each a $\mathbb{F}_0$ linear combinations of $v_1,\dots,v_n$ say

$$x_j=\sum_{i=1}^n c_{ij}v_i$$

where $c_{ij}\in\mathbb{F}_0$, $1\le j\le n$. Likewise x is a $\mathbb{F}_0$ linear combination of $v_1,\dots,v_n:$

$$x=\sum_{i=1}^n d_iv_i.$$

Now set $\alpha=(a_1,a_2,\dots,a_n)^T, C=(c_{ij})$ and $d=(d_1,\dots,d_n)^T$. We then have the matrix equation $C\alpha=ad$. However, since C is non-singular with entries in $\mathbb{F}_0$ and d has entries in $\mathbb{F}_0$ there is a unique solution $\beta\in\mathbb{F}^n$ such that $C\beta=d$. It therefore follows that if $\beta=(b_1,\dots,b_n)^T$ then $a_i=ab_i$. However, in this case, $u=u_1+u_2=a_1x_1+a_2x_2=ab_1x_1+ab_2x_2$. Suppose now that $\mathbb{F}_0$ is a finite field. Then there are integers $b_i', i=1,2$ so that $b_i'.1_{\mathbb{F}}=b_i$ In this case $\langle u_1+u_2\rangle = \langle ab_1x_1+ab_2x_2\rangle = \langle b_1x_1+b_2x_2\rangle = \langle b_1'x_1+b_2'x_2\rangle \in Z$. When $\mathbb{F}_0$ is the rationals there are integers $b_i', i=1,2$ and f such that $b_i=\frac{b_i'}{f}, i=1,2$. Then $\langle v_1+u_2\rangle = \langle ab_1x_1+ab_2x_2\rangle = \langle \frac{a}{f}(b_1'x_1+b_2'x_2)\rangle = \langle b_1'x_1+b_2'x_2\rangle \in Z$, again a contradiction. This completes the proof. ∎

5 The orthogonal geometry $\Omega^+(2n, \mathbb{F})$

In this section we consider the root subgroup geometry of an orthogonal space of type (1) that is non-degenerate of even dimension $2n$ and Witt index n. As in the previous section we will work with an equivalent geometry which allows us to express the points and lines in terms of the underlying orthogonal space rather than subgroups in the orthogonal group. The identification established here will apply in the following two sections as well.

We continue with the notation of sections two and three so that (V, Q) is an orthogonal space of one of the types (1) - (3) over a field $\mathbb{F}$ and has Witt index at least three. Let Π_k denote the set of totally singular subspaces of V of dimension k. Set $\mathcal{P} = \Pi_2$, the totally singular projective lines. These are the points of this equivalent geometry. Two distinct such "points" l, m will be "collinear" in this geometry when $l \cap m \neq 0$ an $m \subset l^\perp$. The "line" on this pair is then the set $\{l' \in \mathcal{P} | l \cap m \subset l' \subset \langle l, m \rangle\}$. We let $\mathcal{L}$ denote the set of lines and Γ the pair $(\mathcal{P}, \mathcal{L})$.

We now specialize, for the remainder of this section, to the case that (V, Q) is of type (1) and dimension of V is $2n \geq 6$. It is well known that the module $\wedge^2(V)$ for the group $\Omega^+(2n, \mathbb{F}) = \{\sigma : V \to V | Q(\sigma(v)) = Q(V), \forall v \in V\}$ is of dimension $2n^2 - n$ and affords an embedding for this geometry (when the characteristic of $\mathbb{F}$ is not two this module is irreducible and isomorphic to the adjoint module of $\Omega^+(2n, \mathbb{F})$). By (1.1) it is clearly the case that the generating rank of $\Omega^+(2n, \mathbb{F})$ is at least $2n^2 - n$.

The main objective of this section is the proof of the following which deals with one of the cases of our main theorem

Theorem 5.1. *For a prime field $\mathbb{F}$ the generating rank of $\Omega^+(2n, \mathbb{F})$ is $2n^2 - n$.*

Proof: For a singular point x let $(x) = \{l \in \mathcal{P} | x \subset l\}$. This is a subspace of $\Gamma = (\mathcal{P}, \mathcal{L})$ and is isomorphic to the polar space of singular points and singular lines in an orthogonal space $\Omega^+(2n-2, \mathbb{F})$. By 2.3 this can be generated by $2n-2$ points, consisting of a set of lines $l_i, m_i, i = 1, 2, \ldots, n-1$ such that the points $l_i/x, m_i/x$ is a hyperbolic basis for $x^\perp/x$.

Consider now the following set, Z, of singular lines from V :

$$\langle x_i, x_j \rangle, \langle y_i, y_j \rangle, \langle x_i, y_j \rangle, i \neq j;$$

$$\langle x_1 + x_2 + \cdots + x_n, y_1 - y_i \rangle, i = 2, \ldots, n; \langle x_1 - y_2, y_1 + x_2 \rangle.$$

The number of such lines is $2 \times \binom{n}{2} + n(n-1) + (n-1) + 1 = 2n(n-1) + n - 1 + 1 = 2n^2 - n$. We will show these lines generate Γ when $\mathbb{F}$ is a prime field. We proceed by induction. Let S denote the subspace of $\mathcal{P}$ spanned by Z.

Assume first that $n = 3$. Let U be a vector space over $\mathbb{F}$ of dimension four with basis u_1, u_2, u_3, u_4. In the space $\bigwedge^2(U)$ let $u_{ij} = u_i \wedge u_j, i < j$. This is a six dimensional space. The map $\tilde{Q} : \wedge^2(U) \to \mathbb{F}$ given by

$$\tilde{Q}(\sum_{i<j} \alpha_{ij} u_{ij}) = \alpha_{12}\alpha_{34} - \alpha_{13}\alpha_{24} + \alpha_{14}\alpha_{23}$$

is a hyperbolic quadratic form on $U_2 = \wedge^2(U)$. Thus, U_2 and V are isomorphic orthogonal spaces and can be identified. Under this identification there is an isomorphism between the geometry of the previous section whose points consisted of pairs (u, H) where u is a projective point in U and H is a hyperplane containing u and the geometry of singular lines of $U_2 = \wedge^2(U)$. It is given by the map which takes (x, H) to $x \wedge H$.

Next note that the basis $u_{14}, u_{23}; u_{24}, -u_{13}; u_{34}, u_{12}$ is a hyperbolic. Therefore we may make the identification $x_i = u_{i4}, i = 1, 2, 3; y_1 = u_{23}, y_2 = -u_{13}, y_3 = u_{12}$. Let $I = \{1, 2, 3, 4\}$ and for $J \subset I^{\{3\}}$ set $U_J = \{u_j | j \in J\}$. Also denote by u the vector $u_1 + u_2 + u_3 + u_4$. From the preceeding section we know that the root subgroup geometry of $SL(4, \mathbb{F})$ is generated by $\{(\langle u_i \rangle, U_J) | i \in J \in I^{\{3\}}\}$ together with any three points which generate $(\langle u \rangle)$, in particular, the points $(\langle u \rangle, \langle u_1 + u_2, u_3, u_4 \rangle)$, $(\langle u \rangle, \langle u_1, u_2 + u_3, u_4 \rangle)$, and $(\langle u \rangle, \langle u_1, u_2, u_3 + u_4 \rangle)$. Under the identification of the u_{ij} with the x_i, y_j the first twelve lines are just the $\langle x_i, x_j \rangle, \langle y_i, y_j \rangle, \langle x_i, y_j \rangle$. For example, $(\langle u_1 \rangle, \langle u_1, u_2, u_3 \rangle)$ is identified with $\langle u_{12}, u_{13} \rangle = \langle y_2, y_3 \rangle$. On the other hand we have the following identification of the other three points:

$$(\langle u \rangle, \langle u_1 + u_2, u_3, u_4 \rangle) \to \langle y_1 - y_2 + x_3, x_1 + x_2 + x_3 \rangle;$$

$$(\langle u \rangle, \langle u_1, u_2 + u_3, u_4 \rangle) \to \langle x_1 - y_2 + y_3, x_1 + x_2 + x_3 \rangle;$$

$$(\langle u \rangle, \langle u_1, u_2, u_3 + u_4 \rangle) \to \langle x_1 - y_2 + y_3, y_1 + x_2 - y_3 \rangle.$$

Thus, in order to prove that Z is a generating set in this case we need to show that the above three points are in S.

Set $x = x_1 + x_2 + x_3$. First note that for a given i that the four lines $\langle x_i, x_j \rangle, \langle x_i, y_j \rangle$, $j \neq i$ generate $(\langle x_i \rangle)$ by an application of (2.2). Consequently, for each i, $\langle x, x_i \rangle \in S$. By assumption $\langle x, y_1 - y_2 \rangle$, $\langle x, y_1 - y_3 \rangle \in Z$ and whence in S. As we have just seen, also $\langle x, x_2 \rangle$, $\langle x, x_3 \rangle \in S$. But these four points span $(\langle x \rangle)$ by again appealing to (2.2) and therefore $(\langle x \rangle) \subset S$. Therefore $\langle y_1 - y_2 + x_3, x_1 + x_2 + x_3 \rangle, \langle x_1 - y_2 + y_3, x_1 + x_2 + x_3 \rangle \in S$. Now by assumption $\langle x_1 - y_2, y_1 + x_2 \rangle \in Z$. Since $(\langle x_3 \rangle) \subset S$, in particular $\langle x_3, x_1 - y_2 \rangle$, $\langle x_3, y_1 + x_2 + x_3 \rangle \in S$. Then $\langle x_1 - y_2, y_1 + x_2 \rangle$ and $\langle x_3, x_1 - y_2 \rangle$ are collinear points in S and so S contains every point on the line spanned by these two points, and hence contains $\langle x_1 - y_2, y_1 + x_2 + x_3 \rangle$. Now S contains $\langle x_1 - y_2, y_1 + x_2 + x_3 \rangle$ and $\langle x_3, y_1 + x_2 + x_3 \rangle$ which are two collinear points and from this it follows that S contains $\langle x_1 - y_2 + y_3, y_1 + x_2 - y_3 \rangle$. This establishes the result in the case $n = 3$.

Now assume $n > 3$ and for every k, $3 \leq k < n$ the result has been established. Again let S denote the subspace spanned by Z. Set $x = x_1 + x_2 + \cdots + x_n$. Now by the argument used above $((\langle x\rangle)) \subset Y$. Consequently, for every $i \neq j, \langle x, y_i - y_j\rangle \in Y$. Set $y = y_i - y_j$. The set of points $\{\langle y, z\rangle | z \in \langle x_1, x_2, \ldots, x_n\rangle \cap y^{\perp}\}$ is a subspace of Γ and a projective space of rank $n-2$ and therefore is generated by any $n-1$ points not contained in a hyperplane. Since the points $\langle x_k, y\rangle, k \neq i, j$ are in S as is $\langle x, y\rangle$, these points generate this projective space and therefore $\langle x_i + x_j, y_i - y_j\rangle \in S$. Now the set of lines $\langle x_k, y\rangle, \langle y_k, y\rangle, k \neq i, j; \langle x_i + x_j, y\rangle, \langle y_i, y_j\rangle$ is a hyperbolic basis for $((\langle y\rangle))$ contained in S and therefore $((\langle y\rangle)) \subset S$.

Suppose $J \subset \{1, 2, \ldots, n\}$. Set $x_J = \sum_{j \in J} x_j$. By what we have shown, $\langle x_J, y_i - y_k\rangle \in Y$ for any pair $\{i, k\} \subset J$. In particular, $\langle x_1 + x_2 + x_3, y_1 - y_i\rangle \in S, i = 2, 3$. Since $\langle x_1 - y_2, y_1 + x_2\rangle \in S$ it follows from the previous case that every singular line contained in $\langle x_i, y_i | i = 1, 2, 3\rangle$ is an element of S. In particular, $\langle x_2 - y_3, y_2 + x_3\rangle, \langle x_1 - y_3, y_1 + x_3\rangle \in S$. Suppose $J \subset I, |J| = n-1$ and let $X_j = \langle x_j | j \in J\rangle, Y_J = \langle y_j | j \in J\rangle, V_J = X_J \oplus Y_J$. Aslo let $X = \langle x_j | 1 \leq j \leq n\rangle, Y = \langle y_j | 1 \leq j \leq n\rangle$. By induction every singular line contained in V_J is an element of S. It is therefore the case that for any $i \neq j, \alpha \in \mathbb{F}, ((\langle x_i + \alpha x_j\rangle)), ((\langle y_i + \alpha y_j\rangle)) \subset S$.

We now claim for any $x \in X, ((\langle x\rangle)) \subset S$ and similarly for a $y \in Y$. Suppose $x = \sum_{i=1}^{n} \alpha_i x_i$. By relabeling the indices if necessary we can assume that $\alpha_1 = 1$ and that $\alpha_i \neq 0, 1 \leq i \leq t, \alpha_i = 0, i > t$. By what has already been shown, the lines $\langle x, x_i\rangle \in S$ for $2 \leq i \leq n$. Also the lines $\langle x, y_i\rangle \in S$ for $i > t$. Additionally, by what we have thus far shown, the lines $\langle x, \alpha_i y_1 - y_i\rangle \in S$, $2 \leq i \leq t$. These form a hyperbolic basis for $((\langle x\rangle))$, hence $((\langle x\rangle)) \subset S$. Similarly, for $y \in Y$, $((\langle y\rangle)) \subset S$.

We now proceed to the general case. Let $z = x + y, x \in X, y \in Y$. Suppose for some $J \subset \{1, 2, \ldots, n\}, |J| = n-1$ that $z \in V_J$. Then by induction every line on z in V_J is contained in S. On the other hand, if $i \notin J$ then $\langle z, x_i\rangle, \langle z, y_i\rangle$ are singular lines and contained in S and together with the lines on z in V_J these generate $((\langle z\rangle))$. We now prove that there must be a $J \subset \{1, 2, \ldots, n\}, |J| = n-1$ such that $x^{\perp} \cap V_J \neq z^{\perp} \cap V_J \neq y^{\perp} \cap V_J$. Suppose to contrary. Let $J_i = \{1, 2, \ldots, n\} \setminus \{i\}, i = 1, 2$. We may suppose $x^{\perp} \cap V_{J_1} = z^{\perp} \cap V_{J_1}$. Since $X_{J_1} \subset x^{\perp}$ it follows that $y \in \langle y_1\rangle$ and therefore, since $y \neq 0$, we may assume that $y = y_1$. It then follows that $x^{\perp} \cap V_{J_2} \neq z^{\perp} \cap V_{J_2}$ and therefore $z^{\perp} \cap V_{J_2} = y^{\perp} \cap V_{J_2}$ and hence $x = \alpha x_2$ for some $\alpha \in \mathbb{F}$. But we have already seen that in this case $((\langle z\rangle)) \subset S$.

So, let J be a proper subset of $\{1, 2. \ldots, n\}$ such that $x^{\perp} \cap V_J \neq z^{\perp} \cap V_J \neq y^{\perp} \cap V_J$ and choose a vector $u \in z^{\perp} \cap V_J \setminus \langle x, y\rangle^{\perp}$. Now let $X_y = X \cap y^{\perp}$ a hyperplane in X. Choose a basis $a_1, a_2, \ldots, a_{n-1} = x$. Also, $Y \cap x^{\perp}$ is a hyperplane. For each $i < n-1$ let $A_i = \langle a_k | 1 \leq k \leq n-1, k \neq i\rangle$. Then $Y \cap A_i^{\perp}$ is a two dimensional space containing y. Let $b_i \in Y \cap A_i^{\perp}$ such that $(a_i, b_i) = 1$. Now the $2n-3$ lines $\langle z, a_i\rangle, \langle z, b_i\rangle$, $1 \leq i \leq n-2$, and $\langle x, y\rangle$, all of which contain z, span a geometric hyperplane in $((\langle z\rangle))$ namely consisting of those points which are collinear with the point $\langle x, y\rangle$. Now the point $\langle z, u\rangle$ is not contained in this subspace of $((\langle z\rangle))$. Since $u \in V_J, ((\langle u\rangle)) \subset S$. Thus, $((\langle z\rangle)) \subset S$ which completes the result. ∎

6 Generating The Root Subgroup Geometry of $\Omega(2n+1,\mathbb{F})$

We continue with the notation of sections two, three and five and let (V,Q) be a non-singular orthogonal space of type $\Omega(2n+1,\mathbb{F})$ where $\mathbb{F}$ is a finite prime field. We set $x_n = z_1, y_n = z_2, z = z_3$.

It is well known that the space $\wedge^2(V)$ for the group $\Omega(2n+1,\mathbb{F})$ of dimension $2n^2+n$ affords an embedding of this geometry (when characteristic of $\mathbb{F}$ is not two it is irreducible and isomorphic to the adjoint module) so that the generating rank of the long root subgroup geometry of $\Omega(2n+1,\mathbb{F})$ is at least $2n^2+n$.

It is the purpose of his section to prove the following part of our main theorem:

Theorem 6.1. *Let $\mathbb{F}$ be a prime field. Then the generating rank of the long root subgroup geometry of $\Omega(2n+1,\mathbb{F}), n \geq 3,$ is $2n^2+n$.*

Proof: Let Z consist of the following singular lines of Z :

$$\langle x_i, x_j\rangle, \langle y_i, y_j\rangle, \langle x_i, y_j\rangle, i \neq j;$$

$$\langle x_1 + x_2 + \cdots + x_n, y_1 - y_i\rangle, 2 \leq i \leq n; \langle x_1 - y_2, y_1 + x_2\rangle;$$

$$\langle x_i, z\rangle, \langle y_i, z\rangle, 1 \leq i < n; \langle x_1 + x_2 + \cdots + x_n, y_1 - z\rangle; \langle y_1 + y_2 + \cdots + y_n, x_1 - z\rangle.$$

Let $X = \langle x_i | 1 \leq i \leq n-1\rangle$, $Y = \langle y_i | 1 \leq i \leq n-1\rangle$ and $V_{n-1} = X \oplus Y$. Further, let $V_n = V_{n-1} \oplus \langle x_n, y_n\rangle, V_x = V_{n-1} \oplus \langle x_n, z\rangle, V_y = V_{n-1} \oplus \langle y_n, z\rangle$.

Let S be the subspace of Γ generated by Z. We show that Z contains every singular line of V. First note that by (5.1) if $u \in V_n$, is a singular vector then $\{l \in (\langle u\rangle) | l \subset V_n\}$ is contained in S. Also note that for any singular vector u that the subspace $(\langle u\rangle)$ is isomorphic to a polar space of type $\Omega(2n-1,\mathbb{F})$. Moreover, for $u \in V_n$ the subspace $\{l \in (\langle u\rangle) | l \subset V_n\}$ is a geometric hyperplane and a maximal subspace of $(\langle u\rangle)$.

We now claim for each of the vectors $x_i, y_i, 1 \leq i \leq n-1$ that $(\langle x_i\rangle), (\langle y_i\rangle) \subset S$. By what we have just stated, it suffices to prove that for each such vector there is a line on this vector in Z but not contained in V_n. Since $\langle x_i, z\rangle \in Z, \langle y_i, z\rangle \in Z, 1 \leq i \leq n-1$ this is satisfied and the claim is established. Also note that since $x_1 + x_2 + \cdots + x_n, y_1 + y_2 + \cdots + y_n \in V_n$ and $\langle x_1 + x_2 + \cdots + x_n, y_1 - z\rangle \in Z, \langle y_1 + y_2 + \cdots + y_n, x_1 - z\rangle \in Z$ it also follows by the same argument that $(\langle x_1 + x_2 + \cdots + x_n\rangle), (\langle y_1 + y_2 + \cdots + y_n\rangle) \subset S$.

We next prove that if $u_1, u_2 \in V_n, u_1 \perp u_2$ and $(\langle u_1\rangle), (\langle u_2\rangle) \subset S$ then for every vector $u \in \langle u_1, u_2\rangle, u \neq 0$ that $(\langle u\rangle) \subset S$. This follows since there are singular vectors in $\langle u_1, u_2\rangle^\perp$ which are not contained in V_n. Suppose w is such a vector. Then the lines $\langle u_i, w\rangle \in S, i = 1,2$. It then follows that $\langle u, w\rangle \in S$ and consequently there are singular lines on u which are elements of S which are not contained in V_n. As a result $(\langle u\rangle) \subset S$.

We can now complete the proof. The subgeometry of the polar space of singular points and singular lines contained in V_n is generated by the points $x_i, y_i, 1 \le i < n$ together with the two points $x_1 + x_2 + \cdots + x_n, y_1 + y_2 + \cdots + y_n$ and it is therefore the case that for every singular vector $u \in V_n, (\langle u\rangle) \subset S$. Since V_n is a hyperplane of V every singular line meets V_n and thus $S = \mathcal{P}$. This completes the proof. ■

7 Generating the Root Subgroup Geometry of $\Omega^-(2n+2,\mathbb{F}), n \geq 3$.

We continue with the notation of sections two, three and five and let (V, Q) be of type $\Omega^-(2n+2,\mathbb{F}), \mathbb{F}$ a prime field.

As in section five and six, it is well known that the space $\wedge^2(V)$ of dimension $2n^2 + 3n + 1$ (which is irreducible when characteristic of $\mathbb{F}$ is not two) affords an embedding of this geometry so that the generating rank of the long root subgroup geometry of $\Omega(2n+1,\mathbb{F})$ is at least $2n^2 + 3n + 1$.

In this section we complete the proof of our main theorem by treating the last case of an orthogonal geometry. This is achieved in

Theorem 7.1. *Let $\mathbb{F}$ be a prime field and $n \geq 3$. Then the generating rank of the long root subgroup geometry of $\Omega^-(2n+2,\mathbb{F})$, is $2n^2 + 3n + 1$.*

Proof: For a subspace M of V let $\mathcal{P}(M) = \{l \in \mathcal{P} | l \subset M\}$. Set $V_{2n-2} = \langle x_1, y_i | 1 \le i \le n-1\rangle, V_{2n-1} = V_{2n-2} \oplus \langle z_1\rangle, V_{2n} = V_{2n_1} \oplus \langle z_2\rangle$ and $U = V_{2n} \oplus \langle z_3\rangle$. U is a non-singular subspace of dimension $2n+1$. Let Z_0 be the following set of singular lines from U :

$$\langle x_i, x_j\rangle, \langle x_i, y_j\rangle, \langle y_i, y_j\rangle, i \neq j;$$

$$\langle x_i, z_j\rangle, \langle y_i, z_j\rangle, 1 \le i \le n-1, j = 1,2,3;$$

$$\langle x_1 + x_2 + \cdots + x_{n-1} + z_1, y_1 - y_i\rangle, 2 \le i \le n-1,$$

$$\langle x_1 + x_2 + \cdots + x_{n-1} + z_1, y_1 - z_2\rangle, \langle x_1 - y_2, y_1 + x_2\rangle;$$

$$\langle x_1 + x_2 + \cdots + x_{n-1} + z_1, y_1 - z_3\rangle; \langle y_1 + y_2 + \cdots + y_{n-1} + z_2, x_1 - z_3\rangle.$$

Note that there are $2n^2+n$ lines in Z_0. By the proof of (6.1) the subspace S_0 of $\mathcal{P}$ generated by Z_0 is $\mathcal{P}(U)$. We now remark that for any singular vector $u \in U$ that $(\langle u\rangle) \cap \mathcal{P}(U) = \mathcal{P}(U \cap u^\perp)$ is a subspace of $\mathcal{P}$ isomorphic to $\Omega(2n-1, \mathbb{F})$ and is a geometric hyperplane in $(\langle u\rangle)$. Therefore, if l is any singular line on u not contained in $\mathcal{P}(U)$ then the subspace spanned by l and $(\langle u\rangle) \cap \mathcal{P}(U)$ is $(\langle u\rangle)$.

Now let Z_1 be the following set of singular lines:

$$\langle x_i, z_4\rangle, \langle y_i, z_4\rangle, 1 \le i \le n-1;$$

$$\langle x_1 + z_1, y_1 - z_4\rangle, \langle x_1 + z_2, y_1 - z_4\rangle, \langle x_1 + z_3, dy_1 - z_4\rangle.$$

There are $2n+1$ lines in Z_1. Set $Z = Z_0 \cup Z_1$ and note that $|Z| = 2n^2 + 3n + 1$. Let S be the subspace spanned by Z. Next note that for each $i, 1 \le i \le n-1$ that $\langle x_i, z_4\rangle$ is a line on x_i not contained in U and so by the argument of the previous paragraph, $(\langle x_i\rangle) \subset S, 1 \le i \le n-1$. Similarly, $(\langle y_i\rangle) \subset S$. By the argument of the proof of Theorem C for every singular point $w \in V_{n-1}$, $(\langle w\rangle) \subset S$.

Now since $\langle x_1 + z_1, y_1 - z_4\rangle$ is a singular line on $x_1 + z_1 \in U$ which is not in $\mathcal{P}(U)$ it follows by the same reasoning that $(\langle x_1 + z_1\rangle) \subset S$. Since the polar space of singular points and lines in V_{2n-1} is generated by the singular points contained in V_{2n-2} together with $x_1 + z_1$ it then follows that for every singular vector in V_{2n}, S contains $(\langle w\rangle)$. Arguing in similar fashion we next get every line on $x_1 + z_2$ is contained in S and then for every singular point $w \in V_{2n}, (\langle w\rangle) \subset S$. In like fashion because the singular line $\langle x_1 + z_3, dy_1 - z_4\rangle \in Z$ we conclude that for every singular vector $w \in U, (\langle w\rangle) \subset S$. We are now done: U is a hyperplane of V. If l is a singular line $l \cap U \neq 0$. If $w \in l \cap U$ then $l \in (\langle u\rangle) \subset S$. ∎

8 Generating Sets of Lie Incidence Geometries

In this section we collect what is known about the generation of Lie incidence geometries. The first result can be found explicitly in [2], [11], and implicitly in [15].

Theorem 8.1. *(a) Let $\mathbb{F}$ be any field.*

(i) The generating rank of the Lie incidence system $A_{n,k}(\mathbb{F})$ is $\binom{n+1}{k}$.

(ii) The generating rank of the Lie incidence geometry $D_{n,n}(\mathbb{F})$ is 2^{n-1}.

(iii) The generating rank of the Lie incidence geometry $E_{6,1}(\mathbb{F})$ is 27 and the generating rank of the Lie incidence geometry $E_{7,1}(\mathbb{F})$ is 56.

(b) Assume the characteristic of $\mathbb{F}$ is not two. Then the generating rank of $B_{n,n}(\mathbb{F})$ is 2^n.

In each instance the generating rank is equal to the embedding rank.

The following results are contained in [5] and [7], respectively:

Theorem 8.2 (Cooperstein). *Let $\mathbb{F}$ be a field, $|\mathbb{F}| > 2$. Then the generating rank of the Lie incidence geometry $C_{n,n}(\mathbb{F})$, which is isomorphic to the dual polar space $DSP(2n, \mathbb{F})$ of symplectic type, is $\binom{2n}{n} - \binom{2n}{n-2}$.*

Theorem 8.3 (Cooperstein). *Let $\mathbb{F}$ be a field, $|\mathbb{F}| > 2$. Then the generating rank of the Lie incidence geometry ${}^2A_{2n-1,n}(\mathbb{F})$, which is isomorphic to the dual polar space $DSU(2n, \mathbb{F})$ of unitary type, is $\binom{2n}{n}$.*

Note that in characteristic two the geometries $B_{n,n}(\mathbb{F})$ and $C_{n,n}(\mathbb{F})$ are identical and so, except for the field of two elements, the generating rank of this geometry is known as well. Andries Brouwer has made the following:

Conjecture 8.4. *The embedding rank of the dual polar space $DSP(2n, 2)$ is $\frac{(2^n+1)(2^{n-1}+1)}{3}$.*

Brouwer has demonstrated that $er(DSP(2n, 2)) \geq \frac{(2n+1)(2^{n-1}+1)}{3}$. The cases $n \leq 5$ of this conjecture are known to be true: When $n = 2$ this incidence geometry is just the (2,2) generalized quadrangle which has embedding rank 5 and generating rank 5. Brouwer [3] has shown that the embedding rank of the dual polar space $DSP(6, 2)$ is 15 and Cooperstein and Shult prove in [10] that the generating rank is also 15. Brouwer has also demonstrated that the embedding rank of $DSP(8, 2)$ is 51. In [6] Cooperstein proves that the generating rank is 51 when $n = 4$ and 187 when $n = 5$, also settling Brouwer's conjecture affirmatively in the latter case. Thus we have

Theorem 8.5. *For $2 \leq n \leq 5$ the generating rank of the dual polar space of type $DSP(2n, 2)$ is $\frac{(2^n+1)(2^{n-1}+1)}{3}$.*

Another general gap in our knowledge exists for the dual polar spaces $DSU(2n, 2)$ of unitary type over the field with two elements. It has been conjectured [13] that the embedding rank of this geometry is $\frac{4^n+2}{3}$. The cases $n = 2, 3$ are known: When $n = 2$ this geometry is the generalized quadrangle with parameters (2,4) which consists of the singular points and lines in the orthogonal space $\Omega^-(6, 2)$ which has embedding rank and generating rank six. On the other hand, Yoshiara in [18] has shown that $DSU(6, 2)$ has embedding rank 22. The generating rank of this geometry has also been obtained (see [8])

Theorem 8.6 (Cooperstein). *The generating rank of $DSU(6, 2)$ is 22.*

Finally, Frohardt and Johnson [12] show that each of the generalized hexagons with parameters (2,2) have embedding rank 14. In [8] the generating ranks of these Lie incidence geometries are determined:

Theorem 8.7 (Cooperstein). *The generating rank of either generalized hexagon with parameters (2,2) is 14.*

Added in proof :

In "A note on Embeddable GF(2)-Geometries" Stefan Heiss has given an elegant counterexample to Conjecture 1.3.

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